Week 5 Notes

mercoledì 6 dicembre 2023 18:49

# LIFE ANNUITIES PAID MORE THAN ONCE PER YEAR

## WHOLE LIFE ANNUITY DUE paid m times per year

CHARACTERISTICS

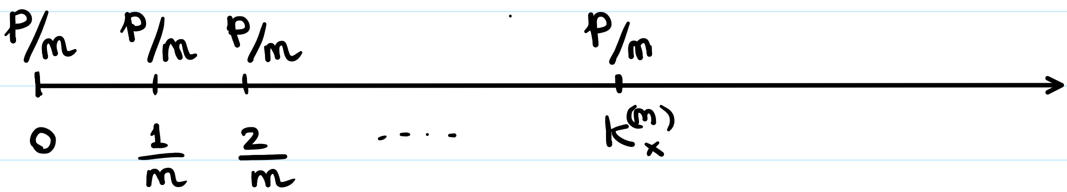
- times of payments $0, \frac{1}{m}, \frac{2}{m}, \dots, K_x^{(m)}$

  where $K_x^{(m)} = \frac{[m T_x]}{m}$

- total number of payments $m \cdot K_x^{(m)} + 1$

- amount paid in 1 year $p$

  amount paid in each installment $\frac{p}{m}$




PRESENT VALUE OF THE FLOW

$$Y = \frac{p}{m} \cdot \sum_{k=0}^{m \cdot K_x^{(m)}} v^{\frac{k}{m}} = p \cdot \frac{1}{m} \frac{1 - v^{K_x^{(m)} + \frac{1}{m}}}{1 - v^{1/m}}$$

$$= p \cdot \frac{1 - v^{K_x^{(m)} + \frac{1}{m}}}{d^{(m)}}$$

ACTUARIAL VALUE of the life annuity

$$E[Y] = p \cdot \frac{1 - E\left[v^{K_x^{(m)} + \frac{1}{m}}\right]}{d^{(m)}}$$

$$\mathbb{E}[Y] = p \cdot \frac{1 \;\cdots\; \mathbb{E}[\cdots]}{d^{(m)}}$$

$$=: \ddot{a}_x^{(m)}$$

Actuarial value of a UNIT whole life annuity due paid m times per year.

If the WHOLE LIFE ANNUITY IS IMMEDIATE

$$Y = p \cdot \left( \frac{1 + v^{K_x^{(m)} + \frac{1}{m}}}{d^{(m)}} - \frac{1}{m} \right)$$

$$\text{ACTUARIAL VALUE} = \mathbb{E}[Y] = p \cdot a_x^{(m)}$$

where $\quad a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}$

It is in general complicated to evaluate $\ddot{a}_x^{(m)}$ because that would need to know periodic survival probabilities which are not provided by life tables.
However, it can be approximated using WHOOLHOUSE'S FORMULAS

2 TERMS WHOOLHOUSE'S FORMULA

$$\ddot{a}_x^{(m)} \sim \ddot{a}_x - \frac{m-1}{2m}$$

3 TERMS WHOOLHOUSE'S FORMULA

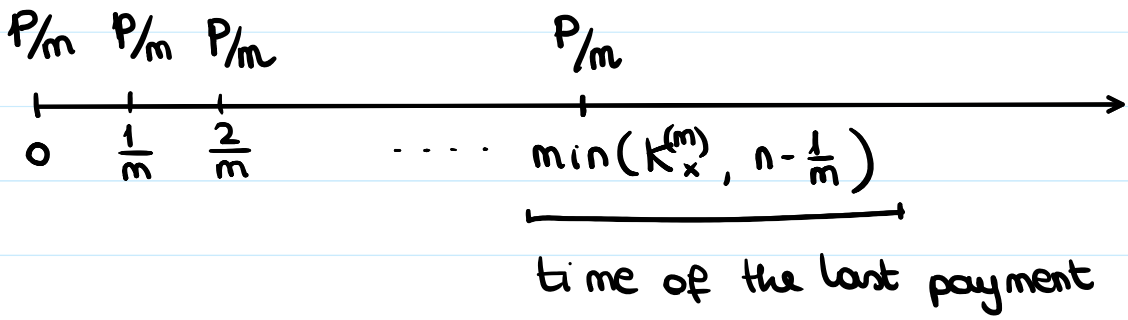
## 3 TERMS WHOOLHOUSE'S FORMULA

$$\ddot{a}_x^{(m)} \sim \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12\,m^2}(\delta - \mu_x)$$

where $\delta = \log(1+i)$ FORCE OF INTEREST

$$\mu_x = -\frac{1}{2}\log({}_2P_x)$$

## TERM ANNUITY DUE PAID m times per year with yearly amount p (each time $\frac{p}{m}$ is paid)




If $K_x^{(m)} \leq n - \frac{1}{m} \Rightarrow K_x^{(m)}$ is the time of the last payment

