The Wisdom of the Crowd in Dynamic Economies*

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Abstract

The Wisdom of the Crowd applied to financial markets asserts that prices represent a consensus belief that is more accurate than individual beliefs. However, a market selection argument implies that prices eventually reflect only the beliefs of the most accurate agent. In this paper, we show how to reconcile these alternative points of view. In markets in which agents naively learn from equilibrium prices, a dynamic Wisdom of the Crowd holds. Market participation increases agents' accuracy, and equilibrium prices are more accurate than the most accurate agent.

Keywords: Wisdom of the Crowd, Heterogeneous Beliefs, Market Selection Hypothesis, Naive Learning.
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1 Introduction

The informational content of prices is a central issue in the analysis of equilibria of competitive markets. In financial markets, in particular, asset prices are often believed to be good predictors of the economic performance of the underlying fundamentals. Three different mechanisms have been proposed as possible explanations for this remarkable property. The rational expectation and the learning-from-price literatures argue that equilibrium prices are accurate because they reveal and aggregate the information of all market participants. The Market Selection Hypothesis, *MSH*, proposes instead that prices become accurate because they eventually reflect only the beliefs of the most accurate agent. The Wisdom of the Crowd, *WOC*, suggests that market prices are accurate because individual, opposite biases are averaged out by the price formation mechanism.

Although these theories aim to explain the same phenomenon, they rest on different and somehow conflicting hypotheses. In the learning-from-price literature, all agents are assumed to agree on the way to interpret information. In equilibrium, when all private information gets revealed, all agents must hold the same belief because they cannot "agree to disagree." Therefore, the MSH and the WOC arguments are void. By contrast, in the MSH and WOC literatures, agents can disagree on how to interpret information about fundamentals. However, existing models of market selection are incompatible with the WOC because they do not allow for belief heterogeneity in the long run: by selecting the most accurate agent, the market destroys all accuracy gains that could be achieved by balancing out agents' opposite biases. Focusing on static settings, the WOC literature struggles to justify the assumption that the consumption-share/belief distribution is such that the opposite biases of agents cancel out.

In this paper, we bring together the contributions of these three branches of literature and provide conditions for the WOC to occur in dynamic economies. We extend the general equilibrium model of market selection of Sandroni (2000) and Blume and Easley (2006) by allowing the beliefs of some agents to depend on an endogenous market consensus, but not in a fully rational way. When (some) agent beliefs follow our rule, the market selects against inaccurate agents but only when their beliefs cannot be used to increase the consensus accuracy. And the WOC occurs in equilibrium, irrespective of the initial consumption-share/beliefs distribution because selection forces endogenously determine a consumption-share dynamics that makes the market consensus more accurate than the most accurate agent in isolation. Furthermore, when some agents have opposite bias, the consensus becomes as accurate as the truth in the limit of these agents relying only on the consensus.

Specifically, we assume that (some) agent beliefs for next-period states are formed by giving constant weight to two different models. The first model is endogenously generated by the market and represents the market *consensus*. The second model, *dogmatic probabilities*, is agent-specific and represents each agent's subjective probabilistic view of the world. We interpret this rule as depicting the beliefs of an agent who tries to find a compromise between his subjective view about fundamentals and the possibility that markets might be accurate after all.[1]

The dynamics of our economy depends crucially on the definition of the consensus. We start our analysis by adopting a notion of consensus, *market probabilities*, that is informationally taxing to calculate but serves as a theoretical benchmark because it makes the dynamics of beliefs and the occurrence of the WOC qualitatively independent of risk attitudes and the aggregate endowment process. Next, we focus on the case in which agents use consensuses that are easy to calculate. We either assume that the aggregate endowment is constant and agents use the *risk-neutral probabilities* for consensus or we allow for changes in the aggregate endowment but require that agents have common CRRA utility and use a modification of the *risk-neutral probabilities* that corrects for aggregate risk bias. In this setting, we characterize how risk attitudes affect agents survival and the WOC. Ceteris paribus, economies with more risk-averse agents generate more accurate risk-neutral probabilities than economies with less risk-averse agents and the WOC occurs under weaker conditions.

[1]This rule of thumb, first introduced by Manski (2006) in the context of static prediction markets, captures the idea that agents' opinions might depend on a market consensus in a way that is not fully rational. Full rationality would require each agent to have a correct model of the world and of how other agents process information — a situation which never occurs in practice. On the contrary, agents in our model settle on a second best: they naively incorporate other agents' opinions by anchoring their beliefs to the market consensus.

The following describes the structure of the paper and our main findings.
First, in Section 2, we introduce the model of the economy, agent beliefs, the market consensuses, beliefs accuracy, and we define the WOC as the situation in which the consensus is more accurate than all dogmatic probabilities.

Second, in Section 3, we focus on the case in which the consensus is the market probability. We show that the WOC emerges when at least two agents with opposite bias sufficiently weigh the consensus in forming their beliefs. In this case, the equilibrium path exhibits long-run heterogeneity, market probabilities never settle down, and the selection forces endogenously generate a consumption-share/belief dynamic that determines the WOC. Moreover, we demonstrate that market accuracy is a virtuous self-fulfilling prophecy. If some agents with opposite bias are almost certain that the consensus is correct, the consensus is indeed almost correct. Selection forces endogenously determine a consumption share dynamic such that, in equilibrium, the consensus almost coincides with the true probability.

Last, in Section 4, we extend our analysis to the case in which agents use the risk-neutral probability for consensus and characterize how risk attitudes affect the risk-neutral consensus accuracy and the beliefs dynamic. We provide sufficient conditions for the occurrence of the WOC and for the self-fulfilling property of the consensus accuracy to occur that take agents' risk attitudes into consideration. Ceteris paribus, economies with more risk-averse agents generate more accurate risk-neutral probabilities than economies with less risk-averse agents and the WOC occurs under weaker conditions.

Throughout the paper we use simulations for illustrative purposes; their length varies to accommodate the different convergence rates; to ease comparison, we use the same typical path for all simulations unless differently specified. Proofs are in Appendices.

1.1 Related literature

A very influential stream of literature argues that asset prices are accurate because financial markets are an efficient aggregator of private information (Grossman, 1976, 1978; Radner, 1979; Grossman and Stiglitz, 1980). Closely related to the literature on information transmission (Aumann, 1976; Geanakoplos and Polemarchakis, 1982), this literature assumes that agents disagree solely due to differences in their private information and provides conditions under which the price formation mechanism reveals all private information to all agents in the market. Because all agents agree on the way to interpret information, and prices instantaneously reveal all available information, in equilibrium all agents must hold the same beliefs and no WOC or selection based on belief heterogeneity can occur. Prices are accurate because they reflect and aggregate all relevant information. It is hard to imagine, however, that most agents active in financial markets can agree on what information is relevant and how to interpret it — *"Ordinary investors have no model or at best a very incomplete model of the behavior of prices, dividends, or earnings of speculative assets" — Shiller (1984).* In fact, there is overwhelming evidence documenting the inability of agents to process information "rationally," even in simple experimental settings (Kahneman, 2011), and that agents who use well established models might be acting irrationally by failing to account for transaction costs (Barber and Odean, 1999) or estimation errors (DeMiguel et al., 2009).

An alternative explanation for market accuracy, the MSH, relies on the evolutionary argument that markets become accurate because they select for accurate agents (Alchian, 1950; Friedman, 1953). According to the MSH, agents with inaccurate beliefs lose their wealth to accurate agents and, eventually, equilibrium prices are accurate because they reflect only the beliefs of the most accurate agent in the economy (Sandroni, 2000). In these models the market identifies the best model but does not work as an aggregator. By selecting for a unique most accurate agent, the market "destroys" all the accuracy gains that could be achieved by pooling the diverse opinions of the agents who vanish and no WOC can occur. Accordingly, market prices can only be as accurate as the most accurate agent (Blume and Easley, 2009), even in the knife-edge cases in which there are multiple survivors (Jouini and Napp, 2011; Massari, 2013). In addition to our model, others in the market selection literature allow for long-run survival of agents with heterogeneous beliefs, but do not explicitely analyze the accuracy of the resulting prices. Survival of agents with heterogeneous beliefs occurs in economies with incomplete markets (Beker and Chattopadhyay, 2010; Cogley et al., 2013; Cao, 2017), ambiguous averse agents (Guerdjikova and Sciubba, 2015), exogenous saving rules (Bottazzi and Dindo, 2014; Bottazzi et al., 2017), and recursive preferences (Borovička, 2015; Dindo, 2015). A model that merges elements of rational learning from prices and selection is Mailath and Sandroni (2003). This model does not endogenously generate WOC because long-run heterogeneity is a consequence of the presence of noise traders.

Finally, the WOC argument (initially proposed by Galton, 1907, and more recently popularized by Surowiecki, 2005), hypothesizes that asset prices are accurate because the opposite, idiosyncratic errors of individual agents are averaged out by the price formation mechanism. The WOC hypothesis has inspired a growing interest in prediction markets (Wolfers and Zitzewitz, 2004; Arrow et al., 2008) and social trading platforms (Chen et al., 2014; Pelster et al., 2017). Within the prediction markets literature, most of the attention has been focused on static settings. However, there is no solid foundation to justify the WOC argument. WOC can occur only if the consumption-shares/beliefs distribution is such that individual mistakes cancel out. The main limitation of WOC is the lack of theoretical arguments supporting this assumption. Further, there is evidence that even if agents were rationally processing private unbiased signals, the aggregate beliefs might be biased nonetheless (Ali, 1977; Manski, 2006; Ottaviani and Sørensen, 2014). Works that also combine dynamic elements such as ours in prediction markets are Kets et al. (2014) and Bottazzi and Giachini (2016). The WOC has also been investigated within other contexts. In the literature of social learning in networks, Golub and Jackson (2010) and Jadbabaie et al. (2012) provide conditions under which agents imitating each other and naively updating their beliefs — using a rule similar to ours — can achieve the same outcome as rational learning models. In the literature on collective problem-solving, Hong and Page (2004) explore the trade-off between opinion diversity and the difficulty in identifying optimal solutions (see also Page, 2007).

Here, we propose a model that combines these points of views and offers a general framework which overcomes their individual shortcomings.

2 The model

We study a standard dynamic stochastic exchange economy with complete markets where agents have heterogeneous beliefs about the realizations of states of nature. Assuming complete markets implies that agents can use contracts to exchange contingent commodities for any date and any state. Since agents have heterogeneous beliefs but are otherwise identical, they assign different evaluations for contingent commodities and use the available contracts to trade on such differences. Market clearing determines equilibrium prices and allocations. At equilibrium, the agent who assigns a higher probability to a certain event takes a long position (in excess of his equilibrium consumption if beliefs were homogeneous) in the contract paying a unit of the consumption good in that event. The agent with a lower probability supplies the contract. We are interested in studying the resulting consumption-share and state-price dynamics and in characterizing their long-run properties. The central question is whether market forces can endogenously generate a measure of consensus which is more accurate than all agents in isolation.

Time is discrete, indexed by t, and begins at date $t = 0$. In each period $t \geq 1$, the economy can be in one of S mutually exclusive states, $\mathcal{S}$. The set of partial histories until t is the Cartesian product $\Sigma^t = \times^t \mathcal{S}$ and the set of all paths is $\Sigma := \times^\infty \mathcal{S}$. $\sigma = (\sigma_1, ...)$ is a representative path, $\sigma^t = (\sigma_1, ..., \sigma_t)$ is a partial history until period t, and $\mathcal{F}_t$ is the σ-algebra generated by the cylinders with base σ^t. By construction $(\mathcal{F}_t)_{t=0}^\infty$ is a filtration and $\mathcal{F}$ is the σ-algebra generated by their union.

We assume that states of nature are i.i.d. with $P_t = P \in \Delta^{|\mathcal{S}|}$ for all $t \geq 1$. With an abuse of notation, P also denote the true measure on $(\Sigma, \mathcal{F})$. For any probability measure ρ on $(\Sigma, \mathcal{F})$, $\rho(\sigma^t) := \rho(\{\sigma_1 \times ... \times \sigma_t \times S \times S \times ...\})$ is the marginal probability of the partial history σ^t while $\rho_t := \rho(\sigma_t|\sigma^{t-1}) = \frac{\rho(\sigma^t)}{\rho(\sigma^{t-1})}$ is the conditional probability of the generic state σ_t given σ^{t-1}, so that $\rho(\sigma^t) = \prod_{\tau=1}^t \rho(\sigma_\tau|\sigma^{\tau-1})$.

Next, we introduce a number of economic variables with time index t. All these variables are adapted to the information filtration $(\mathcal{F}_t)_{t=0}^\infty$.

The economy contains a finite set of agents $\mathcal{I}$. For all paths σ, each agent $i \in \mathcal{I}$ is endowed with a stream of the consumption good, $(e_t^i(\sigma))_{t=0}^\infty$. We take the consumption good in $t = 0$ as the numéraire of the economy. Each agent's objective is to maximize the stream of discounted expected utility he gets from consumption. Expectations are computed according to agent beliefs p^i, a measure on $(\Sigma, \mathcal{F})$. Beliefs are heterogeneous and agents agree to disagree. Naming $q(\sigma^t)$ the date $t = 0$ price of the asset that delivers one unit of consumption in event σ^t and none otherwise, agent i maximization reads:

$$\max_{(c_t^i(\sigma))_{t=0}^{\infty}} \mathrm{E}_{p^i}\left[\sum_{t=0}^{\infty} \beta^t u^i(c_t^i(\sigma))\right] \quad s.t. \quad \sum_{t\geq 0} \sum_{\sigma^t \in \Sigma^t} q(\sigma^t)\left(c_t^i(\sigma) - e_t^i(\sigma)\right) \leq 0.$$

A competitive equilibrium is a sequence of prices and, for each agent, a consumption plan that is preference maximal on the budget set, and such that markets clear in every period: $\forall (t, \sigma), \sum_{i\in\mathcal{I}} e_t^i(\sigma) = \sum_{i\in\mathcal{I}} c_t^i(\sigma)$. Assumptions **A1-A4** below are standard in the market selection literature: **A1-A3** ensure the existence of a competitive equilibrium, while **A4** guarantees that the market selects for the most accurate agent(s) rather than for those that save the most. In Appendix C we give the formal definition of the competitive equilibrium when agents' beliefs depend on the endogenous consensuses and prove its existence.

A1 The payoff functions $u^i : \mathbb{R}_+ \to [-\infty, +\infty]$ are C^1, strictly concave, increasing, and satisfy the Inada condition at 0 — that is, $u^i(c)' \to \infty$ as $c \searrow 0$.

A2 The aggregate endowment is uniformly bounded from above and away from 0:

$$\infty > F > \sup_{t,\sigma} \sum_{i\in\mathcal{I}} e_t^i(\sigma) > \inf_{t,\sigma} \sum_{i\in\mathcal{I}} e_t^i(\sigma) > f > 0.$$

A3 (i) For all agents $i \in \mathcal{I}$ and for all (t, σ), $p^i(\sigma^t) > 0 \Leftrightarrow P(\sigma^t) > 0$.
$(ii) \exists \epsilon > 0$ such that for all agents $i \in \mathcal{I}$ and for all (t, σ), $p^i(\sigma_t | \sigma^{t-1}) > \epsilon$.[2]

A4 All agents have common discount factor: $\forall i \in \mathcal{I}, \beta^i = \beta \in (0, 1)$.

[2]The way we define p^i (Definition 4) ensures that **A3** is satisfied even if p^i are endogenous (Lem. 4).

2.1 Agents accuracy and survival

In this section, we remind the reader of standard definitions and results from the market selection literature. The asymptotic fate of an agent is characterized by his consumption-shares as follows.

Definition 1. *Agent i* vanishes *if* $\lim_{t\to\infty} c_t^i(\sigma) = 0$ *P-a.s., he* survives *if* $\limsup_{t\to\infty} c_t^i(\sigma) > 0$ *P-a.s., he* dominates *if* $\lim_{t\to\infty} \frac{c_t^i(\sigma)}{\sum_{i\in\mathcal{I}} c_t^i(\sigma)} = 1$ *P-a.s..*

Since it became the standard after Blume and Easley (1992), we rank agents' accuracy according to their average (conditional) relative entropies (Kullback-Leibler divergences).

Definition 2. *The* average relative entropy *from p^i to the true probability P is*

$$\bar{d}(P||p^i) := \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} d(P||p^i_\tau),$$

where, for all τ, $d(P||p^i_\tau) := \mathrm{E}_P \left[\ln \frac{P(\sigma_\tau)}{p^i(\sigma_\tau|\sigma^{\tau-1})}\right]$.

The average relative entropy is uniquely minimized at $p^i = P$, strictly convex, and $d(P||\pi) = \bar{d}(P||\pi)$ P-a.s. whenever P and π are i.i.d. measures. We say that

Definition 3. *Agent i is* more accurate *than agent j if* $\bar{d}(P||p^i) < \bar{d}(P||p^j)$, *$P$-a.s.. Agent i is* as accurate as *agent j if* $\bar{d}(P||p^i) = \bar{d}(P||p^j)$, *$P$-a.s..*

This notion of accuracy is commonly adopted in the market selection literature because of its straightforward implications for agents survival. Under **A1-A4**, the pairwise comparison of agents accuracies delivers a sufficient condition for an agent to vanish.

Proposition 1. *(Sandroni, 2000). Under **A1-A4**, agent i vanishes if there exists an agent $j \in \mathcal{I}$ who is more accurate:*

$$\bar{d}(P||p^j) < \bar{d}(P||p^i) \ P\text{-a.s.} \Rightarrow \textit{Agent } i \textit{ vanishes.}$$

This fundamental result, together with known results in probability theory, allows to easily characterize survival of agents with exogenous beliefs. The difficulty we have to overcome is to calculate the accuracy of agents whose beliefs depend on an endogenous measure of consensus.

2.2 Agents beliefs

We assume that beliefs for next-period states of the agents in our economy are formed by giving constant weights to two different models. The first model p^C is endogenous, and it represents the *market consensus*. The second model, *dogmatic probabilities* (π^i), is exogenous and agent specific. When an agent gives zero weight to the consensus, his beliefs are exogenous and we make no assumption about them — aside from ***A3*** and the basic requirement that $\bar{d}(P||\cdot)$ exists. Otherwise, we assume that dogmatic probabilities are i.i.d.[3] and in the strict interior of the simplex, which ensures that **A3** holds (Lemma 4 in Appendix).

