

Exercise 4

Multiple-Objective Optimization

Chick'n-Pick'n: where to expand?

The fast-food chain has **\$2,000,000** to open new outlets. It can choose among three formats:

- ▶ **Lunch counters** in downtown office buildings.
- ▶ **Mall restaurants** with space to eat inside.
- ▶ **Stand-alone restaurants** with a drive-through and indoor seating.

Objectives of the expansion plan

Choose how many outlets of each type to open, to maximize both **annual return** and **jobs created**.

The number of available sites is limited. Each new outlet must be a **whole outlet**, and total start-up cost cannot exceed the budget.

The data for each new outlet

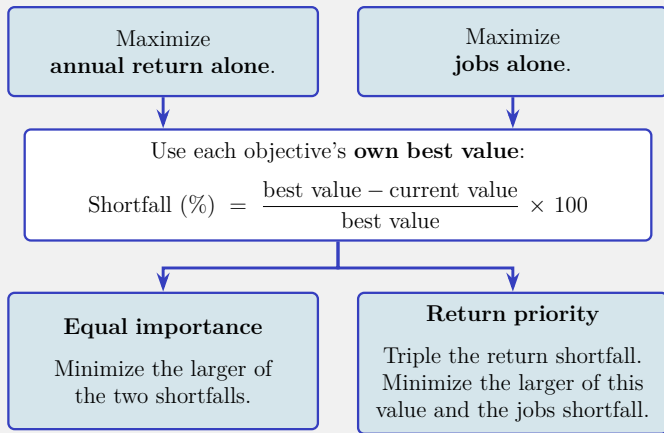
Each column describes **one outlet** of that type.

	Lunch counter	Mall	Stand-alone
Start-up cost (\$)	150,000	275,000	450,000
Annual return (\$)	85,000	125,000	175,000
Jobs created	9	17	35
Available sites	5	7	3

Budget: \$2,000,000. The company may leave some money unspent and some sites unused.

Costs, annual returns and jobs add across the outlets selected. For example, two lunch counters cost \$300,000, provide \$170,000 per year and create 18 jobs.

Our task: choose a plan with two objectives



Compare the resulting plans: outlets, cost, annual return and jobs.

Explain how the priorities change the plan.

Optimization with multiple objective functions

Let L , M and S be the numbers of new **lunch counters**, **mall restaurants** and **stand-alone restaurants**.

$$\begin{aligned}\text{Annual return } R &= 85,000L + 125,000M + 175,000S, \\ \text{Jobs } J &= 9L + 17M + 35S.\end{aligned}$$

Every plan must satisfy

$$\begin{aligned}150,000L + 275,000M + 450,000S &\leq 2,000,000, \\ 0 \leq L \leq 5, \quad 0 \leq M \leq 7, \quad 0 \leq S \leq 3,\end{aligned}$$

with L , M and S **integers**.

The two objectives use the **same decisions** and the **same constraints**.

Enter the outlet data

A1	fx Outlet type			
	A	B	C	D
1	Outlet type	Lunch counter	Mall	Stand-alone
2	Start-up cost (\$)			
3	Annual return (\$)			
4	Jobs created			
5	Available sites			

Enter the labels shown
A1:D1 and A2:A5.

Labels for the outlet types and their data.

Enter the values for each outlet

B2	<i>fx</i>	150000		
	A	B	C	D
1	Outlet type	Lunch counter	Mall	Stand-alone
2	Start-up cost (\$)	150,000	275,000	450,000
3	Annual return (\$)	85,000	125,000	175,000
4	Jobs created	9	17	35
5	Available sites	5	7	3

Enter the values shown in B2:D5.

Costs, returns and jobs are per outlet; site limits cap the outlet counts.

Enter the budget

B9	<i>fx</i>	2000000
	A	B
9	Budget (\$)	2,000,000



A9: Budget (\$)

B9: 2000000

The total start-up cost must stay within this budget.

Choose the outlets and calculate their cost

B7	<i>fx</i>	0			
	A	B	C	D	
7	New outlets	0	0	0	
8					
9	Budget (\$)	2,000,000			
10					

A7: New outlets

B7:D7: enter 0 in each cell.

B7:D7 are the outlet counts Solver will choose.

Calculate the total start-up cost

B10	<i>fx</i>	=SUMPRODUCT(B2:D2;B7:D7)			
	A	B	C	D	
7	New outlets	0	0	0	
8					
9	Budget (\$)	2,000,000			
10	Start-up cost (\$)	0			

A10: Start-up cost (\$)

B10: =SUMPRODUCT(B2:D2;B7:D7)

Add cost $\times$ outlet count for each type. Require $B10 \leq B9$.

Label the objective table

A12:C12		fx	Measure		
	A		B		C
12	Measure		Current		Best alone
13	Annual return (\$)				
14	Jobs				

Enter the labels shown
in A12:C12 and A13:A14.
Leave C13:C14 empty for now.

Compare the current plan with the best result for each objective.

Calculate the annual return

B13	<i>fx</i>	=SUMPRODUCT(B3:D3;B7:D7)		
	A	B	C	
12	Measure	Current	Best alone	
13	Annual return (\$)	0		
14	Jobs			

In B13, enter

=SUMPRODUCT(B3:D3;B7:D7)

Multiply each return by its outlet count, then add the three results.

