ℓ_0-Regularized Huber Loss Function

Deniz Akkaya[1] Mustafa Ç. Pınar[2]

[1]Bilkent University, Turkey, deniz.akkaya@bilkent.edu.tr

[2]Bilkent University, Turkey mustafap@bilkent.edu.tr

Keywords: Huber loss function, sparsity, local minima, global minima

We investigate the solution of the non-convex optimization problem defined using a positive scalar β:

$$\min_{x\in\mathbb{R}^n} F(x) \equiv \Phi(x) + \beta\|x\|_0,$$

where the term $\|x\|_0$ counts the non-zero elements of the vector x, and $\Phi(x) = \sum_{i=1}^{M} \phi_\gamma(r_i(x))$ with $r_i(x) = a_i^T x - d_i$, $i = 1, \ldots, m$. Furthermore, the function $\phi_\gamma(t) : \mathbb{R} \to \mathbb{R}$, known as the Huber loss function [1], is defined using a positive scalar γ as

$$\phi_\gamma(t) = \begin{cases} \frac{t^2}{2\gamma} & \text{if } |t| \leq \gamma \\ |t| - \frac{\gamma}{2} & \text{if } |t| > \gamma. \end{cases}$$

The Huber loss function is one of the central tools of the field of robust statistics (and regression) where an estimator is sought under deviations from the Gaussian nature of the errors in the observations. It is also quite useful in engineering applications as demonstrated in [2]. There are literally hundreds of papers focusing on the sparsity-regularized least squares problem:

$$\min_{x\in\mathbb{R}^n} \|Ax - d\|_2^2 + \beta\|x\|_0 \tag{1}$$

where $A \in \mathbb{R}^{M\times N}$ and $d \in \mathbb{R}^M$, see e.g., [3, 4] where optimality conditions and algorithms are presented for problem (1). To the best of the authors' knowledge, no similar study concerning F is available. We prove that by solving an auxiliary convex optimization problem using the Huber loss function one can find a local minimizer of F. We then give necessary and sufficient conditions under which a local minimizer is strict. We also prove that the set of global minimizers is non-empty. We investigate bounds on the parameter β for which global minimizers are sparse.

References

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