Definition 4. *For all (t, σ), agent i beliefs are given by*

$$p^i(\sigma_t|\sigma^{t-1}) = (1 - \alpha^i)p^C(\sigma_t|\sigma^{t-1}) + \alpha^i \pi^i(\sigma_t), \tag{1}$$

where $\alpha^i \in (0, 1]$ and for $\alpha^i \in (0, 1), \pi^i$ is strictly positive measure on $(\mathcal{S}, 2^{\mathcal{S}})$.

This rule describes the attitude of an agent who partially believes in the WOC. The parameter α^i determines how much agent i believes in the accuracy of the consensus. Having $\alpha^i = 1$ represents the extreme scenario in which agent i ignores the consensus. This is the standard case in the market selection literature, where it is typically assumed that agent beliefs are independent of each other and of equilibrium quantities. Whereas $\alpha^i = 0$ represents the case in which agent i does not give any weight to his dogmatic probabilities because he is certain that markets are accurate — with a similar attitude to the economist who finds a $20 bill lying on the ground and refuses to believe it. The intermediate cases of $\alpha^i \in (0, 1)$ are those that generate the most interesting results.[4]

Definition 4 describes a mental attitude that is consistent with many known biases including anchoring (Shiller, 1999) and herding (Lakonishok et al., 1992). Furthermore,

[3] All results generalize verbatim to the case in which the π^i probabilities are derived via Bayes rule from an i.i.d. prior support. Because the Bayesian posterior generically converges to a unique i.i.d. model (the model with the lowest K-L divergence to the truth, Berk, 1966) and our measure of accuracy (Definition 3) is an average measure, these Bayesian agents can be treated WLOG as agents with i.i.d. beliefs in terms of survival and accuracy.

[4] We rule out $\alpha^i = 0$ because $\alpha^i = 0$ for all $i \in \mathcal{I}$ leads to an indeterminate equilibrium.

the beliefs formation rule of Definition 4 has been used to discuss the effect of agents' partial learning from equilibrium prices in the context of static prediction markets, (Manski, 2006); a similar rule is used in the learning literature on networks by Jadbabaie et al. (2012); and beliefs (1) determine a portfolio that (assuming log utility) coincides with the Fractional-Kelly rule proposed by MacLean et al. (2011) in the portfolio theory literature.

2.3 A definition of the Wisdom of the Crowd

We say that the WOC^C occurs if the market consensus, p^C, is more accurate than the beliefs of the most accurate agent in isolation. Two probabilities play a special role in our definition: the Best Individual Probability (π^{BIP}), which is the most accurate dogmatic probability, and the Best Collective Probability (π^{BCP}), which is the most accurate combination of agents' dogmatic probabilities. Moreover, we say that dogmatic probabilities are diverse when the Best Collective Probability differs from the Best Individual Probability, that is, if it is possible to combine dogmatic probabilities into a prediction that is more accurate than that of all dogmatic probabilities.

Definition 5. *Given a set of dogmatic probabilities* $\{\pi^1, ...\pi^I\}$*:*

- *the* Best Individual Probability *is:* $\pi^{BIP} = \underset{\pi \in \{\pi^1,...,\pi^I\}}{\operatorname{argmin}} \bar{d}(P||\pi)$*;*
- *the* Best Collective Probability *is:* $\pi^{BCP} = \underset{p \in Conv(\pi^1,...,\pi^I)}{\operatorname{argmin}} \bar{d}(P||p)$*;*
- *Agents beliefs are* diverse *if it is possible to achieve accuracy gains by balancing the different opinions of market participants:* $\pi^{BIP} \neq \pi^{BCP}$*.*

Given our definitions of agent beliefs and consensuses (below), when an agent is alone in the market his beliefs, his dogmatic probabilities and the consensus coincide ($p^i = \pi^i = p^C$). Therefore, we can define the WOC as follows.

Definition 6. *The* WOC^C occurs *if* p^C *is more accurate than* π^{BIP}*:*

$$\bar{d}(P||p^C) < \bar{d}(P||\pi^{BIP}), \ P\text{-a.s.}.$$

To gain intuition, consider a two-state, $\mathcal{S} = \{u, d\}$, two-agent, $\mathcal{I} = \{1, 2\}$, economy. The true probability of state u is $P(u) = .5$. Agent 1 is pessimistic about u, while agent 2 is optimistic. Their dogmatic probabilities are $\pi^1(u) = .4$ and $\pi^2(u) = .7$, respectively. Clearly, agent 1 has the most accurate dogmatic probabilities, thus $\pi^{BIP} = \pi^1 = .4$, while the most accurate way to combine the dogmatic probabilities of the two agents is $\frac{2}{3}\pi^1(u) + \frac{1}{3}\pi^2(u) = p^{BCP} = P$. The WOC occurs if market probabilities are more accurate than the dogmatic probability of agent 1 (and thus 2) — in other words, if the market consensus is more accurate than all market participants in isolation.

2.4 Market consensuses

A crucial point of our analysis is the definition of the market consensus p^C. We conduct our analysis using different measures of consensus. The rationale behind these measures is that the consensus obtained in an economy with a unique agent must coincide with the beliefs of the agent. All the measures of consensus we propose coincide in economies with constant aggregate endowment in which all agents have log utility. However, under more general assumptions they are not the same because they are differently affected by agent risk attitudes and fluctuations of the aggregate endowment.

The first measure of consensus we propose is *market probabilities*: p^M.

Definition 7. *For all* (t, σ)*,* market probabilities *are*

$$p^M(\sigma_t|\sigma^{t-1}) = \sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1}) \frac{\bar{c}^i_{t-1}}{\sum_{j\in\mathcal{I}} \bar{c}^j_{t-1}}, \tag{2}$$

where $\bar{c}^i_t = \frac{1}{u^i(c^i_t(\sigma))'}$.

If all agents have log utility and the aggregate endowment is constant, p^M coincides with the risk-neutral probabilities and can be calculated from equilibrium prices alone. In these economies Rubinstein (1974) shows that a representative agent exists and that his unconditional beliefs are $\sum_{i\in\mathcal{I}} p^i(\sigma^t)\frac{c^i_0}{\sum_{j\in\mathcal{I}} c^j_0}$. Lemma 1 shows that p^M makes the analysis of general economies qualitatively equivalent to that of a log economy with no aggregate risk, albeit a distortion of the initial weights.

Lemma 1. *Under **A1-A4**, on a competitive equilibrium for all* (t, σ) *it holds*

$$p^M(\sigma^t) = \sum_{i \in \mathcal{I}} p^i(\sigma^t) \frac{\vec{c}_0^i}{\sum_{j \in \mathcal{I}} \vec{c}_0^j}.$$

For the general case, the calculation of p^M requires knowledge of the preferences and the consumption-shares of all agents. While it is unlikely that an agent in the market would have this degree of information, we use market probability to set a benchmark for the results that follow.

Next, we propose measures of consensus that can be easily calculated from equilibrium prices. When the aggregate endowment is constant, we study the occurrence of the WOC when some of the agents use the risk-neutral probabilities for consensus.

Definition 8. *For all* (t, σ)*, the risk-neutral consensus is*

$$p^{RN}(\sigma_t|\sigma^{t-1}) = \frac{q(\sigma_t|\sigma^{t-1})}{\sum_{\tilde{\sigma}_t} q(\tilde{\sigma}_t|\sigma^{t-1})}, \tag{3}$$

where $q(\sigma_t|\sigma^{t-1}) := \frac{q(\sigma^t)}{q(\sigma^{t-1})}$ *is the equilibrium price of a claim that pays a unit of consumption at period/event* σ_t*, in terms of consumption at period/event* σ^{t-1}*.*

The analysis of economies in which agents rely on the risk-neutral consensus is more complex than it is for agents using p^M because agents' risk attitudes do affects p^{RN} accuracy and thus agents accuracy and survival. We show that ceteris paribus, economies with more risk-averse agents generate more accurate risk-neutral probabilities than economies with less risk-averse agents and the WOC^{RN} occurs under weaker conditions. Lemma 2 express the equilibrium value of p_t^{RN} in a way that facilitates its comparison to p_t^M.

Lemma 2. *Under **A1-A4**, on a competitive equilibrium for all* (t, σ) *it holds*

$$p^{RN}(\sigma_t|\sigma^{t-1}) \propto \sum_{i \in \mathcal{I}} p^i(\sigma_t|\sigma^{t-1}) \frac{\vec{c}_{t-1}^i}{\sum_{j \in \mathcal{I}} \vec{c}_t^j}.$$

The difference between p^M and p^{RN} becomes apparent comparing the weights given to agent beliefs in Definitions 7 with those in Lemma 2 ($\frac{\bar{c}^i_{t-1}}{\sum_{j\in\mathcal{I}}\bar{c}^j_{t-1}} \neq \frac{\bar{c}^i_{t-1}}{\sum_{j\in\mathcal{I}}\bar{c}^j_{t}}$). The first one is state independent because the ratio involves the marginal utility of consumptions in the same period. The second one is state dependent because the ratio compares marginal utilities in two different periods. Moreover, only p^{RN} requires to be normalized.

In an economy with a unique agent and constant aggregate endowment for all $(t,\sigma), \bar{c}_t = \bar{c}_{t-1}$ and both measures satisfy our desiderata to be an unbiased estimator of the beliefs of the agent . However, p^{RN} fails to satisfy this property in economies where the aggregate endowment varies because there are some (t,σ) such that $\bar{c}_t \neq \bar{c}_{t-1}$.

The last measure of market consensus we study can be calculated from prices and aggregate endowment alone and corrects for this bias in economies in which all agents have common CRRA utility function $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ — which can be interpreted as representing the industry standard.

Definition 9. *For all (t,σ), the* γ-adjusted risk-neutral consensus *is*

$$p^{RN}_{\gamma}(\sigma_t|\sigma^{t-1}) = \frac{q(\sigma_t|\sigma^{t-1})\boldsymbol{e}_t(\sigma)^{\gamma}}{\sum_{\tilde{\sigma}_t} q(\tilde{\sigma}_t|\sigma^{t-1})\boldsymbol{e}_t(\tilde{\sigma})^{\gamma}} \tag{4}$$

where $\boldsymbol{e}_t(\sigma) = \sum_{i\in\mathcal{I}} e^i_t(\sigma)$ is the aggregate endowment.

Lemma 3 express the equilibrium value of p^{RN}_{γ} in economies in which all agents have identical CRRA utilities in a way that facilitates its comparison with p^{RN}. It shows that p^{RN}_{γ} is immune to biases due to fluctuations of the aggregate endowment because it is a consumption-share version of the p^{RN} consensus.

Lemma 3. *Under **A1-A4**, if all agents have common CRRA utility with parameter $\gamma \in (0,\infty)$, on a competitive equilibrium for all (t,σ) it holds that*

$$p^{RN}_{\gamma}(\sigma_t|\sigma^{t-1}) \propto \sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1}) \frac{\phi^i_{t-1}(\sigma)^{\gamma}}{\sum_{j\in\mathcal{I}} \phi^j_t(\sigma)^{\gamma}};$$

where $\phi^i_t(\sigma) = \frac{c^i_t(\sigma)}{\sum_{j\in\mathcal{I}} c^j_t(\sigma)}$.

3 Main results related to p^M

In this section, we characterize the accuracy of p^M, we provide necessary conditions and sufficient conditions for the WOC^M to occur and we demonstrate its self-fulfilling property. If a diverse group of agents believes in the accuracy of p^M, market probabilities are indeed accurate.

3.1 Accuracy of p^M

The main advantage of using p^M for consensus, is that risk attitudes do not have a qualitative effect on its accuracy. Proposition 2 characterizes the relative accuracy of p^M with respect to that of agents without solving for the equilibrium and independently of how agents form their beliefs.

Proposition 2. *Under **A1-A4**,*

(*a*) *no agent can be more accurate than* p^M*:*

$$\forall i \in \mathcal{I}, \bar{d}(P||p^i) \geq \bar{d}(P||p^M), \; P\text{-}a.s.;$$

(*b*) *agent* i *survives only if he is as accurate as* p^M*:*

$$\text{Agent } i \text{ survives} \Rightarrow \bar{d}(P||p^i) = \bar{d}(P||p^M), \; P\text{-}a.s..$$

Proof. See Appendix A. □

Proposition 2 simplifies our analysis because standard techniques to approximate market probabilities and agent beliefs accuracy cannot be used when agent beliefs depend on the endogenous consensus. All the results of this section are obtained by combining Propositions 1 and 2, and by taking advantage of the convexity of the relative entropy.

Market probabilities provide a fundamental hedging benefit to the agents. By believing in p^M an agent weakly improves its accuracy irrespectively of his dogmatic beliefs, of the beliefs of the other agents, and of the true probability.

Proposition 3. *Under **A1-A4**, if* $\alpha^i \in (0,1)$ *and* i *uses* p^M *for consensus,*

$$\bar{d}(P||p^i) \leq \bar{d}(P||\pi^i) \; P\text{-}a.s.;$$

with strict inequality if $p_t^M \neq \pi^i$ a positive fraction of periods.

Proof. See Appendix A. $\square$

If π^i is the true model — or is the probability obtained by Bayes rule when the prior support is correctly specified —, agent i's average accuracy is not diminished by mixing with market probabilities because market probability converges to π^i exponentially fast since he dominates. Otherwise, if agent i's subjective probabilistic model of the world is incorrect — or if he cannot learn it because its prior support does not contain the true model —, mixing with the consensus improves agent i's accuracy whenever the consensus is more accurate than his dogmatic beliefs.

Furthermore, p^M is at least as accurate as π^{BIP} and at most as accurate as π^{BCP}, provided that all agents with $\alpha^i \in (0,1)$ use p^M for consensus.

Corollary 1. *Under **A1-A4**, if all agents with $\alpha^i \in (0,1)$ use p^M for consensus, p^M is at least as accurate as π^{BIP} and at most as accurate as π^{BCP}:*

$$\bar{d}(P||\pi^{BCP}) \leq \bar{d}(P||p^M) \leq \bar{d}(P||\pi^{BIP}), \quad P\text{-}a.s..$$

Proof. $\bar{d}(P||p^M) \leq^{By\ Prop.2} \bar{d}(P||p^{BIP}) \leq^{By\ Prop.3} \bar{d}(P||\pi^{BIP})$.
$\bar{d}(P||p^M) \geq \bar{d}(P||\pi^{BCP}) =^{P\text{-a.s.}} \min\limits_{p \in Conv(\pi^1,...,\pi^I)} d(P||p)$ because $\forall(t,\sigma), p_t^M \in^{By\ Lem.5} Conv(\pi^1,...,\pi^I)$. $\square$

Corollary 1 is proven showing that in the long-run either the agent with the most accurate dogmatic probabilities dominates, and market probabilities are as accurate as π^{BIP}, or there is long-run heterogeneity, and market probabilities are a convex combination of the surviving agents' dogmatic probabilities — thus, at most as accurate as π^{BCP} by definition.

3.2 Necessary conditions for WOC^M

When the reference consensus is p^M, we identify two necessary conditions for WOC^M. First, it must be possible to achieve accuracy gains by balancing the different opinions of market participants (diversity). Second, at least some of the agents must believe in market accuracy — which is necessary for long-run heterogeneity. Only under these conditions selection forces can induce a non-degenerate consumption-share distribution that makes market probabilities more accurate than the most accurate agent in isolation.

Proposition 4. *Under **A1-A4**, if all agents use p^M for consensus, WOC^M can occur only if beliefs are diverse and the beliefs of at least one agent depend on p^M.*

Proof. See Appendix A. □

The first requirement (diversity) tells us that the WOC^M cannot occur if all agents share the same bias. For example, in an economy with two states in which all dogmatic probabilities overweight the same state, no WOC^M can occur because the most accurate combination of agent beliefs is the one obtained by giving all wealth to the least biased among the agents (BIP). Furthermore, this condition tells us that the WOC^M cannot occur if there is an agent that knows (or eventually learns) the truth because $P = \pi^{BCP} = \pi^{BIP}$.

The second requirement (relevance of the market consensus) confirms the standard result in the selection literature that WOC cannot occur when agents' beliefs do not depend on endogenous quantities. For example, suppose the market has an optimistic and a pessimistic agent. If the pessimistic agent is less accurate than the optimist, then the pessimist vanishes, and market probabilities become optimistic. Clearly, this is not the best way to make use of agent opinions. A better way would be to redistribute consumption-shares in such a way that market probabilities become accurate by balancing the opposite biases of the two agents. However, this is impossible when agent beliefs are independent of each other because only the most accurate agent survives (Blume and Easley, 2009).

3.3 Sufficient conditions for the WOC^M

While the market might be populated by many agents with arbitrary beliefs and preferences, the next condition shows that to guarantee that the WOC^M occurs it suffices to verify a condition on only two agents. If agent BIP mixes with p^M and if at the prices set by BIP there is an agent with $\alpha^i \in (0,1)$ that is more accurate than BIP, then at least two agents survive and WOC^M occurs.

Proposition 5. *Under **A1-A4**, WOC^M occurs and at least two agents survive if agent BIP relies on p^M with $\alpha^{BIP} \in (0,1)$ and*

$$\exists i \in \mathcal{I} : \bar{d}(P||(1-\alpha^i)\pi^{BIP} + \alpha^i \pi^i) < \bar{d}(P||\pi^{BIP}) \tag{5}$$

Proof. See Appendix A. □

For intuition, consider a log economy with two states, $\mathcal{S} = \{u, d\}$, and two agents $\mathcal{I} = \{BIP, 2\}$. The true probability of state u is $P(u) = .5$. Agent BIP is pessimistic about u, while agent 2 is optimistic. Their dogmatic probabilities are $\pi^{BIP}(u) = .4$ and $\pi^2(u) = .7$, respectively. Because agent beliefs are diverse ($\pi^{BIP} \neq \pi^{BCP} = P$) it is possible to achieve accuracy gains by mixing their opinions.