Calculate the number of jobs

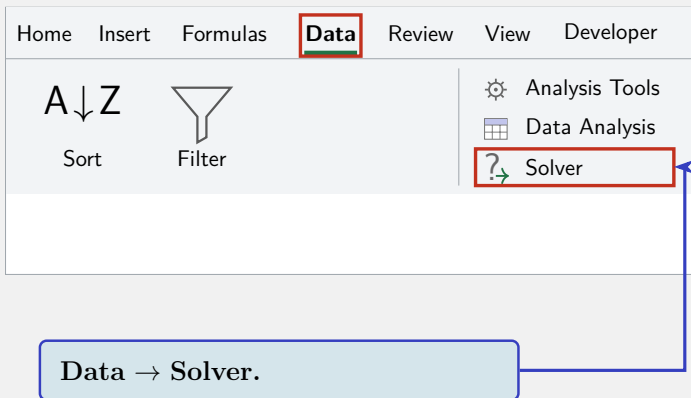
B14	<i>fx</i>	=SUMPRODUCT(B4:D4;B7:D7)	
	A	B	C
12	Measure	Current	Best alone
13	Annual return (\$)	0	
14	Jobs	0	

In B14, enter

=SUMPRODUCT(B4:D4;B7:D7)

Multiply the jobs per outlet by each outlet count, then add the results.

Open Solver for the outlet plan



Use the worksheet containing the outlet data and formulas.

Find the highest possible annual return

Solver Parameters

Set Objective:

To: ☒ Max ☐ Min ☐ Value Of:

By Changing Variable Cells:

Maximize B13 by changing B7:D7.

B13 calculates annual return; B7:D7 contains the outlet numbers.

Add the budget and site limits

Add Constraint

Cell Reference:	Constraint:
\$B\$10	\$B\$9
≤	▼

Add Constraint

Cell Reference:	Constraint:
\$B\$7:\$D\$7	\$B\$5:\$D\$5
≤	▼

Click **Add** in Solver. Enter each constraint shown; click **Add** after each one.

Stay within the **\$2 million budget** and the available sites.

Require whole numbers of outlets

Add Constraint

Cell Reference:		Constraint:
\$B\$7:\$D\$7	int▼ ≤ = ≥ int bin dif	integer
OK	Cancel	Add

Set **B7:D7** to **int**, then click **OK**.

int allows only whole outlet numbers: 0, 1, 2, and so on.

Choose Simplex LP and nonnegative decisions

Solver Parameters

Set Objective: ☒ Max

By Changing Variable Cells:

Subject to the Constraints:

☒ Make Unconstrained Variables Non-Negative

Solving Method:

Tick **Non-Negative**.
Choose **Simplex LP**; open **Options**.

The return, cost and site constraints are linear in the outlet numbers.

Set Integer Optimality to zero

Options

All Methods GRG Nonlinear Evolutionary

Solving with Integer Constraints

☐ Ignore Integer Constraints

Integer Optimality (%):

Cancel **OK**

Leave **Ignore Integer Constraints** unchecked.
Set **Integer Optimality (%) = 0**; click **OK**.

A positive percentage can stop the integer search early.

Solve and keep the outlet plan

The image shows two overlapping dialog boxes from a spreadsheet application. The top dialog, titled "Solver Parameters", has a "Solve" button highlighted with a red rectangle. The bottom dialog, titled "Solver Results", has two radio buttons: "Keep Solver Solution" (selected and highlighted with a red rectangle) and "Restore Original Values". An "OK" button in the bottom right of the "Solver Results" dialog is also highlighted with a red rectangle. A blue arrow points from the "Keep Solver Solution" radio button to a light blue instruction box at the bottom. Another blue arrow points from the "Solve" button in the "Solver Parameters" dialog to the "Keep Solver Solution" radio button.

Solver Parameters

Solve

Solver Results

☒ Keep Solver Solution

☐ Restore Original Values

OK

Click **Solve**. When finished, choose **Keep Solver Solution** → **OK**.

The chosen outlet numbers remain in **B7:D7**.

Copy the best annual return

B13	<i>fx</i>	=SUMPRODUCT(B3:D3;B7:D7)		
	A	B	C	D
7	New outlets	4	5	0
8				
9	Budget (\$)	2,000,000		
10	Start-up cost (\$)	1,975,000		
11				
12	Measure	Current	Best alone	
13	Annual return (\$)	965,000		
14	Jobs	121		

Copy B13.

4 lunch counters and 5 mall restaurants; no stand-alone restaurants.

Save the best annual return

C13	fx		
	A	B	C
12	Measure	Current	Best alone
13	Annual return (\$)	965,000	
14	Jobs	121	

Select C13.
Open **Paste Special**.

Choose **Values**,
then **OK**.

Paste Special

Paste

☐ All

☐ Formulas

☒ Values

☐ Formats

☐ Comments

☐ Validation

Cancel OK

Values copies 965000 into C13; B13 keeps its formula.

Keep the best return for comparison

C13	<i>fx</i>	965000		
	A		B	C
12	Measure		Current	Best alone
13	Annual return (\$)		965,000	965,000
14	Jobs		121	

Keep C13 fixed at 965000
for every remaining solve.

The return shortfall will compare each plan with this best value.

Find the highest possible number of jobs

Change only the objective to **B14**. Keep **Max**, the same changing cells and all the original constraints.

Solver Parameters

Set Objective:

To: ☒ Max ☐ Min

By Changing Variable Cells:

Subject to the Constraints:

B10 <= B9
B7:D7 <= B5:D5
B7:D7 = integer

Add

Change

Delete

Solving Method:

Solve and keep the solution. **C13 remains 965000**; it is a saved value and does not follow the current return in B13.

Copy the best number of jobs

B14	<i>fx</i>	=SUMPRODUCT(B4:D4;B7:D7)		
	A	B	C	D
7	New outlets	4	0	3
8				
9	Budget (\$)	2,000,000		
10	Start-up cost (\$)	1,950,000		
11				
12	Measure	Current	Best alone	
13	Annual return (\$)	865,000	965,000	
14	Jobs	141		

Copy B14.



4 lunch counters and 3 stand-alone restaurants; no mall restaurants.

