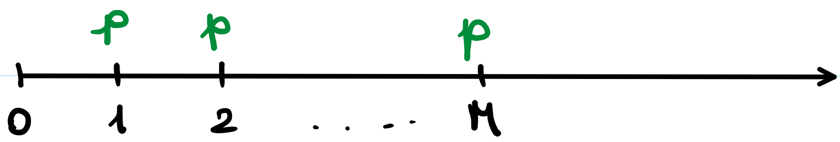
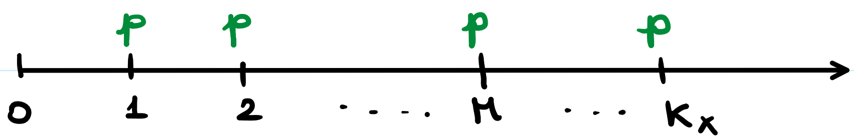
EXERCISE 2

Whole life annuity immediate with M years guarantee

If $K_x \leq M$



If $K_x \geq M+1$



Note that in this contract payments up to time M are done in any case whereas payments after time M+1 (included) are done only upon survival

Q1 : the present value of the benefit is

$$Y = p \underbrace{\sum_{k=1}^{M} v^M}_{\text{deterministic part}} + \underbrace{p \, \mathbb{1}_{K_x \geq M+1} \sum_{k=M+1}^{K_x} v^k}_{\text{stochastic part}}$$

the deterministic part represents the guaranteed amount the stochastic part is the excess to the guarantee which is only paid if the insured life is alive

We can also write Y in the following way which more convenient for the next computations :

by scaling $k = j + M + 1$ we can rewrite the summation as:

$$Y = p \cdot a_{\overline{M|}} + p\, v^{M+1}\, \mathbb{1}_{K_x \geq M+1} \underbrace{\sum_{j=0}^{K_x - M - 1} v}_{\text{For this part}}$$

call J_x the remaining lifetime of a person aged $x+M+1$
It holds that

$$K_x = J_x + M + 1$$

and hence the part in the summation is the PV of a whole life annuity due for a life aged $x+M+1$

Q2:

$$\mathbb{E}[Y] = p\left(a_{\overline{M|}} + v^{M+1} \cdot P(K_x \geq M+1) \cdot \ddot{a}_{x+M+1}\right)$$

$$= p\left(a_{\overline{M}|} + v^{M+1} \cdot {}_{M+1}p_x \quad \ddot{a}_{M+1}\right)$$

(3) take $p = 6000$

$M = 20$

$x = 50$

$i = 5\%$

we get $v = \frac{1}{1+i} = \frac{1}{1.05}$

$$\mathbb{E}[Y] = 6000\left(\frac{1-v^{20}}{d}v + v^{21}\,{}_{21}p_{50} \quad \ddot{a}_{71}\right)$$

We only need to compute $v^{21}\,{}_{21}p_{50}$

$$v^{21}\,{}_{21}p_{50} = v \cdot \boxed{v^{20} \cdot {}_{20}p_{50}} \cdot {}_1p_{70} = v\,{}_{20}E_{50}\,{}_1p_{70}$$

Now all the numbers are in the lifetable

$\ddot{a}_{71} = 11,6803$

${}_{20}E_{50} = 0.34824$

${}_1p_{70} = 1 - 0.01413 = 0.989587$

$\mathbb{E}[Y] = 97,774.36$

EXERCISE 4

Whole life insurance with

$C = 250{,}000$

$x = 60$

Annual premiums of amount P are paid every year starting at $t = 1$, ending at $t = 15$

$i = 5\%$

EXPENSES:

initial	500 €	at $t = 0$
renewal	3% P	at $t = 1, 2, \dots, 15$
Terminal	350	at $t = K_x + 1$

Q1: $L_0^{gross} = Z + E - Y$

$$Z = C \cdot v^{K_{60}+1}$$

$$E = 500 + 0.03\,P \cdot \sum_{k=1}^{\min(K_{60},15)} v^k + 350\, v^{K_{60}+1}$$

$$Y = P \cdot \sum_{k=1}^{\min(K_{60},15)} v^k$$

$$L_0^{gross} = 250\,350\, v^{K_{60}+1} + 500 - 0.97\,P \cdot \sum_{k=1}^{\min(K_{60},15)} v^k$$

Q2: We want to solve the equation

$$\mathbb{E}[L_0^{gross}] = 0$$

$$\mathbb{E}[L_0^{gross}] = 250{,}350 \cdot A_{60} + 500 - 0.97P \cdot a_{60:\overline{15|}}$$

where $a_{60:\overline{15|}}$ is the actuarial value of the unit term annuity immediate with maturity of 15 years

It holds that $a_{60:\overline{15|}} = \ddot{a}_{60:\overline{15|}} - 1 + {}_{15}E_{60}$

(cf. notes from Tuesday)