Figure 1 [top] shows that long-run heterogeneity and WOC^M occurs if both agents give enough weight to market probabilities. With α^{BIP}=α^2=.2, our sufficient condition is satisfied and we have long-run heterogeneity and WOC^M because the dependency of agent beliefs on market probabilities makes it impossible for any agent to dominate. When agent BIP (2) consumption-shares become large, his dogmatic probabilities have a large impact on market probabilities, making his beliefs less accurate than those of agent 2 (BIP). Thus, consumption-shares never find a resting point, market probabilities remain close to P and are more accurate than π^{BIP}. Formally, the consumption-shares are mean-reverting processes around the value $\bar{\phi}^{BIP}$ that determines a market probability $\bar{p}^M$ which makes agents BIP and 2 equally accurate, i.e. $\phi_t^{BIP} \gtreqless \bar{\phi}^{BIP} \Leftrightarrow d(P||p_t^{BIP}) \gtreqless d(P||p_t^2)$. The WOC^M occurs because $\bar{p}^M$ is more accurate than π^{BIP} and π^2, and market probabilities stay close to $\bar{p}^M$ a large enough number of periods. Figure 1 [mid] shows that the WOC^M does not occur if agent 2 does not give enough weight to p^M because only agent BIP survives.[5] Last, Figure 1 [bottom] shows that long-run heterogeneity

[5] With $\alpha^2 = .9$, agent 2 vanishes because he is less accurate than agent BIP for every consumption-share distribution: $\forall c_t^{BIP}, d(P||p_t^2) > d(P||\pi_t^{BIP})$. This can be verified by noticing that agent 2's beliefs are less accurate than agent BIP's even when agent BIP dominates and sets equilibrium prices equal to his dogmatic probabilities π^{BIP}: $p^2|_{p^M=\pi^{BIP}} = .1(.4) + .9(.7) = .67 \Rightarrow d(P||p^2|_{p^M=\pi^{BIP}}) > d(P||\pi^{BIP})$.

is not a sufficient condition for WOC^M. If agent BIP does not rely on the consensus ($\alpha^{BIP} = 1$), we do have long-run heterogeneity, but no WOC^M because agent BIP survives and, by Proposition 2, p^M is as accurate as every agent that survive.

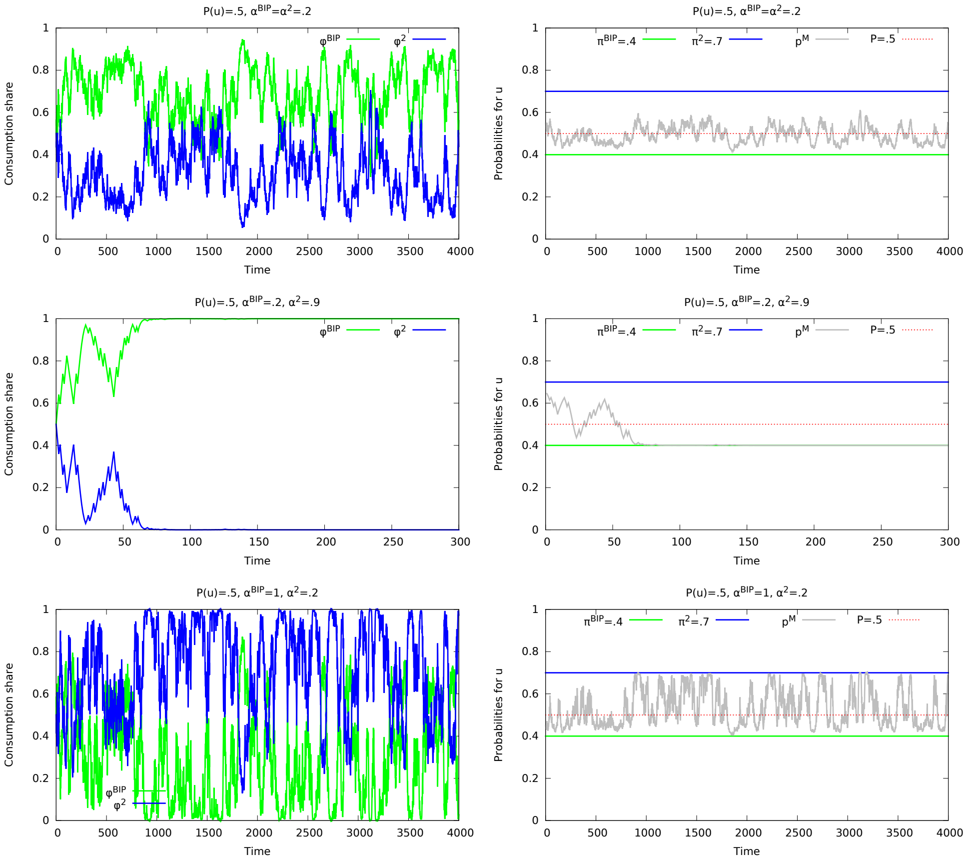

Figure 1: Consumption-shares **[left]** and market probability **[right]** dynamics in two log-economies with identical dogmatic beliefs $[\pi^{BIP}(u), \pi^2(u)] = [.4, .7]$ and different mixing coefficients. **[top]**: with $[\alpha^{BIP}, \alpha^2] = [.2, .2]$, condition (5) holds and the WOC^M occurs. Consumption-shares never find a resting point, and market probabilities are more accurate than π^{BIP}. **[mid]**: with $[\alpha^{BIP}, \alpha^2] = [.2, .9]$, agent 2 doesn't give enough weight to p^M to survive. The WOC^M does not occur, agent BIP dominates, and market probabilities are as accurate as π^{BIP}. **[bottom]**: with $[\alpha^{BIP}, \alpha^2] = [1, .2]$, there is long-run heterogeneity because agent 2 gives enough weight to p^M for survival, but the WOC^M does not occur. p^M is a mean-reverting process with the same accuracy of π^{BIP} because agent BIP survives and his beliefs are independent of p^M.

3.4 Accurate markets: A self-fulfilling prophecy (p^M)

Here we demonstrate that if there is a group of agents in the economy with beliefs around the truth that are (almost) sure that market probabilities are accurate, then market probabilities are indeed (almost) accurate, irrespective of the beliefs of the other agents. By strongly relying on market probabilities, agents generate a virtuous interaction that makes both their beliefs and the market more accurate. In equilibrium, the selection forces endogenously generate a consumption-share/beliefs distribution which determine market probabilities that are (almost) correct even if no agent knows the truth.

Theorem 1. *Let* $(\mathcal{E}_\alpha)$ *be a family of economies that satisfies* ***A1-A4*** *with a subset of agents* $\hat{\mathcal{I}}$ *that relies on* p^M *with* $\alpha^i \in (0, \bar{\alpha}]$ *and such that* $P \in Conv(\hat{\mathcal{I}})$*. Name each economy market probabilities process* $(p^M_{t,\bar{\alpha}})_{t=0}^{\infty}$*, then:*

$$\lim_{\bar{\alpha}\to 0} \bar{d}(P||p^M_{\bar{\alpha}}) = 0, \quad P\text{-}a.s..$$

Proof. See Appendix A. □

Theorem 1 is proven by leveraging the equilibrium condition of Proposition 2, which allows us to look directly at the long run equilibrium outcomes, rather than characterizing the stochastic equilibrium dynamics of the economy. Its validity does not require any assumption on the beliefs of agents in $\mathcal{I} \setminus \hat{\mathcal{I}}$ beside **A3**.

The intuition regarding the equilibrium dynamics goes as follows.[6] The p^M process is characterized by three parameters which depend on $\bar{\alpha}$. These are its drift, its variance, and the threshold, $\hat{p}^M$, that determine a drift change. The effect of $\bar{\alpha}$ on $\hat{p}^M$ is easy to obtain: $\hat{p}^M \to^{\bar{\alpha}\to 0} P$. The theorem holds because for every interval around $\hat{p}^M$, $\bar{\alpha}$ can be chosen small enough to ensure that the market belief process spends most of its periods in that interval. The difficulty in proving the result is that a lower $\bar{\alpha}$ implies a lower variance, but also a weaker mean-reverting drift of the market probability process — the selection forces are weaker because agent beliefs become more similar. Thus, we have to determine which effect dominates when $\bar{\alpha}$ is small. Our result implies that

[6]The proof of Theorem 3 formalizes this intuition verbatim, under stronger assumptions. The proof of Theorem 1 is shorter and more general, but does not give intuition about the equilibrium dynamics.

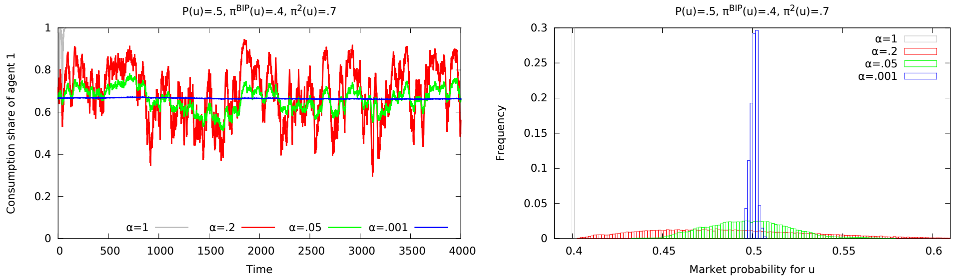

Figure 2: Consumption-share dynamics **[left]** and p^M frequencies **[right]** in four log-economies with true probability $P(u) = .5$, two agents with dogmatic probabilities $\pi^{BIP}(u) = .4$ and $\pi^2(u) = .7$, $\alpha^{BIP} = \alpha^2 = \bar{\alpha}$ and four different values of $\bar{\alpha} = [1,\ .2,\ .05,\ .001]$. The figure shows that a smaller $\bar{\alpha}$ determines frequencies of p^M that are more concentrated around the truth.

the accuracy gain for a more accurate mean-reverting point and a lower variance of the market probability process more than compensates for the accuracy loss due to weaker mean-reverting forces. Although market probabilities might take a long time to reach $\hat{p}^M$ when $\bar{\alpha}$ is small, a low $\bar{\alpha}$ makes p^M accurate because it forces p^M to remain close to $\hat{p}^M$ after reaching it.

Figure 2 illustrates Theorem 1 by showing the consumption-share dynamics and the frequency of market probabilities of four economies that differ only in their value of $\bar{\alpha}$. All economies have two agents with dogmatic probabilities $\pi^{BIP}(u) = .4$ and $\pi^2(u) = .7$, so that $\pi^{BIP} \neq P \in Conv(\pi^{BIP}, \pi^2)$ and $\alpha^{BIP} = \alpha^2 = \bar{\alpha}$. As per Proposition 4, when $\bar{\alpha} = 1$, no WOC occurs: prices are as accurate as π^{BIP}. As per Proposition 5, for $\bar{\alpha}$ low enough, no agent dominates and market probability is more accurate than π^{BIP}. In this specific example, $\bar{\alpha} = 0.2$ is already small enough for agent BIP not to dominate. As per Theorem 1, for $\bar{\alpha} = .001 \approx 0$ the market probabilities distribution becomes concentrated in a small interval around P, which makes p^M almost as accurate as the truth. If agents strongly believe that the market is accurate, then the market is indeed accurate.

4 Main results related to p^{RN} and p^{RN}_γ

In this section we study the long-run property of markets in which (some) agents use either p^{RN} or p^{RN}_γ for market consensus under the following assumptions.

***A5* :** Either there is constant aggregate endowment, or all agents in $\bar{\mathfrak{I}} := BIP \cup \{i \in \mathfrak{I} : \alpha^i \neq 1\}$ have identical CRRA utility and the aggregate endowment is not constant.[7]

Because the results we derive in both settings are identical, we adopt the abuse of notation $p^{RN} = p^{RN}_{\gamma}$ when the aggregate endowment is not constant.[8]

The equilibrium dynamics of an economy in which agents use p^{RN} for consensus differs from that of an economy in which the same agents use p^M for consensus. For example, it is possible that if agents use p^{RN} for consensus there is a dominating agent while, on the same path, long-run heterogeneity would appear if the same agents were to use p^M for consensus. Moreover, p^{RN} does not satisfy the properties of p^M discussed in Section 3: the belief of every surviving agent is typically not as accurate as p^{RN} (see Proposition 6, below), p^{RN}_t might not be a convex combination of agents dogmatic beliefs, and examples show that the weak inequalities $\bar{d}(P||\pi^{BCP}) \leq \bar{d}(P||p^{RN}) \leq \bar{d}(P||\pi^{BIP})$ might fail (see the example in section 4.3).

4.1 Accuracy of p^{RN}

In this section, we characterize the relative accuracy of p^{RN} and p^M, and discuss its dependence on agent risk attitudes and mixing coefficients.

First, we characterize the sign of $\bar{d}(P||p^{RN}) - \bar{d}(P||p^M)$ as a function of risk attitudes, independent of the α^is. Proposition 6 illustrates how the RRA parameters of the surviving agents affect the accuracy of p^{RN}. Ceteris paribus, economies with more risk-averse agents determine (weakly) more accurate risk-neutral probabilities.

Proposition 6. *Under **A1-A5**, let $\hat{\mathfrak{I}}$ be the set of surviving agents, then,*

(a) $\forall i \in \hat{\mathfrak{I}}, \gamma^i \in (0,1] \Rightarrow p^{RN}$ *is at most as accurate as* $p^M : \bar{d}(P||p^{RN}) \geq \bar{d}(P||p^M),\ P$*-a.s.*

(b) $\forall i \in \hat{\mathfrak{I}}, \gamma^i = 1 \Rightarrow p^{RN}$ *is as accurate as* $p^M : \bar{d}(P||p^{RN}) = \bar{d}(P||p^M),\ P$*-a.s.*

(c) $\forall i \in \hat{\mathfrak{I}}, \gamma^i \in [1,\infty) \Rightarrow p^{RN}$ *is at least as accurate as* $p^M : \bar{d}(P||p^{RN}) \leq \bar{d}(P||p^M),\ P$*-a.s.*

with strict inequality if and only if there is long-run heterogeneity in beliefs and at least one among the surviving agent has $\alpha \in (0,1)$.

[7] The reason why we need only to pose assumptions on agents in $\bar{\mathfrak{I}}$ is that Proposition 1 guarantees that the only agent with exogenous beliefs that might survive and have long run effect on the consensus is agent BIP.

[8] In the Appendix we present proofs for the two settings separately, when needed.

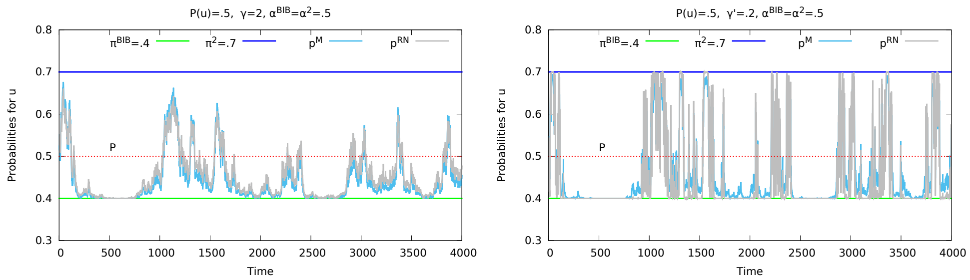

Figure 3: p^{RN} and p^M dynamics on the same path of in two economies in which agents mix using p^{RN}. The economies have two states $S = \{u, d\}$, two agents $\mathcal{I} = \{BIP, 2\}$ with $[\pi^{BIP}(u), \pi^2(u)] = [.4, .7]$, $[\alpha^{BIP}, \alpha^2] = [.5, .5]$ and common γ. **[left]** with $\gamma = 2 > 1$ p^{RN} is closer to the truth than p^M. **[right]** with $\gamma = .2 < 1$, p^{RN} is further from to the truth than p^M.

Proof. See Appendix A $\square$

Figure 3 illustrates Proposition 6. Everything else equal, when $\gamma > (<)1$ the p^{RN} process remains closer to (further away from) the truth than the p^M process calculated in the same economy.

Remark. The validity of the accuracy relations found in Propositions 2 and 6 is independent of the way agents obtain their p^i, but applies only if p^M and p^{RN} are calculated in the same economy. It would be a mistake to derive results related to p^{RN} approximating the results derived for p^M because the dynamic of an economy in which agents use p^M for consensus can differ from that of an economy in which the same agents use p^{RN} for consensus even if all agents have an arbitrarily small value of the α^is.

Second, we provide a bound for the difference between the accuracy of p^{RN} and p^M which depends on the mixing coefficients but is independent of risk attitudes. This difference decreases on the lowest mixing coefficient among the surviving agents. Furthermore, p^{RN} is as accurate as p^M when there is no long-run heterogeneity or all surviving agents have log utility.

Proposition 7. *Under **A1-A5**, let $\hat{\mathcal{I}}$ be the set of surviving agents that use p^{RN} for consensus and $\underline{\alpha} = \operatorname{argmin}_{i \in \hat{\mathcal{I}}} \alpha^i$,*

$$\bar{d}(P||p^{RN}) = \bar{d}(P||p^M) + O(\underline{\alpha}) - |O(\underline{\alpha}^2)| \ P\text{-}a.s.$$

with $\bar{d}(P||p^{RN}) = \bar{d}(P||p^M)$ if one agent dominates, or all agents in $\hat{\mathfrak{I}}$ have $\alpha = 1$, or all agents in $\hat{\mathfrak{I}}$ have $\gamma = 1$.

Proof. See Appendix A $\square$

A useful implication of Proposition 7 is that if one of the agents that survive gives full confidence to p^{RN} ($\alpha^i = 0$), then $p^{RN} = p^M$, irrespective of risk attitudes.