If $K_x^{(m)} \geq n \Rightarrow n - \frac{1}{m}$ is the time of the last payment

## PRESENT VALUE

$$Y = \frac{p}{m} \cdot \sum_{k=0}^{\min(K_x^{(m)}\cdot m,\, nm-1)} v^{\frac{k}{m}} = p \cdot \frac{1}{m} \cdot \frac{1 - v^{\min(K_x^{m} + \frac{1}{m},\, n)}}{1 - v^{1/m}}$$

ACTUARIAL VALUE

$$\mathbb{E}[Y] = p \cdot \ddot{a}^{(m)}_{x:\overline{n}|}$$

where $$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m} \cdot \frac{1 - \mathbb{E}\left[v^{\min\left(K_x^{(m)} + \frac{1}{m},\, n\right)}\right]}{1 - v^{1/m}}$$

ACTUARIAL VALUE of a UNIT term annuity due paid m times per year

If the term annuity is immediate

$$a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m}$$

the problem of computing $\ddot{a}^{(m)}_{x:\overline{n}|}$ is the same as before : we need periodic survival probabilities that are usually not available

Hence we approximate $\ddot{a}^{(m)}_{x:\overline{n}|}$ with a suitable WHOOLHOUSE'S FORMULA for TERM ANNUITIES

2 TERMS

$$\ddot{a}^{(m)}_{x:\overline{n}|} \sim \ddot{a}_{x:\overline{n}|} - \frac{m-1}{12m}\left(1 - {}_nE_x\right)$$

3 TERMS

$$\ddot{a}^{(m)}_{x:\overline{n}|} \sim \ddot{a}_{x:\overline{n}|} - \frac{m-1}{12m}\left(1 - {}_nE_x\right) - \frac{m^2-1}{12m^2}(\delta + \mu_x)(1 - {}_nE_x)$$

# LIFE INSURANCE CASHFLOW

Life insurance cashflow is generated by

BENEFITS paid by the insurer
EXPENSES of the insurer
FROM THE POINT OF VIEW OF THE INSURER
cash outflow

PREMIUMS paid by the policy holder
cash inflow

## PREMIUMS

can be paid in different modes.

- SINGLE PREMIUM: at the issue of the contract. it Can be NET or GROSS depending whether expenses are excluded or included in the calculation
- PERIODIC PREMIUMS: the first of which at the issue of the contract. these can be NET or GROSS
- NON PERIODIC PREMIUMS, again NET or GROSS

To evaluate premiums a premium calculation principle must be applied. We use the EQUIVALENCE PRINCIPLE

**EXPENSES** of the insurer, are typically divided in 3 families :

- INITIAL EXPENSES paid at $t=0$
  Usually quantified as a large percentage of the first premium

- MAINTENANCE / RENEWAL EXPENSES
  fixed small percentage of periodic premiums

- TERMINATION EXPENSES
  Small amount paid with the benefit

## PREMIUMS EVALUATION

We introduce the Loss & profit at issue $L_0$ with the property that

$$L_0 = \begin{cases} \text{loss} & \text{if } L_0 > 0 \\ \text{profit} & \text{if } L_0 < 0 \end{cases}$$

More commonly $L_0$ is called the LOSS AT ISSUE and it is defined as follows:

$$L_0 = PV(\text{cash outflow}) - PV(\text{cash inflow})$$

Depending whether expenses are excluded or included we can distinguish between :

NET LOSS AT ISSUE

$$L_0^{net} = PV(\text{benefit}) - PV(\text{net premiums})$$

## GROSS LOSS AT ISSUE

$$L_0^{gross} = PV(\text{benefit}) + PV(\text{expenses}) - PV(\text{gross premiums})$$

$L_0^{net}$, $L_0^{gross}$ are random variables that depend on the remaining lifetime of the insured life

Under the EQUIVALENCE PRINCIPLE it holds that the EXPECTED LOSS AT ISSUE is zero

In mathematical terms $\mathbb{E}[L_0^*] = 0$
(where $*$ means either net or gross)

Equivalently:

$$\mathbb{E}[PV(\text{cash outflow})] = \mathbb{E}[PV(\text{cash inflow})]$$

Since cash inflow in our setup is made only of premiums we can rephrase the equivalence principle as:
on average premiums allow the insurance company to repay all liabilities

the EQUIVALENCE PRINCIPLE provides the recipe to evaluate premiums

EXAMPLE
Consider the case of NET SINGLE PREMIUM

$P_0$ is paid at $t=0$

According to the EQUIVALENCE PRINCIPLE

$$\mathbb{E}[PV(\text{benefit})] = \mathbb{E}[\underbrace{PV(\text{net single Prem})}_{P_0}]$$