Which results into:

$$\mathbb{E}[L_0^{gross}] = 250{,}350\, A_{60} + 500 - 0.97P\left(\ddot{a}_{60:\overline{15|}} - 1 + {}_{15}E_{60}\right)$$

Hence

$$P = \frac{250{,}350\, A_{60} + 500}{0.97\left(\ddot{a}_{60:\overline{15|}} - 1 + {}_{15}E_{60}\right)}$$

$$\ddot{a}_{60:\overline{15|}} = \sum_{k=0}^{14} v^k \, {}_kp_{60} = \sum_{k=0}^{9} v^k \, {}_kp_{60} + \sum_{k=10}^{14} v^k \, {}_kp_{60}$$

$$= \underbrace{\ddot{a}_{60:\overline{10}|}}_{7.9555} + \underbrace{\sum_{k=10}^{14} v^k \, {}_kp_{60}}_{\text{To be computed}}$$

Recursively we have:

$K=10$: $v^{10} \, {}_{10}p_{60} = {}_{10}E_{60} = 0.57864$

$K=11$ $\quad v^{11} \, {}_{11}p_{60} = {}_{10}E_{60} \cdot v \cdot {}_1p_{70}$

$K=12$ $\quad v^{12} \, {}_{12}p_{60} = v^{11} \, {}_{11}p_{60} \cdot v \cdot {}_1p_{71}$

$K=13$ $\quad v^{13} \, {}_{13}p_{60} = v^{12} \, {}_{12}p_{60} \cdot v \; {}_1p_{72}$

$K=14$ $\quad v^{14} \, {}_{14}p_{60} = v^{13} \, {}_{13}p_{60} \cdot v \cdot {}_1p_{73}$

then $\displaystyle\sum_{K=10}^{14} v^k \, {}_kp_x \sim 2.57$

And finally $\ddot{a}_{60:\overline{15}|} = 7.9555 + 2.57 \sim 10.52$

$${}_{15}E_{60} = v^{15} \cdot {}_{15}p_{60} = v^{14} \cdot {}_{14}p_{60} \; v \cdot {}_1p_{74}$$

alternatively $\quad {}_{15}p_{60} = \dfrac{\ell_{75}}{\ell_{60}}$ and directly compute

$${}_{15}E_{60} = v^{15} \cdot {}_{15}p_{60}$$

then $P = \dfrac{250350 \cdot 0.29028 + 500}{0.97 \cdot 9,65} \sim 7586, ..$

Q3. We want to compute the probability that the contract is profitable at time 0

that is $P(L_0^{gross} < 0)$

We write:

$$L_0^{gross} = 250350\, v^{K_{60}+1} + 500 - 0.97 \cdot 7586 \cdot \sum_{k=1}^{\min(K_{60},15)} v^k$$

$$= 250350\, v^{K_{60}+1} + 500 - 0,97 \cdot 7586 \left(\frac{1 - v^{\min(K_{60}+1,16)}}{d} - 1 \right)$$

$$= 250350\, v^{K_{60}+1} + 0.97 \cdot 7586 \cdot 21 \cdot v^{\min(K_{60}+1,16)}$$

$$+ 500 - 0.97 \cdot 7586 (21 - 1)$$

$$= a\, v^{K_{60}+1} + b\, v^{\min(K_{60}+1,16)} + c$$

where $a = 250350$

$b = 0.97 \cdot 7586 \cdot 21$

$c = 500 - 0.97 \cdot 7586 \cdot 21 \quad \underline{\underline{< 0}}$

Note that

$$L_0^{gross} = \begin{cases} a\,\sigma^{K_{60}+1} + b\,\sigma^{K_{60}+1} + c & \text{If } K_{60} \le 15 \\ a\,\sigma^{K_{60}+1} + b\,\sigma^{16} + c & \text{If } K_{60} \ge 16 \end{cases}$$

Hence

$$P(L_0^g < 0) = P\left((a+b)\,\sigma^{K_{60}+1} + c < 0 \cap K_{60} \le 15\right) +$$

$$+\, P\left(a\,\sigma^{K_{60}+1} + b\,\sigma^{16} + c < 0 \cap K_{60} \ge 16\right)$$

YELLOW:

$$P\Big(K_{60} > \underbrace{\frac{\ln\left(-\frac{c}{a+b}\right)}{\ln(\sigma)} - 1}_{19,81} \cap K_{60} \le 15\Big)$$

$$= P(K_{60} \ge 20 \cap K_{60} \le 15) = 0$$

GREEN

$$P\Big(K_{60} \ge \underbrace{\frac{\ln\left(\frac{-c-b\sigma^{16}}{a}\right)}{\ln(\sigma)} - 1}_{23,4} \cap K_{60} \ge 16\Big)$$

$$P(K_{60} \ge 24 \cap K_{60} \ge 16) = P(K_{60} \ge 24) = {}_{24}P_{60}$$