4.2 Discussion explaining the accuracy of p^{RN}

In this section, we propose two intuitions for the difference in the accuracy of p^M and p^{RN}. The first one is to give economic interpretation to the term determining the accuracy differential between p^M and p^{RN} in the proof of Proposition 6:

$$\bar{d}(P||p^{RN}) = \bar{d}(P||p^M) + \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} \ln \sum_{\tilde{\sigma}_\tau} \frac{q_\tau(\tilde{\sigma}_\tau|\sigma^{\tau-1})}{\beta} \quad P\text{-a.s..}$$

Let us start by noticing that, in every (t-1,σ), $\sum_{\tilde{\sigma}_t} q_t(\tilde{\sigma}_t|\sigma^{t-1})$ is the cost of moving a unit of consumption for sure a period ahead, i.e., the reciprocal of the risk-free rate. The effect of risk attitudes on the risk-free rate follows this intuition. In every period most agents subjectively believe that assets are mispriced and trade for speculative reasons because agents disagree. When agents have log utility ($\gamma = 1$), prices (and thus interest rates) do not affect optimal saving choices (the substitution effect equals the income effect) and the reciprocal of the risk free rate is given by the discount factor: for all $(t,\sigma), \beta = \sum_{\tilde{\sigma}_t} q_t(\tilde{\sigma}_t|\sigma^{t-1})$. However, if $\gamma < (>)1$, the substitution effect is stronger (weaker) than the income effect and each agent optimally chooses to save more (less) aggressively than if they had log utility and a lower (higher) risk-free rate arise: for all $(t,\sigma), \sum_{\tilde{\sigma}_\tau} q_\tau(\tilde{\sigma}_\tau|\sigma^{\tau-1}) > (<)\beta$. When there is heterogeneity a positive fraction of periods, this effect renders p^{RN} less (more) accurate than p^M. In the standard case of exogenous beliefs, this effect is present but either disappears in the short run because an agent dominates, or its magnitude is too small to be captured by an average measure of accuracy (Massari, 2017).[9]

The second interpretation is probabilistic and follows the intuition of Massari (2018). If all agents have identical CRRA utility with parameter γ, treating the p^i as given, by Lemma 8 (pg. 40) for all (t,σ),

$$p^{RN}(\sigma_t|\sigma^{t-1}) = \frac{\left(\sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} \phi^i_{\gamma,t-1}(\sigma)\right)^\gamma}{\sum_{\tilde{\sigma}_t}\left(\sum_{j\in\mathcal{I}} p^j(\tilde{\sigma}_t|\sigma^{t-1})^{\frac{1}{\gamma}} \phi^j_{\gamma,t-1}(\sigma)\right)^\gamma}, \text{ with } \phi^i_{\gamma,t-1}(\sigma) = \frac{p^i(\sigma^{t-1})^{\frac{1}{\gamma}}\phi^i_0}{\sum_{j\in\mathcal{I}} p^j(\sigma^{t-1})^{\frac{1}{\gamma}}\phi^j_0}.$$

Note that with $\gamma = 1$ the equation above coincides with p^M (albeit a change in the time zero consumption-shares), and also coincides with the predictive Bayesian measure from a prior $[c^1_{\gamma,t-1}(\sigma), ..., c^I_{\gamma,t-1}(\sigma)]$ on the models $[p^1, ..., p^I]$. Looking at the effect of γ on the prior weights $c^i_{\gamma,t-1}(\sigma)$, it is apparent that levels of $\gamma > (<)1$ can be thought of as modifying the standard Bayesian procedure in the direction of under (over)-reaction because $\gamma > (<)1$ makes less (more) extreme the differences between the likelihoods of the modes. Next, note that there is long-run heterogeneity only if no model is correct and the truth lies between at least two models. In these situations, slowing down (accelerating) the convergence rate delivers predictions that are more (less) accurate than that obtained via Bayes' rule because they remain closer to (further away from) the truth.

4.3 Necessary conditions for the WOC^{RN}

In this section, we identify necessary conditions for WOC^{RN} to occur which take into consideration the effect of risk attitudes on p^{RN} accuracy. Unlike for WOC^M, we find that if all agents have CRRA utility with $\gamma^i > 1$ diversity is not a necessary condition for WOC^{RN}. It is possible that beliefs are not diverse, $\pi^{BIP} = \pi^{BCP} \neq P$, and p^{RN} is more accurate than π^{BCP} because p^{RN} may not be in $Conv(\pi^1, ..., \pi^I)$, unlike p^M.

The market can make use of agent opinions in a way that is even more accurate than the most accurate convex combination of agents opinions.

For **example**, consider an economy with three states[10] $\mathcal{S} = \{u, m, d\}$ and three agents

[9] The same effect is present with exogenous beliefs when there is long-run heterogeneity, e.g. with recursive preferences see Borovička (2015) and Dindo (2015).

[10] Examples require at least three states to be constructed because in a two-state economy it can be shown that $\forall(t,\sigma), p^{RN} \in Conv(\pi^1, ..., \pi^I)$.

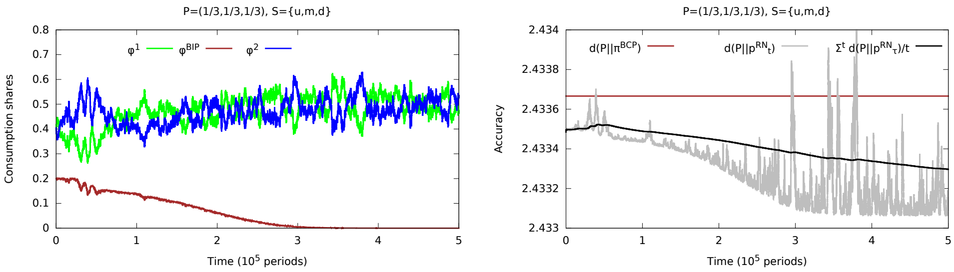

Figure 4: Equilibrium dynamics in a three-state $S = \{u, m, d\}$, three-agent $\mathcal{I} = \{1, BIP, 2\}$ economy with $[\alpha^1, \alpha^{BIP}, \alpha^2] = [.1, 1, .1]$, common $\gamma = 10$, uniform iid P, and in which agent beliefs are as in Equation 6 with $\epsilon = .0001, \pi^1(m) = .6, \pi^2(m) = .4$. Beliefs are not diverse ($\pi^{BIP} = \pi^{BCP}$). However, **[left]** consumption-share dynamics: agent BIP vanishes even if his beliefs is the most accurate combination of other agents beliefs; **[right]** K-L and average K-L dynamics: p^{RN} is more accurate than π^{BCP}.

$\{1, BIP, 2\}$ with common $\gamma > 1$ and $[\alpha^1, \alpha^{BIP}, \alpha^2] = [\alpha, 1, \alpha]$.

$$\text{Let } P = [.\bar{3}, .\bar{3}, .\bar{3}], \text{ and } [\pi^1, \pi^{BIP}, \pi^2]' = \begin{bmatrix} \epsilon & \pi^1(m) - .5\epsilon & \pi^1(d) - .5\epsilon \\ \epsilon & .5 - .5\epsilon & .5 - .5\epsilon \\ \epsilon & \pi^2(m) - .5\epsilon & \pi^2(d) - .5\epsilon \end{bmatrix}. \tag{6}$$

Note that, for $\pi^1(m) > .5 > \pi^2(m)$ beliefs are not diverse: $\pi^{BIP} = \pi^{BCP}$.

$$\text{By Lemma 8 (pg. 40), } \forall(t, \sigma),\ p^{RN}(\sigma_t|\sigma^{t-1}) = \frac{\left(\sum_{i \in \mathcal{I}} p^i(\sigma_t|\sigma^{t-1})^{\frac{1}{\gamma}} \phi^i_{\gamma,t-1}(\sigma)\right)^\gamma}{\sum_{\tilde{\sigma}_t} \left(\sum_{j \in \mathcal{I}} p^j(\tilde{\sigma}_t|\sigma^{t-1})^{\frac{1}{\gamma}} \phi^j_{\gamma,t-1}(\sigma)\right)^\gamma}.$$

Because $\gamma > 1 \Rightarrow$ the denominator is < 1 and $\mathrm{p}^{RN}(u) > \epsilon$ whenever there is long run heterogeneity. Because the relative entropy diverges close to the boundary of the simplex, for ϵ small, a small difference in the direction of $.\bar{3}$ determines a large improvement in $d(P||p_t^{RN})$. Thus, if α is such that there is long run heterogeneity, p^{RN} is more accurate than π^{BIP}, agent BIP vanishes and WOC^{RN} occurs.

Figure 4 illustrates this scenario with $\gamma = 10, \alpha = .1$ and $\epsilon = .0001$ and initial risk adjusted consumption shares [.4,.2,.4].

When $P \in Conv(\pi^1, ..., \pi^I)$, p^{RN} cannot be more accurate than $\pi^{BCP} = P$ and the necessary conditions for WOC^M and WOC^{RN} coincide.

Corollary 2. *Under **A1-A5**, if $P \in Conv(\pi^1, ..., \pi^I)$, WOC^{RN} can occur only if beliefs are diverse and the beliefs of at least one agent depend on p^{RN}.*

Proof. By contradiction, $\pi^{BIP} = \pi^{BCP} =^{\text{by} H_0} P \Rightarrow$ no WOC. The proof of the second mimics Prop.4. $\square$

4.4 Sufficient conditions for the WOC^{RN}

The sufficient conditions for the WOC^{RN} to occur need to take into account how the risk attitudes of the surviving agents affects p^{RN} accuracy.

We start by deriving a sufficient condition for the WOC^{RN} to occur when all agents in $\bar{\mathfrak{I}}$ have CRRA utility with $\gamma^i > 1$. Under this assumption, Proposition 6 guarantees that p^{RN} is at least as accurate as p^M and the sufficient condition we find is weaker than that of Proposition 5. Specifically, Proposition 8 does not require agent BIP beliefs to depend on the consensus.

Proposition 8. *Under **A1-A5**, the WOC^{RN} occurs and at least two agents survive, if all agents $j \in \bar{\mathfrak{I}}$ have CRRA utility with $\gamma^j > 1$ and there is an agent $i \in \bar{\mathfrak{I}}$ such that*

$$\bar{d}(P||(1-\alpha^i)\pi^{BIP} + \alpha^i \pi^i) < \bar{d}(P||\pi^{BIP}). \tag{7}$$

Proof. See Appendix A $\square$

Figure 5 [left] illustrates Proposition 8. For $\gamma = 2 > 1$ and $[\alpha^{BIP}, \alpha^2] = [1, .2]$ condition (7) is satisfied, agent BIP cannot dominate and WOC^{RN} occurs. [right] shows the p^{RN} dynamics on the same path for an economy with the same parameters but in which agent 2 mixes using p^M, rather then p^{RN}. As discussed following Proposition 5, this economy does not generate WOC^M because BIP survives but does not mix. Nevertheless, it does generate WOC^{RN} because there is long run heterogeneity so that p^{RN} is more accurate than p^M (Proposition 6) which is at least as accurate as p^{BIP}(Proposition 3).

More generally, if we do not make assumptions about the preferences of agents in $\bar{\mathfrak{I}}$, we cannot rule out the possibility that the resulting p^{RN} is less accurate than p^M and π^{BIP}. This eventuality makes it harder for the WOC^{RN} to occur when agents rely on

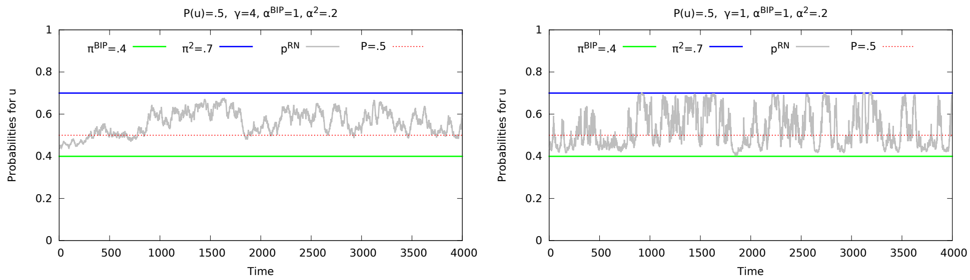

Figure 5: **[left]** p_t^{RN} dynamics in a two-state economy in which agents mix using p^{RN} with parameters $[\pi^{BIP}(u), \pi^2(u)] = [.4, .7]$, $[\alpha^{BIP}, \alpha^2] = [1, .2]$,$\gamma^{BIP} = 2 = \gamma^2$. **[right]** p_t^{RN} dynamics in an economy with the same parameters in which agent 2 mix using p^M, rather than p^{RN}.

p^{RN} rather than p^M. Stronger conditions are needed to prevent the system from entering a dynamic that has long-run heterogeneity but does not deliver an accurate consensus. Furthermore, we are forced to change our proof technique because the equilibrium relation of Proposition 2 and 6 are not accurate enough to guarantee that p^{RN} concentrates around a unique value when some γ^is are smaller than one. Rather than relying on long run properties of the equilibrium, we must now characterize the equilibrium dynamics of the economy. For tractability reasons, we restrict our analysis to economies with two states and common gamma.

Proposition 9. *Consider an economy with two states that satisfies **A1-A5**. If beliefs of agents in $\bar{\mathfrak{I}}$ are diverse, all agents in $\bar{\mathfrak{I}}$ have common CRRA utility with parameter γ, mix using p^{RN} with common α and α is small enough, then the WOC^{RN} occurs.*

Proof. Application of Theorem 3. □

Figure 6 [left] illustrates Proposition 9. With $\gamma = .5 < 1$ and $[\alpha^{BIP}, \alpha^2] = [.2, .2]$ it shows that when all agents have identical α and $P \in Conv(\mathfrak{I})$, α can be chosen small enough for the WOC^{RN} to occur P-a.s. irrespective of risk attitudes. Intuitively, when α is "small enough", the drift and variance conditions on the consumption-shares of an economy that uses p^{RN} for consensus are similar to those of an economy in which the same agents use p^M for consensus. Accordingly, agent 2 survives and p^{RN} is more accurate than π^{BIP} because it belongs to the interior of $Conv(\pi^1, \pi^{BIP})$.

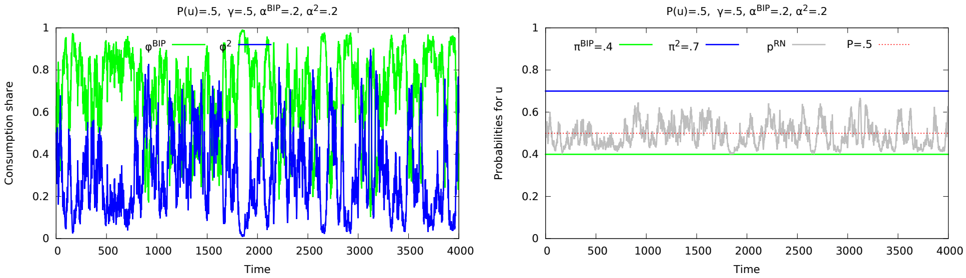

Figure 6: Consumption-shares **[left]** and market probability **[right]** dynamics in a two-state economy with parameters $[\pi^{BIP}(u), \pi^2(u)] = [.4, .7]$, and $\gamma^{BIP} = \gamma^2 = .5$. For $[\alpha^{BIP}, \alpha^2] = [.2, .2]$ the sufficient condition of Proposition 9 is satisfied and the WOC^{RN} occurs.

Discussion

The homogeneity requirement for the values of α in Proposition 9 can be relaxed, but not abandoned. The potential problem is that with $\gamma < 1$ p^{RN} might be less accurate than π^{BIP}, so that it is not guaranteed that believing in market accuracy (weakly) increases agent accuracy (Proposition 3 does not hold). The above observation suggests that without a homogeneity requirement for the value of the α^is, the long-run dynamics of the economy might become path dependent. Figure 7 illustrates the equilibrium consumption-shares and p^{RN} dynamics on two typical paths of the same economy with $\gamma = .5 < 1$ and heterogenous mixing coefficients. It shows that p^{RN} can enter two distinct dynamics. Either [top] the WOC^{RN} occurs because in a finite sample agents BIP and 2 reach a high enough consumption-share to make the dynamics of the system locally independent of the other agents; or [bottom], the WOC^{RN} fails. At the beginning of this path, agents 2 and BIP lose consumption-shares to agents 3 and 4, so that early on $d(P||p^{RN}_t) > d(P||p^M_t)$ and, by giving a lot of weight to p^{RN}, agents BIP and 2 make their beliefs less accurate than those of the other agents and eventually vanish.

To summarise, there is a positive probability for the WOC^{RN} to occur because there is a positive probability that agents BIP and 2 reach a high enough consumption-share to make the dynamics of the system locally independent of agents 3 and 4; however there is also a positive probability for the WOC^{RN} to fail because there is a positive probability that agents 3 and 4 reach a high enough consumption-share to make p^{RN} less accurate than p^M. When this happens, agents BIP and 2 vanish because their beliefs

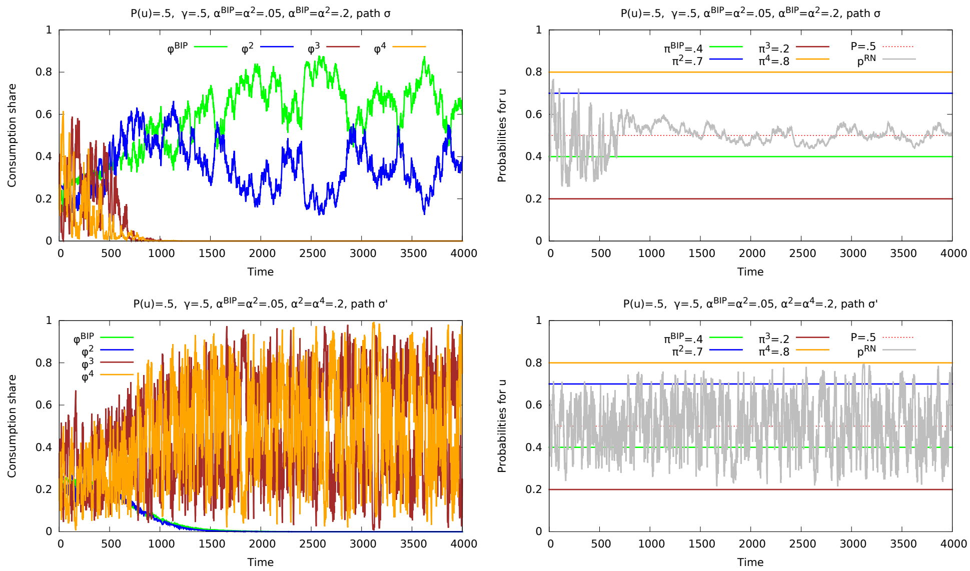

Figure 7: Consumption-shares **[left]** and market probability **[right]** dynamics in two paths generated from the same probability for a two-state, four-agent economy with parameters $[\pi^{BIP}(u), \pi^2(u), \pi^3(u), \pi^4(u)] = [.4, .7, .2, .8]$, $[\alpha^{BIP}, \alpha^2, \alpha^3, \alpha^4] = [.05, .05, .2, .2]$, homogeneous RRA $\gamma = .5$. [Top]: on this path agent BIP and agent 2 dominate and p^{RN} is more accurate than π^{BIP}. [Bottom]: on this other path agent BIB and agent 2 vanish and p^{RN} is less accurate than π^{BIP}.

are less accurate than p^M (by the contrapositive of Proposition 2, b)) since they give a lot of weight to an inaccurate consensus.

4.5 Accurate markets: A self-fulfilling prophecy (RN)

Here we give conditions under which the self-fulfilling prophecy discussed in Section 3.4 holds when agents use p^{RN} for market consensus. As for our sufficient conditions, risk attitudes have an effect on the occurrence of the WOC^{RN}, so that we either need assumptions about agents utilities, or to impose the same restrictions of Proposition 9 to conduct our analysis. If all agents in $\bar{\mathcal{I}}$ have CRRA utility with $\gamma^i > 1$, the self-fulfilling prophecy condition using p^{RN} coincides with that of Theorem 1.