Save the best number of jobs

C14	fx		
	A	B	C
12	Measure	Current	Best alone
13	Annual return (\$)	865,000	965,000
14	Jobs	141	

Select C14.
Open **Paste Special**.

Choose **Values**,
then **OK**.

Paste Special

Paste

☐ All

☐ Formulas

☒ Values

☐ Formats

☐ Comments

☐ Validation

Cancel OK

Values copies 141 into C14; B14 keeps its formula.

Keep the best values for comparison

C14	<i>fx</i>	141	
	A	B	C
12	Measure	Current	Best alone
13	Annual return (\$)	865,000	965,000
14	Jobs	141	141

Keep C13:C14 fixed
for every remaining solve.



The shortfalls will compare each plan with these two best values.

Why we need a compromise

Each objective has a unique best plan. The two best plans differ.

Plan chosen for	Annual return (\$)	Jobs
Highest annual return	965,000	121
Most jobs	865,000	141

The best values cannot both be reached

The plan with the highest return creates 20 fewer jobs. The plan with the most jobs gives up \$100,000 of annual return.

We need a rule for deciding how much of one objective to give up for the other.

Measure each shortfall from its own best value

A **shortfall** is how far a plan falls below the best possible result for one objective. We divide by that objective's best value:

$$\text{Return shortfall} = \frac{965,000 - \text{current annual return}}{965,000},$$

$$\text{Jobs shortfall} = \frac{141 - \text{current jobs}}{141}.$$

For the plan with the most jobs, the return shortfall is

$$\frac{965,000 - 865,000}{965,000} \approx \mathbf{10.36\%},$$

and the jobs shortfall is **0%**.

10.36% of what? Of the best possible annual return, \$965,000.

Add the column headings

D12:F12		fx Shortfall				
	A	B	C	D	E	F
12	Measure	Current	Best alone	Shortfall	Weight	Weighted %
13	Annual return (\$)	865,000	965,000			
14	Jobs	141	141			

Enter the headings

D12: Shortfall

E12: Weight

F12: Weighted %

Headers for shortfalls, importance weights and weighted shortfalls.

Calculate the return shortfall

D13	<i>fx</i>	=(C13-B13)/C13				
	A	B	C	D	E	F
12	Measure	Current	Best alone	Shortfall	Weight	Weighted %
13	Annual return (\$)	865,000	965,000	0.103627		
14	Jobs	141	141			

In D13, enter
$$=(C13-B13)/C13$$

The \$100,000 return gap is **10.36% of the best return** (\$965,000).

Copy the shortfall formula

D14	$fx = (C14 - B14) / C14$					
	A	B	C	D	E	F
12	Measure	Current	Best alone	Shortfall	Weight	Weighted %
13	Annual return (\$)	865,000	965,000	0.103627		
14	Jobs	141	141	0		

Copy D13 down to D14.

The formula uses row 14.

The current plan already creates the maximum **141 jobs**, so the jobs shortfall is **zero**.

Format the shortfalls

D13:D14		fx =(C13-B13)/C13				
	A	B	C	D	E	F
12	Measure	Current	Best alone	Shortfall	Weight	Weighted %
13	Annual return (\$)	865,000	965,000	10.36%		
14	Jobs	141	141	0.00%		

Format D13:D14

Percentage, **2 decimals**.

Percentage formatting displays the shortfall as **10.36%**; the value used in calculations stays unchanged.

Set equal importance

E13:E14		fx	1			
	A	B	C	D	E	F
12	Measure	Current	Best alone	Shortfall	Weight	Weighted %
13	Annual return (\$)	865,000	965,000	10.36%	1	
14	Jobs	141	141	0.00%	1	

Enter 1 in E13 and E14.

The two objectives have equal importance.

Equal weights make the two percentage shortfalls **equally important**.

Calculate the weighted shortfall

F13	<i>fx</i>	=D13*E13				
	A	B	C	D	E	F
12	Measure	Current	Best alone	Shortfall	Weight	Weighted %
13	Annual return (\$)	865,000	965,000	10.36%	1	0.103627
14	Jobs	141	141	0.00%	1	

In F13, enter

=D13*E13

Shortfall × weight.

We multiply the return shortfall by its importance weight. With weight **1**, it remains **10.36%**.

Copy the weighted formula

F14	fx =D14*E14					
	A	B	C	D	E	F
12	Measure	Current	Best alone	Shortfall	Weight	Weighted %
13	Annual return (\$)	865,000	965,000	10.36%	1	0.103627
14	Jobs	141	141	0.00%	1	0

Copy F13 down to F14.

The formula uses row 14.

The weighted jobs shortfall is **zero**: the plan already achieves the maximum number of jobs.

Format the weighted shortfalls

F13:F14		<i>fx</i>	=D13*E13			
	A	B	C	D	E	F
12	Measure	Current	Best alone	Shortfall	Weight	Weighted %
13	Annual return (\$)	865,000	965,000	10.36%	1	10.36%
14	Jobs	141	141	0.00%	1	0.00%

Format F13:F14

Percentage, **2 decimals**.

Keep full formula precision.

These percentages combine shortfalls and importance weights. Solver will minimize the **larger one**.

The minimum worst-shortfall compromise

$$R(L, M, S) = 85,000L + 125,000M + 175,000S,$$

$$J(L, M, S) = 9L + 17M + 35S.$$

Choose L , M , S and q , an upper limit for both weighted shortfalls. The weights w_R, w_J are fixed.

$$\begin{aligned} & \min_{L, M, S, q} \quad q \\ \text{subject to} \quad & w_R \frac{965,000 - R(L, M, S)}{965,000} \leq q, \\ & w_J \frac{141 - J(L, M, S)}{141} \leq q, \\ & L, M, S \in \mathbb{Z}_{\geq 0}, \quad q \geq 0. \end{aligned}$$

Label the limit and the objective

A17:A18		<i>fx</i>	Shortfall limit q	
	A		B	
17	Shortfall limit q			
18	Objective = q			

Enter the labels:

A17: Shortfall limit q

A18: Objective = q

These labels identify the limit q and the objective cell.