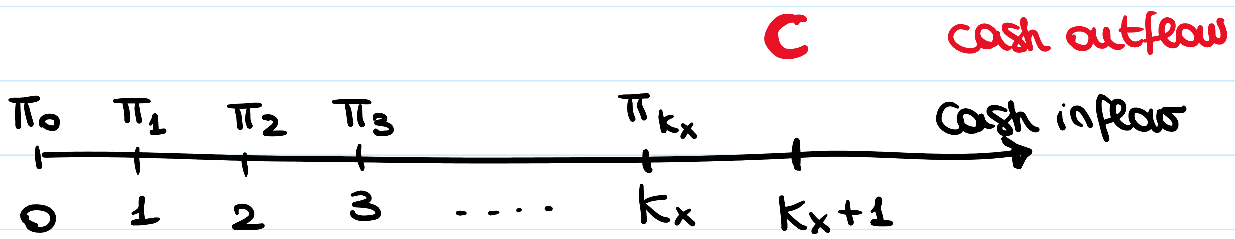
$$\mathbb{E}[PV(\text{benefit})] = P_0$$

NET SINGLE PREMIUM is equal to the actuarial value of the contract

PERIODIC PAYMENTS

Consider a whole life insurance with sum insured $C$, paid at the end of the year of death

Suppose that premiums are paid once per year of amounts $\pi_0, \pi_1, \pi_2, \dots$ as far as the insured life is alive




GOAL: Quantify the premiums

we need to solve the equation $\mathbb{E}[L_0^{net}]=0$

$$L_0^{net} = PV(\text{benefit}) - PV(\text{net premiums})$$

$$L_0^{net} = PV(\text{benefit}) - PV(\text{net premiums})$$

BENEFIT

$$Z = C \cdot v^{K_x+1}$$

NET PREMIUMS

$$Y = \Pi_0 + \Pi_1 \cdot v + \Pi_2 v^2 + \dots + \Pi_{K_x} v^{K_x}$$

$$= \sum_{k=0}^{K_x} \Pi_k \cdot v^k$$

$$L_0^{net} = Z - Y = C v^{K_x+1} - \sum_{k=0}^{K_x} \Pi_k \cdot v^k$$

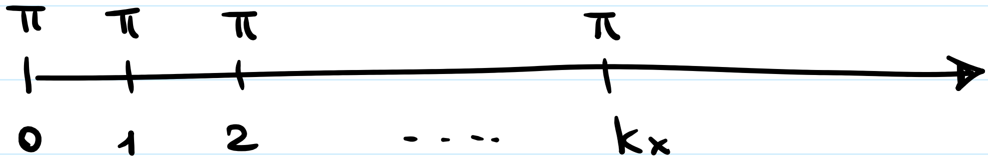
$$\mathbb{E}[L_0^{net}] = 0 \iff C \underbrace{\mathbb{E}[v^{K_x+1}]}_{A_x} - \mathbb{E}\Big[\sum_{k=0}^{K_x} \Pi_k v^k\Big] = 0$$

$$\iff C A_x = \mathbb{E}\Big[\sum_{k=0}^{K_x} \Pi_k v^k\Big]$$

Typical arrangements $\Pi_k = \Pi$ $\forall k$

or $\Pi_k$ deterministic function of $k$

We now focus on the 1st case: constant periodic premiums



We can interprete the flow of premiums as a whole life annuity

$$Y = \pi \cdot \sum_{k=0}^{k_x} v^k = \pi \, \frac{1 - v^{k_x+1}}{d}$$

$$L_0^{net} = C \, v^{k_x+1} - \pi \cdot \frac{1 - v^{k_x+1}}{d}$$

$$\mathbb{E}[L_0^{net}] = 0 \iff C \cdot \underbrace{\mathbb{E}[v^{k_x+1}]}_{A_x} = \pi \cdot \underbrace{\frac{1 - \mathbb{E}[v^{k_x+1}]}{d}}_{\ddot{a}_x}$$

$$\pi = \frac{C \cdot A_x}{\ddot{a}_x}$$

$C = 100{,}000 \qquad x = 50$

$$\pi = \frac{100{,}000 \cdot 0.18931}{17.0245}$$

↑ annual net premium

NET PREMIUMS should satisfy

$$\mathbb{E}\left[PV(\text{benefit})\right] = \mathbb{E}\left[PV(\text{Premiums}^{\text{net}})\right]$$

GROSS PREMIUMS should satisfy

$$\mathbb{E}\left[L_0^{\text{gross}}\right] = 0$$

$$\mathbb{E}\left[PV(\text{benefit})\right] + \mathbb{E}\left[PV(\text{expenses})\right] - \mathbb{E}\left[PV(\text{prem}^{\text{gross}})\right] = 0$$