Theorem 2. *Let $(\mathcal{E}_\alpha)$ be a family of economies that satisfies **A1-A5** with a subset of agents $\hat{\mathcal{I}}$ that relies on p^{RN} with $\alpha^i \in (0, \bar{\alpha}]$ and such that $P \in Conv(\hat{\mathcal{I}})$. Name each economy risk-neutral probabilities process $(p^{RN}_{t,\bar{\alpha}})^\infty_{t=0}$; then, if all agents in $\bar{\mathcal{I}}$ have CRRA* *utilities,*

$$\forall i \in \bar{\mathfrak{I}}, \gamma^i \geq 1 \Rightarrow \lim_{\bar{\alpha}\to 0} \bar{d}(P||p^{RN}_{\bar{\alpha}}) = 0, \quad P\text{-}a.s..$$

Proof. See Appendix B. □

As argued before Proposition 9, the self-fulfilling prophecy property of p^{RN} when $\gamma < 1$ requires tighter conditions to prevent those dynamics in which agents in $\bar{\mathfrak{I}}$ vanish.

Theorem 3. *Let $(\mathcal{E}_\alpha)$ be a family of two-state economies that satisfies **A1-A5** such that $P \in Conv(\bar{\mathfrak{I}})$, all agents in $\bar{\mathfrak{I}}$ mix using p^{RN} with common α and γ, and name each economy risk-neutral probabilities process $(p^{RN}_{t,\alpha})_{t=0}^{\infty}$; then,*

$$\lim_{\alpha\to 0} \quad \bar{d}(P||p^{RN}_{\alpha}) = 0 \quad P\text{-}a.s..$$

Proof. See Appendix B. □

5 Conclusion

We provide conditions under which the MSH and the WOC can be reconciled in a dynamic economy where agents naively learn from an endogenous measure of consensus. Moreover, we show that if a group of agents strongly believe in market accuracy and their beliefs can be combined to obtain the truth a virtuous self-fulling prophecy occurs. Although no agent knows the truth, and the initial consumption-share/beliefs distribution might be severely skewed away from the truth, market selection forces endogenously generate consumption-share/beliefs dynamics which determine a consensus that is almost as accurate as the truth.

A Appendix

We make use of the symbols $\asymp$ and $O(\cdot)$ with the following meanings:

$$f(x) = O(g(x)) \ if \ \limsup_x \left|\frac{f(x)}{g(x)}\right| < \infty.$$

$$f(x) \asymp g(x) \ if \ \forall x, f(x) > 0, g(x) > 0 \ and \ \begin{cases} \limsup_x \frac{f(x)}{g(x)} < \infty \\ \liminf_x \frac{f(x)}{g(x)} > 0 \end{cases}.$$

Proof of Lemma 1

Proof.

$$
\begin{aligned}
\forall(t,\sigma),\ p^M(\sigma^t) &= \prod_{\tau=1}^{t} p^M(\sigma_\tau|\sigma^{\tau-1}) \\
&= \left(\sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})\frac{\bar{c}^i_{t-1}(\sigma)}{\sum_{j\in\mathcal{I}}\bar{c}^j_{t-1}(\sigma)}\right)\prod_{\tau=1}^{t-1} p^M(\sigma_\tau|\sigma^{\tau-1}) \\
&=^{(a)} \left(\sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})p^i(\sigma_{t-1}|\sigma^{t-2})\frac{\bar{c}^i_{t-2}(\sigma)}{\sum_{j\in\mathcal{I}}\bar{c}^j_{t-2}(\sigma)}\right)\frac{1}{p^M(\sigma_{t-1}|\sigma^{t-1})}\prod_{\tau=1}^{t-1} p^M(\sigma_\tau|\sigma^{\tau-1}) \\
&= \sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})p^i(\sigma_{t-1}|\sigma^{t-2})\frac{\bar{c}^i_{t-2}(\sigma)}{\sum_{j\in\mathcal{I}}\bar{c}^j_{t-2}(\sigma)}\prod_{\tau=1}^{t-2} p^M(\sigma_\tau|\sigma^{\tau-1}) \\
&\quad\vdots \\
&= \sum_{i\in\mathcal{I}}\prod_{\tau=1}^{t} p^i(\sigma_\tau|\sigma^{\tau-1})\frac{\bar{c}^i_0}{\sum_{j\in\mathcal{I}}\bar{c}^j_0} \\
&= \sum_{i\in\mathcal{I}} p^i(\sigma^t)\frac{\bar{c}^i_0}{\sum_{j\in\mathcal{I}}\bar{c}^j_0}
\end{aligned}
$$

(a) : by the FOC, for all $(t,\sigma), \forall i\in\mathcal{I}, \bar{c}^i_{t-1}(\sigma) = \frac{\beta p^i(\sigma_{t-1}|\sigma^{t-2})\bar{c}^i_{t-2}(\sigma)}{q(\sigma_{t-1}|\sigma^{t-2})}$
$\Rightarrow \frac{\bar{c}^i_{t-1}(\sigma)}{\sum_{j\in\mathcal{I}}\bar{c}^j_{t-1}(\sigma)} = \frac{p^i(\sigma_{t-1}|\sigma^{t-2})\bar{c}^i_{t-2}(\sigma)}{\sum_{j\in\mathcal{I}} p^j(\sigma_{t-1}|\sigma^{t-2})\bar{c}^j_{t-2}(\sigma)} = \frac{p^i(\sigma_{t-1}|\sigma^{t-2})\bar{c}^i_{t-2}(\sigma)}{p^M(\sigma_{t-1}|\sigma^{t-1})}\frac{1}{\sum_{j\in\mathcal{I}}\bar{c}^j_{t-2}(\sigma)}$.

□

Proof of Lemma 2

Proof. From the FOC, for all (t,σ),

$$\forall i\in\mathcal{I},\quad \bar{c}^i_t(\sigma)q(\sigma_t|\sigma^{t-1}) = \beta p^i(\sigma_t|\sigma^{t-1})\bar{c}^i_{t-1}(\sigma),$$

summing over i and rearranging,

$$
\begin{aligned}
q(\sigma_t|\sigma^{t-1}) &= \sum_{i\in\mathcal{I}}\beta p^i(\sigma_t|\sigma^{t-1})\frac{\bar{c}^i_{t-1}(\sigma)}{\sum_{j\in\mathcal{I}}\bar{c}^i_t(\sigma)} \\
\Rightarrow p^{RN}(\sigma_t|\sigma^{t-1}) &:= \frac{q(\sigma_t|\sigma^{t-1})}{\sum_{\tilde{\sigma}_t} q(\tilde{\sigma}_t|\sigma^{t-1})} \propto \sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1})\frac{\bar{c}^i_{t-1}(\sigma)}{\sum_{j\in\mathcal{I}}\bar{c}^j_t(\sigma)}.
\end{aligned}
$$

□

Proof of Lemma 3

Proof. In every equilibrium, $\forall(t,\sigma)$,

$$
\begin{aligned}
p^{RN}_{\gamma}(\sigma_t|\sigma^{t-1}) &:= \frac{q(\sigma_t|\sigma^{t-1})\boldsymbol{e}_t(\sigma)^{\gamma}}{\sum_{\tilde{\sigma}_t} q(\tilde{\sigma}_t|\sigma^{t-1})\boldsymbol{e}_t(\tilde{\sigma})^{\gamma}} \\
&\propto \sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1}) \frac{\bar{c}^i_{t-1}(\sigma)}{\sum_{j\in\mathcal{I}} \bar{c}^j_t(\sigma)} \frac{\boldsymbol{e}_t(\sigma)^{\gamma}}{\boldsymbol{e}_{t-1}(\sigma)^{\gamma}} \\
&= \sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1}) \frac{c^i_{t-1}(\sigma)^{\gamma}}{\sum_{j\in\mathcal{I}} c^j_t(\sigma)^{\gamma}} \frac{\left(\sum_{j\in\mathcal{I}} c^j_t\right)^{\gamma}}{\left(\sum_{j\in\mathcal{I}} c^j_{t-1}\right)^{\gamma}} \\
&= \sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1}) \frac{c^i_{t-1}(\sigma)^{\gamma}}{\left(\sum_{j\in\mathcal{I}} c^j_{t-1}\right)^{\gamma}} \frac{1}{\sum_{j\in\mathcal{I}} \frac{c^j_t(\sigma)^{\gamma}}{\left(\sum_{k\in\mathcal{I}} c^k_t\right)^{\gamma}}} \\
&= \sum_{i\in\mathcal{I}} p^i(\sigma_t|\sigma^{t-1}) \frac{\phi^i_{t-1}(\sigma)^{\gamma}}{\sum_{j\in\mathcal{I}} \phi^j_t(\sigma)^{\gamma}}
\end{aligned}
$$

□

Lemma 4. *Under **A1, A2** and **A4-(A5)**, if agent beliefs are as in Definition 4 with* $p^C = p^M(p^{RN})$ *and* $\forall i \in \mathcal{I}, \alpha^i \in (0,1]$ *then **A3** is satisfied.*

Proof. By Definition 4, $p^i(\sigma_t|\sigma^{t-1}) = (1-\alpha^i)p^C(\sigma_t|\sigma^{t-1}) + \alpha^i \pi^i(\sigma_t)$ with π^i is strictly positive $\forall\, i \in \mathcal{I}$. Therefore, for all $(t,\sigma), p^i(\sigma_t|\sigma^{t-1}) > 0$. □

Lemma 5. *Under **A1-A4**, if agent beliefs are as in Definition 4 with* $p^C = p^M$*, then* $\forall(t,\sigma), \forall j \in \mathcal{I} \cup M, p^j(\sigma_t|\sigma^{t-1}) \in Conv(\pi^1, ..., \pi^I)$.

Proof. Substituting $p^i(\sigma_t|\sigma^{t-1})$ (Definition 4) in Definition 7,

$$\forall(t,\sigma),\ p^M(\sigma_t|\sigma^{t-1}) = \sum_{i\in\mathcal{I}} \left[(1-\alpha^i)p^M(\sigma_t|\sigma^{t-1}) + \alpha^i\pi^i(\sigma_t)\right] \frac{\bar{c}^i_{t-1}(\sigma)}{\sum_{j\in\mathcal{I}} \bar{c}^j_{t-1}(\sigma)}.$$

$$\text{Rearranging, } \forall(t,\sigma),\ p^M(\sigma_t|\sigma^{t-1}) = \sum_{i\in\mathcal{I}} \pi^i(\sigma_t) \frac{\alpha^i \bar{c}^i_{t-1}(\sigma)}{\sum_{j\in\mathcal{I}} \alpha^j \bar{c}^j_{t-1}(\sigma)} \in Conv(\pi^1, ..., \pi^I).$$

$\forall i \in \mathcal{I} : \alpha^i \in (0,1), p^i(\sigma_t|\sigma^{t-1}) \in Conv(\pi^1, ..., \pi^I)$ because is the convex combination of two points in $Conv(\pi^1, ..., \pi^I)$. □

Proof of Proposition 2

Proof. ***(a)*** Let $\bar{\phi}_0^i := \frac{\bar{c}_0^i}{\sum_{j\in\mathcal{I}} \bar{c}_0^j}$

$$
\begin{aligned}
p^M(\sigma^t) &=^{Lem.1} \sum_{i\in\mathcal{I}} p^i(\sigma^t)\bar{\phi}_0^i \\
\Rightarrow\quad & \forall i \in \mathcal{I}, \quad \ln p^M(\sigma^t) \geq \ln p^i(\sigma^t) + \ln \bar{\phi}_0^i, \\
\Rightarrow\quad & \frac{1}{t}\ln\frac{P(\sigma^t)}{p^M(\sigma^t)} \leq \frac{1}{t}\ln\frac{P(\sigma^t)}{p^i(\sigma^t)} - \frac{1}{t}\ln\bar{\phi}_0^i \\
\Rightarrow\quad & \lim_{t\to\infty}\left[\frac{1}{t}\left[\sum_{\tau=1}^{t}\ln\frac{P(\sigma_\tau)}{p^M(\sigma_\tau|\sigma^{\tau-1})} - \sum_{\tau=1}^{t} d(P||p_\tau^M)\right] + \frac{1}{t}\sum_{\tau=1}^{t} d(P||p_\tau^M)\right] \\
& \leq \lim_{t\to\infty}\left[\frac{1}{t}\left[\sum_{\tau=1}^{t}\ln\frac{P(\sigma_\tau)}{p^i(\sigma_\tau|\sigma^{\tau-1})} - \sum_{\tau=1}^{t} d(P||p_\tau^i)\right] + \frac{1}{t}\sum_{\tau=1}^{t} d(P||p_\tau^i) - \frac{1}{t}\ln\bar{\phi}_0^i\right] \\
\Rightarrow\quad & \bar{d}(P||p^M) \leq \bar{d}(P||p^i) \quad P\text{-a.s., by the SLLNMD.}
\end{aligned}
$$

The last implication follows from the Strong Law of Large Number for Martingale Differences (SLLNMD)(see also Sandroni, 2000) that guarantees that for $j = i, M$,

$$
\lim_{t\to\infty}\frac{1}{t}\left[\sum_{\tau=1}^{t}\ln\frac{P(\sigma_\tau)}{p^j(\sigma_\tau|\sigma^{\tau-1})} - \sum_{\tau=1}^{t} d(P||p_\tau^j)\right] = 0, P\text{-a.s.}
$$

(b): We proceed by proving the contrapositive statement: $\bar{d}(P||p^M) < \bar{d}(P||p^i)$ P-a.s. $\Rightarrow$ agent i vanishes — the opposite inequality is ruled out by (a).

$$
\bar{c}_t^i(\sigma) = \frac{\beta^t p^i(\sigma^t)}{q(\sigma^t)}\bar{c}_0^i \asymp^{\text{by Massari (2017), Th.1}} \frac{p^i(\sigma^t)}{\sum_{i\in\mathcal{I}} p^i(\sigma^t)}\bar{c}_0^i \asymp^{\text{by Lem.1}} \frac{p^i(\sigma^t)}{p^M(\sigma^t)}\bar{c}_0^i
$$

$$
\begin{aligned}
\Rightarrow \lim_{t\to\infty}\frac{1}{t}\ln\bar{c}_t^i(\sigma) &= \lim_{t\to\infty}\frac{1}{t}\ln\frac{p^i(\sigma^t)}{p^M(\sigma^t)} + \frac{1}{t}\ln\bar{c}_0^i \\
&= \lim_{t\to\infty}\frac{1}{t}\left[\ln\frac{P(\sigma^t)}{p^M(\sigma^t)} - \ln\frac{P(\sigma^t)}{p^i(\sigma^t)}\right] \\
&= \bar{d}(P||p^M) - \bar{d}(P||p^i) \qquad P\text{-a.s., by the SLLNMD}
\end{aligned}
$$

$$
\begin{aligned}
\text{Therefore, } \bar{d}(P||p^M) < \bar{d}(P||p^i)\ P\text{-a.s.} &\Rightarrow \lim_{t\to\infty}\frac{1}{t}\ln\bar{c}_t^i(\sigma) < 0, P\text{-a.s.} \\
&\Rightarrow \ln\bar{c}_t^i(\sigma) \to -\infty, P\text{-a.s.} \\
&\Rightarrow \frac{1}{u(c_t^i)'} \to 0\ P\text{-a.s.} \\
&\Rightarrow c_t^i \to 0\ P\text{-a.s.} \quad \text{by } \mathbf{A1} \\
&\Rightarrow \text{agent } i \text{ vanishes.}
\end{aligned}
$$

□

Proof of Proposition 3

Proof. $\forall(t,\sigma)$,

$$\begin{aligned} d(P||p_t^i) &= d(P||(1-\alpha^i)p_t^M + \alpha^i\pi^i)) \\ &\leq^{(a)} (1-\alpha^i)d(P||p_t^M) + \alpha^i d(P||\pi^i) && \text{; by strict convexity of } d(P||\cdot) \\ \Rightarrow \quad & \bar{d}(P||p^i) \leq (1-\alpha^i)\bar{d}(P||p^M) + \alpha^i\bar{d}(P||\pi^i) && \text{; summing and averaging over } t \\ \Rightarrow \quad & \bar{d}(P||p^i) \leq \bar{d}(P||\pi^i) \ P\text{-a.s.} && \text{; because } \bar{d}(P||p^M) \leq^{by\ Prop.2} \bar{d}(P||p^i) \end{aligned}$$

Moreover, $p_t^M \neq \pi^i$ a positive fraction of periods $\Rightarrow \bar{d}(P||p^i) < \bar{d}(P||\pi^i)$ because inequality (a) is strict unless $p_t^M(\sigma) = \pi^i$. □

Proof of Proposition 4

Proof. $WOC^M \Rightarrow$ beliefs must be diverse. We prove the contrapositive statement:

$$\pi^{BCP} = \pi^{BIP} \Rightarrow \bar{d}(P||p^M) \geq \bar{d}(P||\pi^{BIP}) \ P\text{-a.s.} \Rightarrow \text{no WOC}^M.$$

$\forall(t,\sigma), p_t^M \in^{By\ Lem.5} Conv(\pi^1,...,\pi^I)$ and $\pi^{BCP} := \underset{p\in Conv(\pi^1,...,\pi^I)}{\text{argmin}} d(P||p)$.
Thus, for every choice of $\alpha^i \in (0,1]$, $\forall\sigma, \bar{d}(P||p^M) \geq \bar{d}(P||\pi^{BCP}) =^{\text{By } H_0} \bar{d}(P||\pi^{BIP})$.

$WOC^M \Rightarrow \exists i : \alpha^i \in (0,1)$. We prove the contrapositive statement:

$$\forall i \in \mathcal{I}, \alpha^i = 1 \Rightarrow \bar{d}(P||p^M) = \bar{d}(P||\pi^{BIP}) \ P\text{-a.s.} \Rightarrow \text{no WOC}^M.$$

$\forall i \in \mathcal{I}, \alpha^i = 1 \Rightarrow$ agent beliefs are independent of each other.
Therefore, agent BIP survives (by Prop.1) and p^M is as accurate as π^{BIP} (by Prop.2).

□

Proof of Proposition 5

Proof. The condition on p^i is sufficient to guarantee that agent BIP does not have unitary consumption shares a positive fraction of periods — otherwise, agent i would be more accurate than agent BIP, violating Proposition 1.
Therefore, $p_t^M \neq \pi^{BIP}$ a positive fraction of periods and the result follows because

$$\bar{d}(P||p^M) \leq^{\text{Prop.2}} \bar{d}(P||p^{BIP}) <^{\text{by Prop.3}} \bar{d}(P||\pi^{BIP}).$$

□

The following two Lemmas are needed for the proof of Proposition 6. In these proofs we omit the conditioning notation for prices and probabilities and adopt the more compact notation: for $j \in \mathcal{I} \cup RN, p^i(\sigma_t|\sigma^{t-1}) := p^i(\sigma_t|)$ and $q(\sigma_t|\sigma^{t-1}) := q(\sigma_t|)$.