Give the limit a starting value

B17	fx	1
	A	B
17	Shortfall limit q	1
18	Objective = q	

Enter 1 in B17.

This is the starting value.

Solver will choose q , the upper limit for both weighted shortfalls.

Create the objective formula

B18	<i>fx</i>	=B17	
	A		B
17	Shortfall limit q		1
18	Objective = q		1

In B18, enter

=B17

Solver needs an objective formula: minimizing **B18** minimizes q .

Display the limit as a percentage

B17:B18		<i>fx</i>	1
	A		B
17	Shortfall limit q		100.00%
18	Objective = q		100.00%

Format B17:B18

Percentage, **2 decimal places.**

q can have decimals; only the outlet counts must be whole numbers.

Solve for the smallest common limit

Set **B18** to **Min**. Change **B7:D7** and **B17**. Add the constraints below.

Solver Parameters

Set Objective:

To: ☐ Max ☒ Min

By Changing Variable Cells:

Subject to the Constraints:

B10 <= B9
B7:D7 <= B5:D5
B7:D7 = integer
F13:F14 <= B17

Add

Change

Delete

Solving Method:

Keep **Simplex LP**, nonnegativity and **Integer Optimality (%) = 0**. **B17 is not an integer cell**. Solve and keep the solution.

Equal importance: the selected expansion

With **E13 = 1** and **E14 = 1**, the solution is:

Lunch counters	5
Mall restaurants	1
Stand-alone restaurants	2
Start-up cost	\$1,925,000
Annual return	\$900,000
Jobs created	132
Largest weighted shortfall	6.74%

Return shortfall: $\frac{65,000}{965,000} \approx \mathbf{6.74\%}$.

Jobs shortfall: $\frac{9}{141} \approx \mathbf{6.38\%}$.

The larger shortfall, **6.74%**, improves on both extreme plans.

Give annual return three times the importance

E13		fx	3				
	A		B	C	D	E	F
12	Measure		Current	Best alone	Shortfall	Weight	Weighted %
13	Annual return (\$)		900,000	965,000	6.74%	3	20.21%
14	Jobs		132	141	6.38%	1	6.38%

Change E13 to 3.

Keep **E14 = 1.**

Use 3, not 3%.

A 1% shortfall counts as **3% for return** and **1% for jobs**.

Read the updated weighted shortfalls

F13	<i>fx</i>	=D13*E13				
	A	B	C	D	E	F
12	Measure	Current	Best alone	Shortfall	Weight	Weighted %
13	Annual return (\$)	900,000	965,000	6.74%	3	20.21%
14	Jobs	132	141	6.38%	1	6.38%

F13:F14 update on their own.

Keep both formulas.

The plan is still unchanged: **\$900,000** annual return and **132 jobs**.

Run Solver again with the new weights

Solver Parameters

Set Objective:

To: ☐ Max ☒ Min

By Changing Variable Cells:

Subject to the Constraints:

Solving Method:

Click **Solve**, then select
Keep Solver Solution.

Minimize the **largest weighted shortfall** using the new weights.

The plan with more importance on return

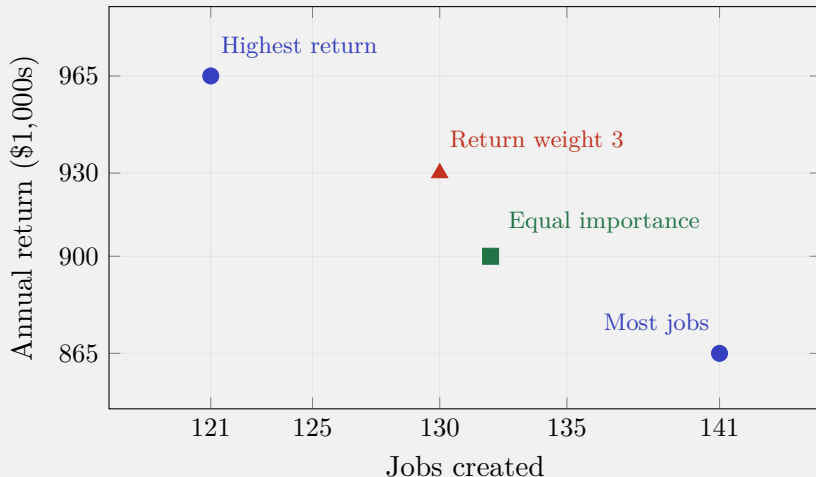
With **E13 = 3** and **E14 = 1**, the solution becomes:

Lunch counters	3
Mall restaurants	4
Stand-alone restaurants	1
Start-up cost	\$2,000,000
Annual return	\$930,000
Jobs created	130

	Actual shortfall	Weight	Weighted shortfall
Annual return	3.63%	3	10.88%
Jobs	7.80%	1	7.80%

B17 and B18 display 10.88%, the largest weighted shortfall.
The formulas use unrounded values.

Plot the trade-off between return and jobs



Each dot is a possible whole-outlet plan. Along these four choices, gaining more jobs means giving up some annual return.