Lemma 6. *Under **A1-A5**, if agents' utilities are CRRA and the aggregate endowment is constant, for all* (t,σ)*,*

$$\begin{array}{l} \forall i, \gamma^i \geq 1 \Rightarrow \frac{1}{\beta}\sum_{\sigma_t} q(\sigma_t|\sigma^{t-1}) \leq 1 \\ \forall i, \gamma^i \leq 1 \Rightarrow \frac{1}{\beta}\sum_{\sigma_t} q(\sigma_t|\sigma^{t-1}) \geq 1 \end{array},$$

with equality if and only if ether $\gamma^i = 1$ *for all agents or all agents have identical beliefs.*

Proof. On every equilibrium path $\forall(t,\sigma)$ and for all i,

$$c_t^i(\sigma) = \left(\frac{\beta p^i(\sigma_t|)}{q(\sigma_t|)}\right)^{\frac{1}{\gamma^i}} c_{t-1}^i(\sigma).$$

Multiplying left and right by $\frac{q(\sigma_t|)}{\beta}$,

$$\frac{q(\sigma_t|)}{\beta} c_t^i(\sigma) = p^i(\sigma_t|)^{\frac{1}{\gamma^i}} \left(\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^i}} c_{t-1}^i(\sigma).$$

Summing left and right over all the agents,

$$\frac{q(\sigma_t|)}{\beta} \sum_{i\in\mathcal{I}} c_t^i(\sigma) = \sum_{i\in\mathcal{I}} p^i(\sigma_t|)^{\frac{1}{\gamma^i}} \left(\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^i}} c_{t-1}^i(\sigma).$$

Dividing left and right by the aggregate endowment (which is constant over t)

$$\frac{q(\sigma_t|)}{\beta} = \sum_{i\in\mathcal{I}} p^i(\sigma_t|)^{\frac{1}{\gamma^i}} \left(\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^i}} \phi_{t-1}^i,$$

where $[\phi_{t-1}^1, ..., \phi_{t-1}^I]$ is the consumption shares distribution in $(t-1, \sigma^{t-1})$.
Summing left and right over the states:

$$\sum_{\sigma_t} \frac{q(\sigma_t|)}{\beta} = \sum_{i\in\mathcal{I}} \sum_{\sigma_t} p^i(\sigma_t|)^{\frac{1}{\gamma^i}} \left(\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^i}} \phi_{t-1}^i.$$

Multiplying the right-hand side by $\frac{\prod_{k\in\mathcal{I}}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^k}}}{\prod_{j\in\mathcal{I}}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^j}}} = 1$ we can express the left-hand side as a function of the risk-neutral probabilities.

$$\sum_{\sigma_t} \frac{q(\sigma_t|)}{\beta} = \sum_{i\in\mathcal{I}} \sum_{\sigma_t} p^i(\sigma_t|)^{\frac{1}{\gamma^i}} p^{RN}(\sigma_t|)^{1-\frac{1}{\gamma^i}} \phi_{t-1}^i \frac{\prod_{k\in\mathcal{I}}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^k}}}{\prod_{j\neq i}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^j}}}. \tag{8}$$

- Let us focus on the case in which $\forall i, \gamma^i \geq 1$.

  Let $i^* := \mathrm{argmax}_{i\in\mathcal{I}}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^i}}$, so that $\forall k, i \in \mathcal{I}$, $\frac{\prod_{k\neq i^*}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^k}}}{\prod_{j\neq i}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^j}}} \leq 1$.

  It follows that

$$\begin{aligned}
\sum_{\sigma_t} \frac{q(\sigma_t|)}{\beta} &= \sum_{i\in\mathcal{I}} \sum_{\sigma_t} p^i(\sigma_t|)^{\frac{1}{\gamma^i}} p^{RN}(\sigma_t|)^{1-\frac{1}{\gamma^i}} \phi_{t-1}^i \left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^{i^*}}} \frac{\prod_{k\neq i^*}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^i}}}{\prod_{j\neq i}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^j}}} \\
&\leq \sum_{i\in\mathcal{I}} \sum_{\sigma_t} p^i(\sigma_t|)^{\frac{1}{\gamma^i}} p^{RN}(\sigma_t|)^{1-\frac{1}{\gamma^i}} \phi_{t-1}^i \left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^{i^*}}}.
\end{aligned}$$

Rearranging,

$$\begin{aligned}
\left(\sum_{\sigma_t} \frac{q(\sigma_t|)}{\beta}\right)^{\frac{1}{\gamma^{i^*}}} &\leq \sum_{i\in\mathcal{I}}\sum_{\sigma_t} p^i(\sigma_t|)^{\frac{1}{\gamma^i}} p^{RN}(\sigma_t|)^{1-\frac{1}{\gamma^i}} \phi^i_{t-1} \qquad (9)\\
&\leq^{(a)} \sum_{i\in\mathcal{I}}\sum_{\sigma_t} \left(\frac{1}{\gamma^i}p^i(\sigma_t|) + \left(1-\frac{1}{\gamma^i}\right)p^{RN}(\sigma_t|)\right)\phi^i_{t-1} = 1\\
\Rightarrow \sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta} \leq 1.
\end{aligned}$$

$(a) : \forall i \in \mathcal{I}, \gamma^i \geq 1 \Rightarrow \forall \sigma_t,\ p^i(\sigma_t|)^{\frac{1}{\gamma^i}} p^{RN}(\sigma_t|)^{1-\frac{1}{\gamma^i}} \leq \frac{1}{\gamma^i}p^i(\sigma_t|) + \left(1-\frac{1}{\gamma^i}\right)p^{RN}(\sigma_t|)$, because strict concavity of log ensures that

$$\begin{aligned}
\ln\left(p^i(\sigma_t|)^{\frac{1}{\gamma^i}} p^{RN}(\sigma_t|)^{1-\frac{1}{\gamma^i}}\right) &= \frac{1}{\gamma^i}\ln p^i(\sigma_t|) + \left(1-\frac{1}{\gamma^i}\right)\ln p^{RN}(\sigma_t|)\\
&\leq \ln\left(\frac{1}{\gamma^i}p^i(\sigma_t|) + \left(1-\frac{1}{\gamma^i}\right)p^{RN}(\sigma_t|)\right).
\end{aligned}$$

- Let's focus on the case in which $\forall i, \gamma^i \leq 1$.

  Let $i^{**} := \operatorname{argmin}_{i\in\mathcal{I}}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^i}}$; thus $\forall k, i \in \mathcal{I}$, $\dfrac{\prod_{k\neq i^{**}}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^k}}}{\prod_{j\neq i}\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^j}}} \geq 1$.

  Proceeding as above, we obtain the opposite inequality:

$$\left(\sum_{\sigma_t}\frac{q(\sigma_t|)}{\beta}\right)^{\frac{1}{\gamma^{i^{**}}}} \geq \sum_{i\in\mathcal{I}}\sum_{\sigma_t} p^i(\sigma_t|)^{\frac{1}{\gamma^i}} p^{RN}(\sigma_t|)^{1-\frac{1}{\gamma^i}}\phi^i_{t-1}. \qquad (10)$$

The result follows by showing that

$$\gamma^i \leq 1\ \ \forall i \Rightarrow \ln\sum_{i\in\mathcal{I}}\sum_{\sigma_t} p^i(\sigma_t|)^{\frac{1}{\gamma^i}} p^{RN}(\sigma_t|)^{1-\frac{1}{\gamma^i}}\phi^i_{t-1} \geq 0.$$

For convenience, let $\forall i, \eta_i := \frac{1}{\gamma^i}$; so that $\forall i, \eta_i \in (1,\infty)$.

$$\begin{aligned}
\ln\sum_{i\in\mathcal{I}}\sum_{\sigma_t} p^i(\sigma_t|)^{\frac{1}{\gamma^i}} p^{RN}(\sigma_t|)^{1-\frac{1}{\gamma^i}}\phi^i_{t-1} &= \ln\sum_{i\in\mathcal{I}}\sum_{\sigma_t}\frac{p^i(\sigma_t|)^{\eta_i}}{p^{RN}(\sigma_t|)^{\eta_i-1}}\phi^i_{t-1}\\
&\geq^{(a)} \sum_{i\in\mathcal{I}}\phi^i_{t-1}\ln\sum_{\sigma_t}\frac{p^i(\sigma_t|)^{\eta_i}}{p^{RN}(\sigma_t|)^{\eta_i-1}}\\
&= \sum_{i\in\mathcal{I}}(\eta_i-1)\phi^i_{t-1}\left(\frac{1}{\eta_i-1}\ln\sum_{\sigma_t}\frac{p^i(\sigma_t|)^{\eta_i}}{p^{RN}(\sigma_t|)^{\eta_i-1}}\right)\\
&=^{(b)} \sum_{i\in\mathcal{I}}(\eta_i-1)\phi^i_{t-1}D_{\eta^i}(p^i_t||p^{RN}_t)\\
&\geq^{(c)} 0.
\end{aligned}$$

(a): By concavity of log.

(b): Recognizing the definition of the Rényi divergence ($D_{\eta^i}(p_t^i||p_t^{RN})$) between p_t^i and p_t^{RN} (Rényi, 1961; Van Erven and Harremos, 2014).
(c): Rény divergence is weakly positive, it equals 0 iff $p^i = p^{RN}$ (Van Erven and Harremos, 2014).

An inspection of Equation (8) shows that equality holds if and only if $\gamma^i = 1$ for all agents — which implies that $\sum_{\sigma_t} \frac{q(\sigma_t|)}{\beta} = \sum\limits_{i\in\mathfrak{I}} \sum_{\sigma_t} p^i(\sigma_t|) = 1$ — or all agents have identical beliefs — $\forall i, p_t^i = p_t = p_t^{RN} \Rightarrow \forall i, \left(\sum_{\sigma_t} \frac{q(\sigma_t|)}{\beta}\right)^{1-\frac{1}{\gamma^i}} = \sum\limits_{i\in\mathfrak{I}} \sum_{\sigma_t} p_t(\sigma_t|)^{\frac{1}{\gamma^i}} p_t(\sigma_t|)^{1-\frac{1}{\gamma^i}} \phi_{t-1}^i = 1.$ $\square$

Lemma 7. *Under **A1-A5** , if all agents have identical CRRA utility then, for all (t,σ):*

$$\begin{array}{c} \forall i, \gamma^i \geq 1 \Rightarrow \frac{1}{\beta}\sum_{\sigma_t} q(\sigma_t|\sigma^{t-1}) \left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma} \leq 1 \\ \forall i, \gamma^i \leq 1 \Rightarrow \frac{1}{\beta}\sum_{\sigma_t} q(\sigma_t|\sigma^{t-1}) \left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma} \geq 1 \end{array};$$

with equality if and only if ether $\gamma = 1$ for all agents or all agents have identical beliefs.

Proof. This proof mimics that of Lemma 6. On every equilibrium path $\forall(t,\sigma)$ and for all i,

$$c_t^i(\sigma) = \left(\frac{\beta p^i(\sigma_t|)}{q(\sigma_t|)}\right)^{\frac{1}{\gamma}} c_{t-1}^i(\sigma).$$

Multiplying both sides by $\frac{q(\sigma_t|)}{\beta}\left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma-1}$ we have

$$\frac{q(\sigma_t|)}{\beta}\left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma-1} c_t^i(\sigma) = p^i(\sigma_t|)^{\frac{1}{\gamma}} \left(\frac{q(\sigma_t|)}{\beta}\left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma}\right)^{1-\frac{1}{\gamma}} c_{t-1}^i(\sigma).$$

Summing left and right over agents, i,

$$\frac{q(\sigma_t|)}{\beta}\left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma-1} \sum_{i\in\mathfrak{I}} c_t^i(\sigma) = \sum_{i\in\mathfrak{I}} p^i(\sigma_t|)^{\frac{1}{\gamma}} \left(\frac{q(\sigma_t|)}{\beta}\left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma}\right)^{1-\frac{1}{\gamma}} c_{t-1}^i(\sigma).$$

Noticing that $\boldsymbol{e}_t(\sigma) = \sum\limits_{i\in\mathfrak{I}} c_t^i(\sigma)$ and $\boldsymbol{e}_{t-1}(\sigma) = \sum\limits_{i\in\mathfrak{I}} c_{t-1}^i(\sigma)$, simplifyng and rearranging

$$\frac{q(\sigma_t|)}{\beta}\left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma} = \sum_{i\in\mathfrak{I}} p^i(\sigma_t|)^{\frac{1}{\gamma}} \left(\frac{q(\sigma_t|)}{\beta}\left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma}\right)^{1-\frac{1}{\gamma}} \phi_{t-1}^i(\sigma)$$

where $[\phi_{t-1}^1, ..., \phi_{t-1}^I]$ is the consumption shares distribution in $(t-1, \sigma^{t-1})$.
Summing left and right over the states:

$$\sum_{\sigma_t} \frac{q(\sigma_t|)}{\beta}\left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma} = \sum_{i\in\mathfrak{I}} \sum_{\sigma_t} p^i(\sigma_t|)^{\frac{1}{\gamma}} \left(\frac{q(\sigma_t|)}{\beta}\left(\frac{\boldsymbol{e}_t(\sigma)}{\boldsymbol{e}_{t-1}(\sigma)}\right)^{\gamma}\right)^{1-\frac{1}{\gamma}} \phi_{t-1}^i(\sigma).$$

Multiplying both sides by $\left(\sum_{\sigma_t} \frac{q(\sigma_t|)\left(\frac{e_t(\sigma)}{e_{t-1}(\sigma)}\right)^\gamma}{\beta}\right)^{1-\frac{1}{\gamma}}$,

$$\left[\sum_{\sigma_t} \frac{q(\sigma_t|)\left(\frac{e_t(\sigma)}{e_{t-1}(\sigma)}\right)^\gamma}{\beta}\right]^{\frac{1}{\gamma}} = \sum_{i\in\mathcal{I}}\sum_{\sigma_t} p^i(\sigma_t|)^{\frac{1}{\gamma}} p^{RN}(\sigma_t|)^{1-\frac{1}{\gamma}}\phi^i_{t-1}. \tag{11}$$

The rest of the proof is now identical to that of Lemma 6, substituting Equation (11) into Equations (9) and (10) to study the cases $\gamma \geq 1$, $\gamma \leq 1$, respectively. □

Proof of Proposition 6

Proof. Let's start from the case of constant aggregate endowment.

$$\begin{aligned}
\text{Note that } \forall(t,\sigma),\ \ln p^{RN}(\sigma^t) &= \ln\prod_i p^{RN}(\sigma_t|) = \ln\prod_i \frac{q(\sigma_t|)}{\sum_{\sigma_t} q(\sigma_t|)} \\
&= \ln\frac{q(\sigma^t)}{\beta^t} - \sum_{\tau=1}^t \ln\left(\frac{1}{\beta}\sum_{\sigma_t} q(\sigma_t|)\right) \\
\text{by Massari (2017), Th.1} &\asymp \ln\left(\sum_i p^i(\sigma^t)\right) - \sum_{\tau=1}^t \ln\left(\frac{1}{\beta}\sum_{\sigma_t} q(\sigma_t|)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\bar{d}(P||p^M) - \bar{d}(P||p^{RN}) &= \lim_{t\to\infty}\frac{1}{t}\left(\ln p^{RN}(\sigma^t) - \ln p^M(\sigma^t)\right) \qquad P\text{-a.s., by the SLLNMD} \\
&= \lim_{t\to\infty}\frac{1}{t}\left(\ln\sum_i p^i(\sigma^t) - \frac{1}{t}\sum_{\tau=1}^t \ln\left(\frac{1}{\beta}\sum_{\sigma_\tau} q(\sigma_\tau|)\right) - \ln p^M(\sigma^t)\right) \\
&=^{\text{By Lem.1}} \lim_{t\to\infty}\frac{1}{t}\left(\ln\sum_i p^i(\sigma^t) - \frac{1}{t}\sum_{\tau=1}^t \ln\left(\frac{1}{\beta}\sum_{\sigma_\tau} q(\sigma_\tau|)\right) - \ln\sum_i p^i(\sigma^t)\right) \\
&= -\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^t \ln\left(\frac{1}{\beta}\sum_{\sigma_\tau} q(\sigma_\tau|)\right) \\
\text{and Lemma 6} \Rightarrow &\begin{cases} \geq 0 \text{ if } \forall i, \gamma^i \in [1,\infty) \\ \leq 0 \text{ if } \forall i, \gamma^i \in (0,1] \end{cases};
\end{aligned}$$

where inequalities are strict if and only if there is long-run heterogeneity a positive fraction of periods (that is if and only if at least one surviving agent has $\alpha \in (0,1)$ Massari, 2017) and not all the surviving agents have log utility (by Lemma 6).

- The proof of the case of common CRRA utility and aggregate risk, is obtained by repeating the same steps but replacing $\left(\frac{1}{\beta}\sum_{\sigma_t} q(\sigma_t|)\right)$ and Lemma 6 with $\left(\frac{1}{\beta}\sum_{\sigma_t} q(\sigma_t|)\left(\frac{e_t(\sigma)}{e_{t-1}(\sigma)}\right)^\gamma\right)$ and Lemma 7, respectively.

□

Proof of Proposition 7

Proof. $\forall i \in \hat{\mathfrak{I}}, \bar{d}(P||p^M) - \bar{d}(P||p^i) =^{\text{By Prop. 2}} 0$
$\Rightarrow^{\text{by Eq.15},Lem.9}\ \forall i \in \hat{\mathfrak{I}}, \bar{d}(P||p^M) - \bar{d}(P||p^{RN}) + \lim_{t\to\infty} \alpha^i \frac{1}{t}\sum_{\tau=1}^{t} E\left[\frac{\pi^i}{p^{RN}_\tau} - 1\right] - |O((\alpha^i)^2)| = 0;$
and the result follows by choosing $i = \text{argmin}_{i\in\hat{\mathfrak{I}}}\, \alpha^i$.
Furthermore, if agent i dominates $p^{RN} \to \pi^i \Rightarrow \bar{d}(P||p^i) - \bar{d}(P||p^M) = 0$.
Last, Massari (2017) has shown that $\bar{d}(P||p^i) - \bar{d}(P||p^M) = 0$ if all surviving agents have α=1. □

Proof of Proposition 8

Proof. The condition on p^i is sufficient to guarantee that agent BIP does not dominate — otherwise, agent i would be more accurate than agent BIP, violating Proposition.1. With long-run heterogeneity, $\bar{d}(P||p^{RN}) <^{Prop.6,(c)} \bar{d}(P||p^M)$ and the result follows because:

$$\bar{d}(P||p^{RN}) <^{Prop.6,(c)} \bar{d}(P||p^M) \leq^{Prop.2} \bar{d}(P||p^{BIP}) \leq \bar{d}(P||\pi^{BIP}),$$

where the last inequality follows because, $\forall(t,\sigma)$,

$$\begin{aligned}
d(P||p_t^{BIP}) &= d(P||(1-\alpha^{BIP})p_t^{RN} + \alpha^{BIP}\pi^{BIP})) \\
&\leq (1-\alpha^{BIP})d(P||p_t^{RN}) + \alpha^{BIP} d(P||\pi^{BIP}) && \text{; by strict convexity of } d(P||\cdot) \\
\Rightarrow \bar{d}(P||p^{BIP}) &\leq (1-\alpha^{BIP})\bar{d}(P||p^{RN}) + \alpha^{BIP}\bar{d}(P||\pi^{BIP}) && \text{; summing and averaging over } t \\
\Rightarrow \bar{d}(P||p^{BIP}) &\leq^{Prop.6,(c)} (1-\alpha^{BIP})\bar{d}(P||p^{RN}) + \alpha^{BIP}\bar{d}(P||\pi^{BIP}) \\
\Rightarrow \bar{d}(P||p^{BIP}) &\leq^{Prop.2} (1-\alpha^{BIP})\bar{d}(P||p^{BIP}) + \alpha^{BIP}\bar{d}(P||\pi^{BIP}) \\
\Rightarrow \bar{d}(P||p^{BIP}) &\leq \bar{d}(P||\pi^{BIP}).
\end{aligned}$$

□

Lemma 8. *Under **A1-A5** , if all agents have identical CRRA utility, then:*

$$p^{RN}(\sigma_t|\sigma^{t-1}) = \frac{\left(\sum_{i\in\mathfrak{I}} \left((1-\alpha)p^{RN}(\sigma_t|\sigma^{t-1}) + \alpha\pi^i(\sigma_t)\right)^{\frac{1}{\gamma}} \phi^i_{t-1}(\sigma)\right)^\gamma}{\sum_{\tilde{\sigma}_t}\left(\sum_{j\in\mathfrak{I}} \left((1-\alpha)p_t^{RN}(\tilde{\sigma}_t|\sigma^{t-1}) + \alpha\pi^j(\tilde{\sigma}_t)\right)^{\frac{1}{\gamma}} \phi^j_{t-1}(\sigma)\right)^\gamma}.$$

Proof. In every equilibrium $\forall i \in \mathfrak{I}$ and $\forall(t,\sigma)$ the FOC is $\frac{(c_t^i(\sigma))^\gamma}{(c_{t-1}^i(\sigma))^\gamma} = \frac{\beta p^i(\sigma_t|\sigma^{t-1})}{q(\sigma_t|\sigma^{t-1})}$; rearranging,

$$\left(q(\sigma_t|\sigma^{t-1})\right)^{\frac{1}{\gamma}} c_t^i(\sigma) = \left(\beta p^i(\sigma_t|\sigma^{t-1}))\right)^{\frac{1}{\gamma}} c_{t-1}^i(\sigma).$$

Summing over agents $\left(\sum_{i\in\mathfrak{I}} c_t(\sigma) = \boldsymbol{e}_t(\sigma)\right)$, and taking the power γ gives

$$q(\sigma_t|)\boldsymbol{e}_t(\sigma)^\gamma = \left(\sum_{i\in\mathfrak{I}} \left(\beta p^j(\sigma_t|\sigma^{t-1}))\right)^{\frac{1}{\gamma}} c_{t-1}^j(\sigma)\right)^\gamma.$$

Using Definition 9, we obtain

$$p_\gamma^{RN}(\sigma_t|\sigma^{t-1}) := \frac{q(\sigma_t|\sigma^{t-1})\boldsymbol{e}_t(\sigma)^\gamma}{\sum_{\tilde{\sigma}_t} q(\tilde{\sigma}_t|\sigma^{t-1})\boldsymbol{e}_t(\tilde{\sigma})^\gamma} = \frac{\left(\sum_{i\in\mathfrak{I}}(\beta p^i(\sigma_t|\sigma^{t-1}))^{\frac{1}{\gamma}} c_{t-1}^i(\sigma)\right)^\gamma}{\sum_{\tilde{\sigma}_t}\left(\sum_{i\in\mathfrak{I}}(\beta p^i(\tilde{\sigma}_t|\sigma^{t-1}))^{\frac{1}{\gamma}} c_{t-1}^i(\sigma)\right)^\gamma},$$

and the result follows substituting each p^i with its Definition 4. The same result for p^{RN} follows by noticing that when $\boldsymbol{e}_t(\sigma) = \boldsymbol{e}$ for all (t, σ) Definition 9 turns into Definition 8. ☐

B Theorems 1,2,3

Lemma 9. *Under **A1-A4 (A5)**, if* $\exists\hat{\mathfrak{I}} \subset \mathfrak{I} : P \in Conv(\hat{\mathfrak{I}}), \forall i \in \hat{\mathfrak{I}}, \alpha^i \in (0, \bar{\alpha})$ *with* $\bar{\alpha} = \max_{i\in\hat{\mathfrak{I}}}\{\alpha^i\} < 1$, *then*
a), *all agents in* $\hat{\mathfrak{I}}$ *use* p^M *for consensus,*$\Rightarrow \exists\boldsymbol{\gamma} \in \Delta^{|\hat{\mathfrak{I}}|}$:

$$\bar{\alpha}\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i}{\alpha^i}\bar{d}(P||p^M) - \bar{\alpha}\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i}{\alpha^i}\bar{d}(P||p^i) = \lim_{t\to\infty}\frac{\bar{\alpha}}{t}\sum_{\tau=1}^{t} E\left[\frac{P}{p_t^M} - 1\right] - |O(\bar{\alpha})^2|, \tag{12}$$

b), *all agents in* $\hat{\mathfrak{I}}$ *use* p^{RN} *for consensus,*$\Rightarrow \exists\boldsymbol{\gamma} \in \Delta^{|\hat{\mathfrak{I}}|}$*:*

$$\bar{\alpha}\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i}{\alpha^i}\bar{d}(P||p^M) - \bar{\alpha}\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i}{\alpha^i}\bar{d}(P||p^i) = \bar{\alpha}\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i}{\alpha^i}\bar{d}(P||p^M) - \bar{\alpha}\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i}{\alpha^i}\bar{d}(P||p^{RN}) + \lim_{t\to\infty}\frac{\bar{\alpha}}{t}\sum_{\tau=1}^{t} E\left[\frac{P}{p_\tau^{RN}} - 1\right] - |O(\bar{\alpha}^2)|. \tag{13}$$

Proof. **a)** By assumption, $\forall i \in \mathfrak{I}, \forall(t, \sigma), p_t^i = p_t^M + \alpha^i(\pi^i - p^M)$. Taylor expanding around 1:

$$E\ln\frac{p_t^i}{p_t^M} = E\ln\left(1 + \alpha^i\left(\frac{\pi^i}{p_t^M} - 1\right)\right) = \alpha^i E\left[\frac{\pi^i}{p_t^M} - 1\right] - |O((\alpha^i)^2)|.$$

So that
$$\begin{aligned}\bar{d}(P||p^M) - \bar{d}(P||p^i) &= \lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^{t}\left[E\ln\frac{p_\tau^i}{p_\tau^M}\right] \\ &= \lim_{t\to\infty}\frac{\alpha^i}{t}\sum_{\tau=1}^{t} E\left[\frac{\pi^i}{p_\tau^M} - 1\right] - |O((\alpha^i)^2)|\end{aligned} \tag{14}$$

Let $\boldsymbol{\gamma} = [\gamma^1, ..., \gamma^I] \in \Delta^{|\hat{\mathfrak{I}}|}$ be such that $\forall s \in S, \sum_{i\in\hat{\mathfrak{I}}}\gamma^i\pi^i(s) = P(s)$ — this vector exists because we assumed $P \in Conv(\hat{\mathfrak{I}})$—, and let $\boldsymbol{\gamma}_\alpha = [\frac{\gamma^1\bar{\alpha}}{\alpha^1}, ..., \frac{\gamma^I\bar{\alpha}}{\alpha^I}]$.

Equation (14) holds for all agents in $\hat{\mathfrak{I}}$, therefore it holds for their $\boldsymbol{\gamma}_\alpha$ wighted sum:

$$\begin{aligned}\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i\bar{\alpha}}{\alpha^i}\bar{d}(P||p^M)-\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i\bar{\alpha}}{\alpha^i}\bar{d}(P||p^i) &= \sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i\bar{\alpha}}{\alpha^i}\lim_{t\to\infty}\alpha^i\frac{1}{t}\sum_{\tau=1}^{t}E\left[\frac{\pi^i}{p^M}_\tau-1\right]-|O((\alpha^i)^2)| \\ &= \lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^{t}\bar{\alpha}\sum_{\gamma^i}E\left[\gamma^i\frac{\pi^i}{p^M_t}-\gamma^i\right]-\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i\bar{\alpha}}{\alpha^i}|O((\alpha^i)^2)| \\ &= \lim_{t\to\infty}\frac{\bar{\alpha}}{t}\sum_{\tau=1}^{t}E\left[\frac{P}{p^M_t}-1\right]-|O(\bar{\alpha}^2)|\end{aligned}$$

b): By assumption, $p^i_t=p^{RN}_t+\alpha(\pi^i-p^{RN})$; performing a Taylor expansion around 1:

$$\begin{aligned}&E\ln\frac{p^i_t}{p^{RN}_t}=E\ln\left(1+\alpha^i\left(\frac{\pi^i}{p^{RN}_t}-1\right)\right)=E\ln 1+\alpha^i E\left[\frac{\pi^i}{p^{RN}_t}-1\right]-|O((\alpha^i)^2)| \\ &\Rightarrow -\bar{d}(P||p^i)=-\bar{d}(P||p^{RN})+\lim_{t\to\infty}\frac{\alpha^i}{t}\sum_{\tau=1}^{t}E\left[\frac{\pi^i}{p^{RN}_\tau}-1\right]-|O((\alpha^i)^2)| \quad P\text{-a.s.};\end{aligned}$$

which implies

$$\bar{d}(P||p^M)-\bar{d}(P||p^i)=\bar{d}(P||p^M)-\bar{d}(P||p^{RN})+\lim_{t\to\infty}\frac{\alpha^i}{t}\sum_{\tau=1}^{t}E\left[\frac{\pi^i}{p^{RN}_\tau}-1\right]-|O((\alpha^i)^2)| \quad P\text{-a.s.} \tag{15}$$

and Equation (13) is obtained taking the $\boldsymbol{\gamma}_\alpha$ weighted sum as in point ***a***.

□

Proof of Theorem 1

Proof. In equilibrium, the following inequalities must hold P-a.s.

$$\begin{aligned}&\forall i\in\hat{\mathfrak{I}},\bar{d}(P||p^M)-\bar{d}(P||p^i)\leq^{\text{By Prop.2 a)}}0 \\ &\Rightarrow\forall\boldsymbol{\gamma}\in\Delta^{|\hat{\mathfrak{I}}|},\bar{\alpha}\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i}{\alpha^i}\bar{d}(P||p^M)-\bar{\alpha}\sum_{i\in\hat{\mathfrak{I}}}\frac{\gamma^i}{\alpha^i}\bar{d}(P||p^i)\leq 0 \\ &\Rightarrow^{\text{By Lem.9,a)}}\lim_{t\to\infty}\frac{\bar{\alpha}}{t}\sum_{\tau=1}^{t}E\left[\frac{P}{p^M_t}-1\right]\leq|O(\bar{\alpha})^2| \\ &\Rightarrow\ \bar{d}(P||p^M_{\bar{\alpha}})\to^{\bar{\alpha}\to 0}0;\end{aligned}$$

The last implication holds because $p^M=P$ is the only minimizer for both the continuous positive functions $d(P||p^M_t)$ and $E\left[\frac{P}{p^M_t}-1\right]$. □

Proof of Theorem 2

Proof. In equilibrium, the following inequalities must hold P-a.s.

$$\forall i \in \hat{\mathcal{I}}, \bar{d}(P||p^M) - \bar{d}(P||p^i) \leq^{\text{By Prop.2 a)}} 0$$

$$\Rightarrow \forall \boldsymbol{\gamma} \in \Delta^{|\hat{\mathcal{I}}|}, \bar{\alpha} \sum_{i\in\hat{\mathcal{I}}} \frac{\gamma^i}{\alpha^i} \bar{d}(P||p^M) - \bar{\alpha} \sum_{i\in\hat{\mathcal{I}}} \frac{\gamma^i}{\alpha^i} \bar{d}(P||p^i) \leq 0$$

$$\Rightarrow^{\text{By Lem.9, b)}} \left(\bar{\alpha} \sum_{i\in\mathcal{I}^*} \frac{\gamma^i}{\alpha^i} \bar{d}(P||p^M) - \bar{\alpha} \sum_{i\in\hat{\mathcal{I}}} \frac{\gamma^i}{\alpha^i} \bar{d}(P||p^{RN}) + \lim_{t\to\infty} \frac{\bar{\alpha}}{t} \sum_{\tau=1}^{t} E\left[\frac{P}{p_\tau^{RN}} - 1 \right] \right) \leq |O(\bar{\alpha}^2)|$$

$$\Rightarrow^{\text{By Prop.6, a)}\ \bar{d}(P||p^M) - \bar{d}(P||p^{RN}) \geq 0} \lim_{t\to\infty} \frac{\bar{\alpha}}{t} \sum_{\tau=1}^{t} E\left[\frac{P}{p_\tau^{RN}} - 1 \right] \leq |O(\bar{\alpha}^2)|$$

$$\Rightarrow \lim_{\alpha\to 0} \bar{d}(P||p_{\bar{\alpha}}^{RN}) = 0.$$

The last implications holds because $p^{RN} = P$ is the only minimizer of the continuous non-negative functions $E\left[\frac{P}{p_t^{RN}} - 1\right] \geq 0$ and $\bar{d}(P||p^{RN}) \geq 0$. ☐

Preliminaries for the proof of Theorem 3

Our proof is an application of Theorem 3.1 of Hajek (1982) which places two conditions on the drift and the variance of a stochastic process (mean reverting in our case) which guarantees that the process spend most of its time close to a boundary (its mean-reverting point in our case). We start by illustrating the results in Hajek (1982) that are relevant to our proof.
Let $(Y_t)_{t=0}^{\infty}$ be a sequence of real valued adopted random variables, with drift $\mathrm{E}[Y_{t+1} - Y_t|\mathcal{F}_t]$ such that conditions $\boldsymbol{C1}, \boldsymbol{C2}$ below are satisfied.

$\boldsymbol{C1.}$ $\exists \epsilon_0 > 0 : E\left[Y_{t+1} - Y_t + \epsilon_0 |\, Y_t > a, \mathcal{F}_{t-1}\right] < 0;$

$\boldsymbol{C2.}$ $\exists Z < \infty : \forall(t,\sigma), \left[|Y_{t+1} - Y_t||\, \mathcal{F}_t\right] < Z$ and $Ee^{\lambda Z} = D < \infty$, for some $\lambda > 0$.

Hajek (1982) Lemma 2.1 (see below) allows us to translate conditions $\boldsymbol{C1}, \boldsymbol{C2}$ into higher order conditions ($\boldsymbol{D1}, \boldsymbol{D2}$) that are used to derive the following

Theorem 3.1, Hajek (1982):

Assume conditions $\boldsymbol{D1}, \boldsymbol{D2}$ on $(Y_t, \mathcal{F}_t)_{t\geq 0}$, for any $\epsilon' > 0$ exist constants K and δ with $\delta \in (0,1)$:

$$P\left[\frac{1}{t} \sum_{\tau=1}^{t} I_{Y_\tau < b} \leq \left(1 - \frac{1}{1-\rho} D e^{\eta(a-b)}\right)(1-\epsilon') \right] \leq K\delta^t \tag{16}$$

Next, we map our problem into Hajek (1982) framework.
To ease notation we focus on state u and define $P := P(\{\sigma_{t+1} = u\})$, $\pi^{BCP} := \pi^{BCP}(\{\sigma_{t+1} = u\})$, $\pi^i := \pi^i(\{\sigma = u\})$, $p_{t+1}^M := p_{t+1}^M(\{\sigma_{t+1} = u|\sigma^t\})$ and $c_t^i := c^i(\sigma_t)$.
WLOG, we focus on a two-agent $(l,\, r)$ economy with $\pi^l < P < \pi^r$.[11]
Let $Y_t := \gamma \ln \frac{\phi_t^r}{\phi_t^l}$; because the economy has two agents $Y_t = \gamma \ln \frac{\phi_t^r}{1-\phi_t^r}$ pins down the consumption

[11]The proof generalize to more than two agents by replacing π^r and π^l by $\pi_t^R := \sum_{p_t^i > P} \pi^i \phi_{t-1}^i(\sigma)$ and $\pi_t^L := \sum_{p_t^i < P} \pi^i \phi_{t-1}^i(\sigma)$; $\phi_t^r(\sigma)$ and $\phi_t^l(\sigma)$ with $\phi_t^R(\sigma) := \sum_{p_t^i > P} \phi_{t-1}^i(\sigma)$ and $\phi_t^L(\sigma) := \sum_{p_t^i < P} \phi_{t-1}^i(\sigma)$ and providing a bound on the difference $\pi_t^R - \pi_t^L$ that holds uniformly for every consumption-share distribution within agents in L and R.

shares of both agents at once, $\forall(\sigma,t), \phi_t^r = \phi(Y_t) = \frac{e^{\frac{Y_t}{\gamma}}}{1+e^{\frac{Y_t}{\gamma}}} = 1-\phi^l$; and we can write p_t^{RN} as a function of Y_t alone, so that the dynamic of $(Y_t)_{t=0}^{\infty}$ is directly informative for the dynamic of $d(P||p_t^{RN})_{t=0}^{\infty}$. Using Lemma 8 (pg. 40), we obtain

$$p_{t+1}^{RN} = \frac{\left((p_{t+1}^r)^{\frac{1}{\gamma}}\phi(Y_t) + (p_{t+1}^l)^{\frac{1}{\gamma}}(1-\phi(Y_t))\right)^{\gamma}}{\left((p_{t+1}^r)^{\frac{1}{\gamma}}\phi(Y_t) + (p_{t+1}^l)^{\frac{1}{\gamma}}(1-\phi(Y_t))\right)^{\gamma} + \left((1-p_{t+1}^r)^{\frac{1}{\gamma}}\phi(Y_t) + (1-p_{t+1}^l)^{\frac{1}{\gamma}}(1-\phi(Y_t))\right)^{\gamma}},$$

where $p_{t+1}^{RN}(Y_t)$ exists and is analytic by the analytic implicit function theorem.
The equilibrium conditions, $\forall(t,\sigma), \left(\frac{c_t^r}{c_t^r}\right)^{\gamma} = \frac{p_t^r}{p_t^l}\left(\frac{c_{t-1}^r}{c_{t-1}^l}\right)^{\gamma}$ imply that our $(Y_t)_{t=0}^{\infty}$ process evolves according to the following dynamics

$$Y_t = Y_{t-1} + I_{\sigma_t=u}\ln\frac{(1-\alpha)p_t^{RN}(Y_{t-1})+\alpha\pi^r}{(1-\alpha)p_t^{RN}(Y_{t-1})+\alpha\pi^l} + I_{\sigma_t=d}\ln\frac{(1-\alpha)(1-p_t^{RN}(Y_{t-1}))+\alpha(1-\pi^r)}{(1-\alpha)(1-p_t^{RN}(Y_{t-1}))+\alpha(1-\pi^l)}.$$

Next we characterize the drift and variance of $(Y_t)_{t=0}^{\infty}$ as a function of α. We use Hajek (1982) theorem 3.1 to show α can be chose small enough to guaranty that Y_t spends most of the periods arbitrarily close to the value $\bar{Y} : p^{RN}(\bar{Y}) = P$.