The communication tower problem

Nonlinear and multiple-objective optimization

The communication tower problem

Call Me Communications must place a new tower to connect four existing towers. Their coordinates, in kilometres, are:

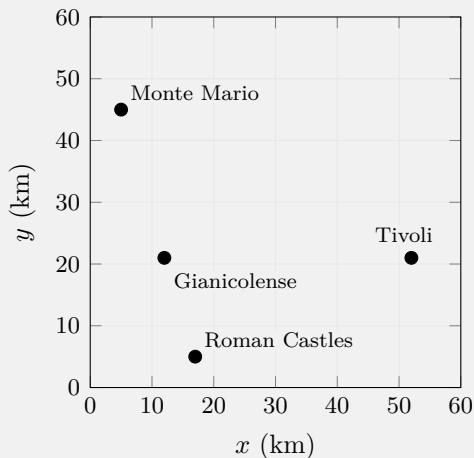
Existing tower	x	y
Monte Mario	5	45
Gianicolense	12	21
Roman Castles	17	5
Tivoli	52	21

The new tower must be **no more than 40 km from each** existing tower. Use Euclidean distances.

1. Find the location that minimizes the **average of the four distances**.
2. Then find the location that minimizes the **longest of the four distances**.
3. For each solution, identify the closest existing tower.

The four locations and the decision

We choose the new tower's two coordinates, x and y . Moving that point changes all four connection distances.



Both axes use the same kilometre scale.

Enter the tower table headings

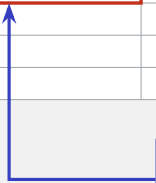
A1:D1		<i>fx</i>	Tower			
	A		B	C	D	
1	Tower		x (km)	y (km)	Distance (km)	
2						
3						
4						
5						
6						
7						
8						

Enter the headings shown
in A1:D1.

Column D will show each tower's distance from the new tower.

Enter the tower names

A2:A5		<i>fx</i>	Monte Mario		
	A		B	C	D
1	Tower		x (km)	y (km)	Distance (km)
2	Monte Mario				
3	Gianicolense				
4	Roman Castles				
5	Tivoli				
6					
7					
8					



Enter the names shown
in **A2:A5**.

Rows 2–5 identify the existing towers.

Enter the tower coordinates

B2:C5		<i>fx</i>	5	
	A	B	C	D
1	Tower	x (km)	y (km)	Distance (km)
2	Monte Mario	5	45	
3	Gianicolense	12	21	
4	Roman Castles	17	5	
5	Tivoli	52	21	
6				
7				
8				

Enter the coordinates shown
in **B2:C5**.

These coordinates are fixed; all distances are measured in kilometres.

Set the starting location

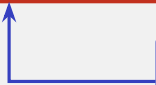
B7:C7		<i>fx</i>	27	
	A	B	C	D
1	Tower	x (km)	y (km)	Distance (km)
2	Monte Mario	5	45	
3	Gianicolense	12	21	
4	Roman Castles	17	5	
5	Tivoli	52	21	
6				
7	New tower	27	30	
8				

Enter the starting location
shown in **A7:C7**.

Solver will change **B7:C7** to choose the new tower's location.

Set the connection limit

B8	<i>fx</i>	40		
	A	B	C	D
1	Tower	x (km)	y (km)	Distance (km)
2	Monte Mario	5	45	
3	Gianicolense	12	21	
4	Roman Castles	17	5	
5	Tivoli	52	21	
6				
7	New tower	27	30	
8	Link limit (km)	40		



Enter the label and limit
shown in **A8:B8**.

No existing tower may be more than **40 km** from the new tower.

Calculate the distance to Monte Mario

D2	<i>fx</i>	=SQRT(((\$B\$7-B2)^2+(\$C\$7-C2)^2)		
	A	B	C	D
1	Tower	x (km)	y (km)	Distance (km)
2	Monte Mario	5	45	26.62705391
3	Gianicolense	12	21	
4	Roman Castles	17	5	
5	Tivoli	52	21	
6				
7	New tower	27	30	
8	Link limit (km)	40		

In D2, enter

=SQRT(((\$B\$7-B2)^2+(\$C\$7-C2)^2)

Euclidean distance: $d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$, in kilometres.

Keep the new tower references when copying

D2	<i>fx</i>	=SQRT(((\$B\$7-B2)^2+(\$C\$7-C2)^2)		
	A	B	C	D
1	Tower	x (km)	y (km)	Distance (km)
2	Monte Mario	5	45	26.62705391
3	Gianicolense	12	21	
4	Roman Castles	17	5	
5	Tivoli	52	21	
6				
7	New tower	27	30	
8	Link limit (km)	40		

\$B\$7 and **\$C\$7** keep referring to these cells when copied.

Solver can still change the values in **B7:C7**.

Copy the distance formula

D3	<i>fx</i>	=SQRT(((\$B\$7-B3)^2+(\$C\$7-C3)^2)		
	A	B	C	D
1	Tower	x (km)	y (km)	Distance (km)
2	Monte Mario	5	45	26.62705391
3	Gianicolense	12	21	17.49285568
4	Roman Castles	17	5	26.92582404
5	Tivoli	52	21	26.57066051
6				
7	New tower	27	30	
8	Link limit (km)	40		

Copy D2 down to D5.
B2/C2 change with the row.

Each copy uses the tower coordinates from its own row.

Format the distances

D2:D5		<i>fx</i>	=SQRT(((\$B\$7-B2)^2+(\$C\$7-C2)^2)		
	A	B	C	D	
1	Tower	x (km)	y (km)	Distance (km)	
2	Monte Mario	5	45	26.6271	
3	Gianicolense	12	21	17.4929	
4	Roman Castles	17	5	26.9258	
5	Tivoli	52	21	26.5707	
6					
7	New tower	27	30		
8	Link limit (km)	40			

Format D2:D5

Number, 4 decimal places.

Formatting changes the display; calculations keep full precision.

Measure the average and the longest connection

Enter the two formulas below. Display four decimals, but keep the full values in the cells.

B10	<i>fx</i>	=AVERAGE(D2:D5)		
	A		B	C
10	Average distance		24.4041	
11	Maximum distance		26.9258	

B10: =AVERAGE(D2:D5)

B11: =MAX(D2:D5)

The average describes the four connections together. The maximum describes the **longest single connection**. At the starting location, all four are below 40 km.

Minimize the average with GRG Nonlinear

Choose **Data** → **Solver**. Minimize the average in B10 by changing the new tower's coordinates.

Solver Parameters

Set Objective:

To: ☐ Max ☒ Min

By Changing Variable Cells:

Subject to the Constraints:

Solving Method:

Select **GRG Nonlinear**: the distance formulas contain squares and square roots. Keep nonnegative variables; use **no integer constraints**. Coordinates may contain decimals.

Set the accuracy and solve

In **Solver Options**, set **Constraint Precision** to **0.00000001**.
On the **GRG Nonlinear** tab, set **Convergence** to **0.00000001**.

What these settings do

Constraint Precision controls how closely Solver respects the limits. Convergence controls when changes in the objective are considered small enough to stop.

Choose **Solve**, then **Keep Solver Solution**. Use these settings throughout this exercise.

GRG finds a numerical answer. Compare it with the checks that follow; a stopping message alone is not a proof of the best possible location.

The location with the smallest average distance

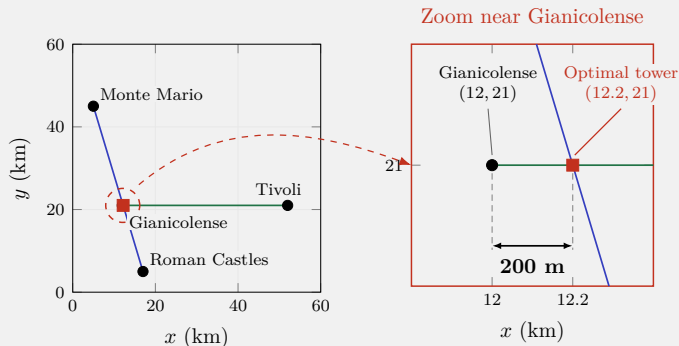
The solution is $x = 12.2$, $y = 21$. The average connection is **20.4403 km**.

Connection to	Distance (km)
Monte Mario	25.0567
Gianicolense	0.2000
Roman Castles	16.7045
Tivoli	39.8000

Gianicolense is the closest tower. Tivoli is the farthest, but its 39.8 km connection respects the 40 km limit.

Copy **B10** to **C10** using **Paste Special** → **Values**. C10 stores the best average, including its undisplayed decimals.

A geometric check of the smallest average



The segments meet at $(12.2, 21)$, **200 m to the right of Gianicolense** at $(12, 21)$. This point minimizes the total length of each pair of connections:

$$\text{best average} = \frac{\sqrt{12^2 + 40^2} + 40}{4} = 20.440306509 \dots$$

Restore the starting coordinates

B7:C7		<i>fx</i>	27	
	A	B	C	D
1	Tower	x (km)	y (km)	Distance (km)
2	Monte Mario	5	45	26.6271
3	Gianicolense	12	21	17.4929
4	Roman Castles	17	5	26.9258
5	Tivoli	52	21	26.5707
6				
7	New tower	27	30	
8	Link limit (km)	40		

Set B7 = 27.
Set C7 = 30.



Use the same worksheet. The graph varies x while $y = 30$ km.

Label the graph columns

H1:I1	fx x (km)	
	H	I
1	x (km)	B11 (km)
2		
3		
4		
5		
:		
183		

Enter the headings:

H1: x (km)

I1: B11 (km)

B11 reports the longest connection distance, in kilometres.

Enter the first x value

H3	fx	25.6
	H	I
1	x (km)	B11 (km)
2		
3	25.6	
4		
5		
:		
183		

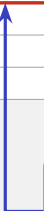


In H3, enter 25.6.

The trial values begin in row 3; leave **H2** empty.

Set the increment

H4	f_x	=H3+0.01
	H	I
1	x (km)	B11 (km)
2		
3	25.6	
4	25.61	
5		
:		
183		



In H4, enter
=H3+0.01

Each trial increases x by **0.01 km**.

Complete the x values

H4:H183		f_x	=H3+0.01
	H		I
1	x (km)		B11 (km)
2			
3	25.6		
4	25.61		
5	25.62		
⋮	⋮		
183	27.4		

Copy H4 down to H183.

H3:H183 contains **181 values**, from **25.60** to **27.40 km**.

Link the table to B11

I2	<i>fx</i>	=B\$11
	H	I
1	x (km)	B11 (km)
2		26.9258
3	25.6	
4	25.61	
5	25.62	
⋮	⋮	
183	27.4	

In I2, enter
=B\$11



Leave **H2** and **I3:I183** empty; Excel will fill the results next.

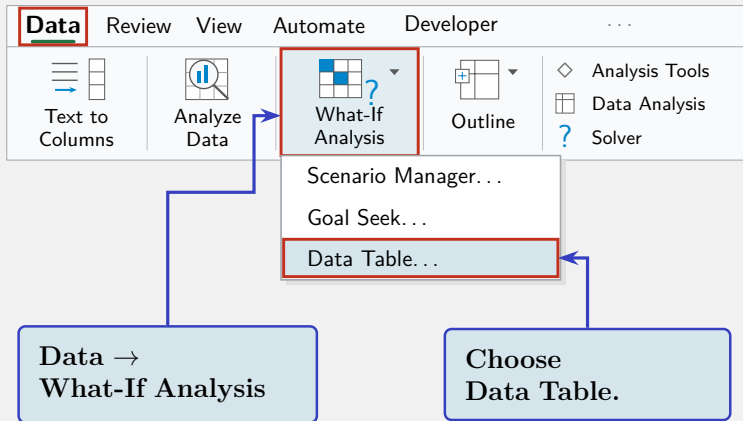
Select the data table range

H2:I183		<i>fx</i>	
	H		I
1	x (km)		B11 (km)
2			26.9258
3	25.6		
4	25.61		
5	25.62		
⋮	⋮		
183	27.4		

Select H2:I183.

Include the formula in **I2** and the blank results cells. Exclude row 1.

Open Data Table



Data Table calculates the same formula for many input values.

Calculate B11 for every x

Data Table repeats a calculation for a list of input values.

The image shows the 'Data Table' dialog box in Excel. It has two input fields: 'Row input cell:' which is empty, and 'Column input cell:' which contains '\$B\$7'. The '\$B\$7' cell is highlighted with a red border. Below these fields are 'Cancel' and 'OK' buttons. The 'OK' button is highlighted with a blue border. A blue arrow points from a text box on the left to the 'Column input cell' field. Another blue arrow points from a text box on the right to the 'OK' button.

Data Table

Row input cell:

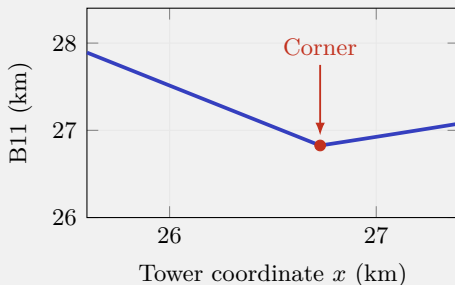
Column input cell:

Column input cell: \$B\$7
Our values run down column H,
so **Row input cell** stays empty.

Click OK.

Excel temporarily puts each H value in **B7**, recalculates **B11**, and records the result in column I. **C7 stays at 30.**

Why not minimize B11 directly?



Introduce an auxiliary variable

GRG uses slopes; corners can make its search less reliable.

Minimizing z (auxiliary) with every distance $\leq z$ avoids MAX in the objective and keeps **the same optimization problem**.

Label the distance limit and the objective

A13:A14		f_x	Distance bound z
	A		B
11	Maximum distance		26.9258
12			
13	Distance bound z		
14	Objective for bound		

Enter the labels:

A13: Distance bound z

A14: Objective for bound

These labels identify the adjustable limit z and its objective formula.

Give the longest distance an adjustable upper limit

B13	fx	40
	A	B
11	Maximum distance	26.9258
12		
13	Distance bound z	40
14	Objective for bound	

In B13, enter 40.



Solver will reduce z , with all four distances at most z .

Link the objective to the distance limit

B14	f_x	=B13
	A	B
11	Maximum distance	26.9258
12		
13	Distance bound z	40
14	Objective for bound	40

In B14, enter
=B13



B14 equals z . This gives GRG a smooth formula to minimize.

Keep the actual longest distance in B11

B11	f_x	=MAX(D2:D5)
	A	B
11	Maximum distance	26.9258
12		
13	Distance bound z	40
14	Objective for bound	40

In B11, keep
=MAX(D2:D5)

B11 is the actual longest distance; B13 is the adjustable limit.

Minimize the longest connection

Change the objective to **B14**. Add **B13** to the changing cells. Replace the previous constraint with these two.

Solver Parameters

Set Objective:

To: ☐ Max ☒ Min

By Changing Variable Cells:

Subject to the Constraints:

\$D\$2:\$D\$5 <= \$B\$13

\$B\$13 <= \$B\$8

Add

Change

Delete

Solving Method:

The first constraint puts all four distances below z . The second keeps z within 40 km. At the minimum, z **equals the longest connection**; any larger value could be reduced.

The location with the shortest possible longest link

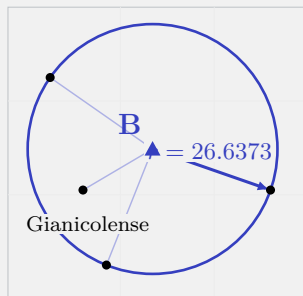
The solution is approximately $x = 26.8417$, $y = 29.7525$. The longest connection is **26.6373 km**.

Connection to	Distance (km)
Monte Mario	26.6373
Gianicolense	17.2303
Roman Castles	26.6373
Tivoli	26.6373

The average is **24.2856 km**. Gianicolense is again the closest tower.

Copy **B11** to **C11** using **Paste Special** → **Values**. C10 and C11 now store the two separate best values. Keep both fixed.

A geometric check of the smallest maximum



Find the **smallest circle containing all four towers**. Place the new tower at its **centre, B**: its radius is the **longest connection**. Here, three towers are on the boundary; Gianicolense is inside and has a shorter connection. **No smaller circle contains all four towers.**

New tower (x, y)	Average (km)	Longest (km)
A : (12.2, 21)	20.4403	39.8000
B : (26.8417, 29.7525)	24.2856	26.6373

A question with two equally important objectives

Management wants a compromise between short connections **on average** and a short **longest connection**.

Keep the 40 km limit.

GOAL
Find the minimum worst-shortfall compromise.

Label the shortfall columns

A16:D16		fx	Objective			
	A		B	C	D	
16	Objective		Current	Best	Shortfall	
17						
18						

Enter the headings:

A16: Objective B16: Current

C16: Best D16: Shortfall

Headings for current values, separate best values and shortfalls.

Label the distance measures

A17:A18		f_x	Average		
	A		B	C	D
16	Objective		Current	Best	Shortfall
17	Average				
18	Maximum bound				

Enter the labels:

A17: Average

A18: Maximum bound

The second row uses z , the upper limit on every connection.

Link the current distances

B17:B18		<i>fx</i>	=B10	
	A	B	C	D
16	Objective	Current	Best	Shortfall
17	Average	24.2856		
18	Maximum bound	26.6373		

Enter the formulas:

B17: =B10

B18: =B13

B17 follows the average; B18 follows the adjustable limit z in B13.

Link the separate best distances

C17:C18		<i>fx</i>	=C10	
	A	B	C	D
16	Objective	Current	Best	Shortfall
17	Average	24.2856	20.4403	
18	Maximum bound	26.6373	26.6373	

Enter the formulas:

C17: =C10

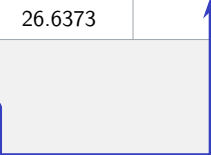
C18: =C11

C10 and C11 keep the separate best values, pasted as fixed numbers.

Calculate the average distance shortfall

D17	<i>fx</i>	=(B17-C17)/C17		
	A	B	C	D
16	Objective	Current	Best	Shortfall
17	Average	24.2856	20.4403	0.1881207
18	Maximum bound	26.6373	26.6373	

In D17, enter
$$=(B17-C17)/C17$$

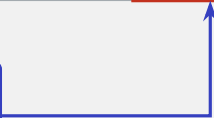


The average is 18.81% above its separate best.

Calculate the longest distance shortfall

D18	<i>fx</i>	$= (B18 - C18) / C18$		
	A	B	C	D
16	Objective	Current	Best	Shortfall
17	Average	24.2856	20.4403	0.1881207
18	Maximum bound	26.6373	26.6373	0

In D18, enter
 $= (B18 - C18) / C18$



Here z equals its separate best, so the shortfall is zero.

Display the shortfalls as percentages

D17:D18	fx	$=(\text{B17}-\text{C17})/\text{C17}$		
	A	B	C	D
16	Objective	Current	Best	Shortfall
17	Average	24.2856	20.4403	18.81%
18	Maximum bound	26.6373	26.6373	0.00%

Select D17:D18. Press Cmd+1.

Format Cells

Number	Alignment	Font	Border	Fill	Protection
Category:		Sample			
Number		18.81%			
Currency					
Percentage		Decimal places: 2 ▼			
		Cancel OK			

Percentage
2 decimals
Then OK.

Both weights are 1, so no extra multiplication is needed.

Label the shortfall limit and objective

A20:A21		f_x	Shortfall bound q
	A		B
20	Shortfall bound q		
21	Objective for shortfall		

Enter the labels:

A20: Shortfall bound q

A21: Objective for shortfall

The same q will limit both shortfalls, in **D17** and **D18**.

Set a starting shortfall limit

B20	f_x	0.2
	A	B
20	Shortfall bound q	0.2
21	Objective for shortfall	

In B20, enter 0.20.

Both q in **B20** and z in **B13** are numbers Solver may change.

Link the objective to the shortfall limit

B21	f_x	=B20
	A	B
20	Shortfall bound q	0.2
21	Objective for shortfall	0.2

In B21, enter
=B20



Minimizing B21 lowers q ; both shortfalls must stay at or below q .

Format the shortfall limit as a percentage

B20:B21		f_x	0.2
	A		B
20	Shortfall bound q		20.00%
21	Objective for shortfall		20.00%

Format Cells

Number	Alignment	Font	Border	Fill
Category:	Sample			
General	20.00%			
Number				
Currency				
Percentage				

Decimal places: 2

Cancel OK

Select B20:B21.
Press **Cmd+1**.

Percentage
2 decimal places
Click **OK**.

0.20 is displayed as 20.00%; calculations use the same value.

Solve for the equal-weight compromise

Minimize **B21**. Change the coordinates, the distance bound z and the shortfall bound q . Keep GRG Nonlinear and nonnegative variables.

Solver Parameters

Set Objective:

To: ☐ Max ☒ Min

By Changing Variable Cells:

Subject to the Constraints:

\$D\$2:\$D\$5 <= \$B\$13

\$B\$13 <= \$B\$8

\$D\$17:\$D\$18 <= \$B\$20

Add

Change

Delete

Solving Method:

B8 stays **40**. C10 and C11 stay at their separate best values. Start from the preceding solution and solve.

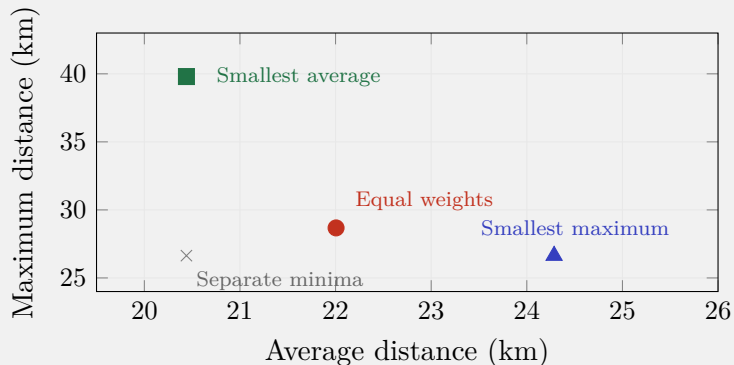
The equal-weight compromise: the answer

The selected location is approximately $x = 23.3930$, $y = 22.9989$.

Measure	Separate best	Compromise	Shortfall
Average distance	20.4403	22.0053	7.6562%
Maximum distance	26.6373	28.6767	7.6562%

See the compromise between the two distances

Each point represents a tower location. Moving towards the lower-left would improve both measures.



The red point limits both increases to **7.6562%**. The two separate minima cannot be achieved at the same location.

Compare the connection limits with VBA

The connection equipment can be configured for different maximum distances. A stricter limit may require a different tower location.

COMPARE ALL THE CONNECTION LIMITS

For **each whole-kilometer limit from 27 to 40 km**, find the location that minimizes the average distance, requiring each of the four connections to be no longer than the selected limit.

Use a VBA **For** loop for the **14 optimizations**. Record the coordinates, average and maximum distance.