Lemma 10. *Under the assumptions of Theorem 3, $\forall(t,\sigma)$:*

$$E\left[Y_{t+1}-Y_t|\,Y_t,\mathcal{F}_{t-1}\right] = \alpha\frac{(\pi^r-\pi^l)}{p_{t+1}^{RN}(1-p_{t+1}^{RN})}\left(P-p_{t+1}^{RN}\right)+O(\alpha^2) \tag{17}$$

Proof. The equilibrium condition implies that for all (t,σ),

$$\begin{aligned}
Y_{t+1}|_{\sigma_{t+1}=u} - Y_t &= \ln\left(\frac{(1-\alpha)p_{t+1}^{RN}+\alpha\pi^r}{(1-\alpha)p_{t+1}^{RN}+\alpha\pi^l}\right) = \ln\left(1+\alpha\frac{(\pi^r-p_{t+1}^{RN})}{p_{t+1}^{RN}}\right) - \ln\left(1+\alpha\frac{(\pi^l-p_{t+1}^{RN})}{p_{t+1}^{RN}}\right);\\
&=^{\text{Taylor expanding around 1}}\ \alpha\frac{(\pi^r-p_{t+1}^{RN})}{p_{t+1}^{RN}} - \alpha\frac{(\pi^l-p_{t+1}^{RN})}{p_{t+1}^{RN}} + O(\alpha^2);\\
Y_{t+1}|_{\sigma_{t+1}=d} - Y_t &= \ln\left(\frac{(1-\alpha)(1-p_{t+1}^{RN})+\alpha(1-\pi^r)}{(1-\alpha)(1-p_{t+1}^{RN})+\alpha(1-\pi^l)}\right) = \ln\left(\frac{1+\alpha\frac{(p_{t+1}^{RN}-\pi^r)}{1-p_{t+1}^{RN}}}{(1+\alpha\frac{(p_{t+1}^{RN}-\pi^l)}{1-p_{t+1}^{RN}}}\right)\\
&=^{\text{Taylor expanding around 1}}\ \alpha\frac{(p_{t+1}^{RN}-\pi^r)}{1-p_{t+1}^{RN}} - \alpha\frac{(p_{t+1}^{RN}-\pi^l)}{1-p_{t+1}^{RN}} + O(\alpha^2).
\end{aligned}$$

Computing the expected value

$$\begin{aligned}
\mathrm{E}[Y_{t+1}-Y_t|\mathcal{F}_t] = &\alpha\left(P\left(\frac{\pi^r-\pi^l}{p_{t+1}^{RN}}\right)+(1-P)\left(\frac{\pi^l-\pi^r}{1-p_{t+1}^{RN}}\right)\right)+O(\alpha^2)\\
=&\alpha(\pi^r-\pi^l)\left(\left(\frac{P}{p_{t+1}^{RN}}+\frac{1-P}{1-p_{t+1}^{RN}}\right)\right)+O(\alpha^2)\\
=&\alpha\frac{(\pi^r-\pi^l)}{p_{t+1}^{RN}(1-p_{t+1}^{RN})}\left(P-p_{t+1}^{RN}\right)+O(\alpha^2)
\end{aligned}$$

□

Lemma 11. *Under the assumptions of Theorem 3, there exists a* $\bar{\alpha} > 0 : \forall \alpha \in (0, \bar{\alpha}]$,

$$\textbf{\textit{C1}}: \qquad E\left[Y_{t+1} - Y_t |\, Y_t > \bar{Y} + \sqrt{\bar{\alpha}}, \mathcal{F}_{t-1}\right] \leq -|O(\alpha)O(\bar{\alpha}^{.5})| + |O(\alpha)O(\bar{\alpha})| + |O(\alpha^2)|;$$

for $\bar{Y} := \gamma \ln \frac{\bar{\phi}^r}{\bar{\phi}^l}$*, with* $\bar{\phi}^l, \bar{\phi}^r : p^{RN}(\bar{Y}) = P$.

Proof. First, note that $\bar{Y} = \gamma \ln \frac{\bar{\phi}^r}{\bar{\phi}^l}$ is well defined. $P \in Conv(\pi^l, \pi^r) \Rightarrow \exists \bar{\phi}^l, \bar{\phi}^r : p^{RN} = P$ because by the implict function theorem $p^{RN}(\phi^r)$, is continuous and $p^{RN}(\phi^r = 0) = \pi^l < P$ and $p^{RN}(\phi^r = 1) = \pi^r > P$. Note that $\bar{\phi}^r$, and thus $\bar{Y}$, may depend on α, but not on $\bar{\alpha}$. Next, Lemma 10 guarantees that

$$E\left[Y_{t+1} - Y_t |\, Y_t > \bar{Y} + \sqrt{\bar{\alpha}}, \mathcal{F}_{t-1}\right] = \alpha \frac{(\pi^r - \pi^l)}{p^{RN}_{t+1}(1 - p^{RN}_{t+1})} \left(P - p^{RN}_{t+1}|_{Y_t > \bar{Y} + \sqrt{\bar{\alpha}}}\right) + O(\alpha^2);$$

so that the result follows by showing that $\left(P - p^{RN}_{t+1}|_{Y_t > \bar{Y} + \sqrt{\bar{\alpha}}}\right) \leq -|O(\sqrt{\bar{\alpha}})| + O(\bar{\alpha})$.

Because of Markovianity, the above does not depend on how p^{RN}_{t+1} and Y_t are determined, and we can drop the time indexes.
We want to show that $p^{RN}(Y;\alpha)$ has a strictly positive and finite derivative in Y for every α small enough, $\alpha \in (0, \bar{\alpha}]$. Then, a Taylor expansion guarantees that for all ϵ small enough $Y > \bar{Y} + \epsilon \Rightarrow p^{RN} > P + |O(\epsilon)| + O(\epsilon^2)$. Taking $\epsilon = \sqrt{\bar{\alpha}}$ concludes the proof.

To calculate $\frac{dp^{RN}}{dY}$ we use the implicit function theorem. For our purposes, let

$$F(Y, p^{RN}) := (p^{RN} - 1)\left((p^r)^{\frac{1}{\gamma}}\phi(Y) + (p^l)^{\frac{1}{\gamma}}(1 - \phi(Y))\right)^{\gamma} + p^{RN}\left((1 - p^r)^{\frac{1}{\gamma}}\phi(Y) + (1 - p^l)^{\frac{1}{\gamma}}(1 - \phi(Y))\right)^{\gamma}.$$

so that $\forall Y \in (-\infty, +\infty)$, the solutions of $F(Y, p^{RN}) = 0$ identify $p^{RN}(Y)$, in particular $F(\bar{Y}, P) = 0$.
By the implicit function theorem, on the solutions of $F(Y, p^{RN}) = 0$

$$\frac{dp^{RN}}{dY} = -\frac{\frac{\partial F(Y,p^{RN})}{\partial Y}}{\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}}.$$

Next we sketch the calculations that show that $\exists \alpha^* : \forall \alpha \in (0, \alpha^*]$, $\frac{dp^{RN}}{dY} \in (0, \infty)$.

- Numerator: $\frac{\partial F(Y,p^{RN})}{\partial Y} \leq 0$, with equality iff $\alpha = 0$

$$\begin{aligned}\frac{\partial F(Y, p^{RN})}{\partial Y} &= -(1 - p^{RN})\gamma\left((p^r)^{\frac{1}{\gamma}}\phi(Y) + (p^l)^{\frac{1}{\gamma}}(1 - \phi(Y))\right)^{\gamma-1}\left((p^r)^{\frac{1}{\gamma}} - (p^l)^{\frac{1}{\gamma}}\right)\phi'(Y) + \\ &- p^{RN}\gamma\left((1 - p^r)^{\frac{1}{\gamma}}\phi(Y) + (1 - p^l)^{\frac{1}{\gamma}}(1 - \phi(Y))\right)^{\gamma-1}\left((1 - p^l)^{\frac{1}{\gamma}} - (1 - p^r)^{\frac{1}{\gamma}}\right)\phi'(Y),\end{aligned}$$

where $\phi'(Y) = \frac{1}{\gamma}\phi(Y)\,(1 - \phi(Y)) > 0$.
Therefore, $\frac{\partial F(Y,p^{RN})}{\partial Y} \leq 0$ because $\pi^r > \pi^l \Rightarrow p^r > p^l \Rightarrow \frac{\partial F(Y,p^{RN})}{\partial Y} \leq 0$;
and $\frac{\partial F(Y,p^{RN})}{\partial Y} = 0$ iff $p^r = p^l \Leftrightarrow \alpha = 0$.

- Denominator: for α small, $\frac{\partial F(Y,p^{RN})}{\partial p^{RN}} \geq 0$, with equality iff $\alpha = 0$
  Note that for $i = l, r$, $p^i = (1 - \alpha)p^{RN} + \alpha\pi^i$,

$$\begin{aligned}
\frac{\partial F(Y,p^{RN})}{\partial p^{RN}} &= \left((p^r)^{\frac{1}{\gamma}}\phi + (p^l)^{\frac{1}{\gamma}}(1-\phi)\right)^{\gamma} \\
&+ \left((1-p^r)^{\frac{1}{\gamma}}\phi + (1-p^l)^{\frac{1}{\gamma}}(1-\phi)\right)^{\gamma} \\
&- (1-p^{RN})\left((p^r)^{\frac{1}{\gamma}}\phi + (p^l)^{\frac{1}{\gamma}}(1-\phi)\right)^{\gamma-1}\left((p^r)^{\frac{1}{\gamma}-1}\phi + (p^l)^{\frac{1}{\gamma}-1}(1-\phi)\right)(1-\alpha) \\
&- p^{RN}\left((1-p^r)^{\frac{1}{\gamma}}\phi + (1-p^l)^{\frac{1}{\gamma}}(1-\phi)\right)^{\gamma-1}\left((1-p^r)^{\frac{1}{\gamma}-1}\phi + (1-p^l)^{\frac{1}{\gamma}-1}(1-\phi)\right)(1-\alpha).
\end{aligned}$$

Note that $\alpha = 0 \Rightarrow p^l = p^r = p^{RN} \Rightarrow \left.\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}\right|_{\alpha=0} = 0$.
Moreover, Lemma 15 shows that $\left.\frac{\partial}{\partial\alpha}\left(\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}\right)\right|_{\alpha=0} = 1$, so that a Taylor series expansion of $\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}$ in $\alpha = 0$ leads to the conclusion that for α small $\frac{\partial F(Y,p^{RN})}{\partial p^{RN}} > 0$.

$$\begin{aligned}
\frac{\partial F(Y,p^{RN})}{\partial p^{RN}} &= \left.\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}\right|_{\alpha=0} + \frac{\partial}{\partial\alpha}\left.\left(\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}\right)\right|_{\alpha=0}(\alpha) + \frac{1}{2}\frac{\partial^2}{\partial\alpha\partial\alpha}\left.\left(\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}\right)\right|_{\alpha=0}(\alpha^2) \\
&= 0 + \alpha + O(\alpha^2)
\end{aligned}$$

- Ratio: $\frac{dp^{RN}}{dY} = -\frac{\frac{\partial F(Y,p^{RN})}{\partial Y}}{\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}} > 0$ for all α small.

The above calculations show that for α small enough $\frac{dp^{RN}}{dY} = -\frac{\frac{\partial F(Y,p^{RN})}{\partial Y}}{\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}} > 0$. However, both derivatives are null at $\alpha = 0$.[12] To show that the ratio is uniformly strictly positive for all α small, we use l'Hôpital's rule to analyze its limit behaviour.

$$\lim_{\alpha\to 0}\frac{dp^{RN}}{dY} = \lim_{\alpha\to 0} -\frac{\frac{\partial}{\partial\alpha}\left(\frac{\partial F(Y,p^{RN})}{\partial Y}\right)}{\frac{\partial}{\partial\alpha}\left(\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}\right)} = -\frac{\left.\frac{\partial}{\partial\alpha}\left(\frac{\partial F(Y,p^{RN})}{\partial Y}\right)\right|_{\alpha=0}}{\left.\frac{\partial}{\partial\alpha}\left(\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}\right)\right|_{\alpha=0}} = \frac{1}{\gamma}(\pi^r - \pi^l)\phi(Y)\left(1-\phi(Y)\right) > 0,$$

where the last equality follows from Lemma 15 which show that
$\left.\frac{\partial}{\partial\alpha}\left(\frac{\partial F(Y,p^{RN})}{\partial Y}\right)\right|_{\alpha=0} = -\frac{1}{\gamma}(\pi^r - \pi^l)\phi(Y)\left(1-\phi(Y)\right)$ and $\left.\frac{\partial}{\partial\alpha}\left(\frac{\partial F(Y,p^{RN})}{\partial p^{RN}}\right)\right|_{\alpha=0} = 1$.

To conclude the proof, note that for $\bar{\alpha}$ small enough we can set a bound b:[13]

$$b = \underset{\alpha\in[0,\bar{\alpha}]}{\operatorname{argmin}}\left\{\frac{dp^{RN}(Y)}{dY} \text{ for } Y \in \left[\bar{Y}(\alpha), \bar{Y}(\alpha) + \sqrt{\bar{\alpha}}\right]\right\};$$

so that, the Taylor expansion of $p^{RN}(Y)$ guarantees that for all $\alpha \in (0, \bar{\alpha}]$

$$Y > \bar{Y} + \sqrt{\bar{\alpha}} \Rightarrow \left(P - p^{RN}_{t+1}\big|_{Y_t > \bar{Y} + \sqrt{\bar{\alpha}}}\right) \leq -b(\sqrt{\bar{\alpha}}) + O(\bar{\alpha}).$$

□

[12] The function $p^{RN}(Y)$, and thus its derivatives, are not defined for $\alpha = 0$. The following equations are conducted on the continuous extension of p^{RN} at $\alpha = 0$.

[13] The existence and positivity of such a minimum is guaranteed by the continuity and positivity of the argument in the closed interval $[0, \bar{\alpha}]$ using the Weirstrass theorem.

Lemma 12. *Under the assumptions of Theorem 3, exists a $\bar{\alpha} > 0 : \forall \alpha \in (0, \bar{\alpha}]$:*

$$\textbf{\textit{C2}} \qquad \exists k' < \infty : \forall (t, \sigma), [Y_{t+1} - Y_t | \, Y_t, \mathcal{F}_{t-1}] \le Z = \alpha k'$$
$$\textit{and} \;\; E(e^{\lambda Z}) \le (e^{\lambda \alpha k'}) = D < \infty \qquad \textit{with } \lambda = \frac{1}{\alpha}.$$

Proof. Lemma 10 shows that $\max_{\sigma,t} [Y_{t+1} - Y_t | \, Y_t, \mathcal{F}_{t-1}] = |O(1)|\alpha$. Thus,

$$\exists k' < \infty : \forall (t, \sigma), [Y_{t+1} - Y_t | \, Y_t, \mathcal{F}_{t-1}] \le Z = \alpha k'$$
$$\text{and} \;\; E(e^{\lambda Z}) \le (e^{\lambda \alpha k'}) = D < \infty \qquad \text{with } \lambda = \frac{1}{\alpha}.$$

□

*Now we use Hajek (1982) Lemma 2.1 to translate conditions **C1, C2** to **D1, D2***
***Lemma 2.1, Hajek (1982)**: Choose constants η, ρ:*

$$0 < \eta \le \lambda; \; \eta < \frac{\epsilon_0}{c} : \; \rho = 1 - \epsilon_0 \eta + c\eta^2$$

with $c = \frac{Ee^{\lambda Z} - (1 - \lambda E(Z))}{\lambda^2}$; and consider the following conditions:

$$\textbf{\textit{D1}} : \; E\left[e^{\eta(Y_{t+1} - Y_t)} \middle| \, Y_t > a, \mathcal{F}_{t-1} \right] < \rho,$$
$$\textbf{\textit{D2}} : \; E\left[e^{\eta(Y_{t+1} - a)} \middle| \, Y_t \le a, \mathcal{F}_{t-1} \right] < D;$$

*then **C1, C2** and $\rho < 1 \Rightarrow$ **D1** and **C2** $\Rightarrow$ **D2**.*

Lemma 13. *Under the assumptions of Theorem 3, exists a $\bar{\alpha} > 0 : \forall \alpha \in (0, \bar{\alpha}]$, **D1, D2** hold for this parameter choice.*

$$\lambda = \frac{1}{\bar{\alpha}}, a = \bar{Y} + \bar{\alpha}^{.5}, \epsilon_0 = |O(\bar{\alpha}^{1.5})|, \eta = \bar{\alpha}^{-.4}, \rho = 1 - |O(\bar{\alpha}^{1.1})| + |O(\bar{\alpha}^{1.2})| < 1.$$

Proof. Because $\alpha \in (0, \bar{\alpha}]$, Lemmas 11 and 12 guaranty that this parameter choice satisfies ***C1, C2***. Moreover, $\lambda = \frac{1}{\bar{\alpha}}$ and Lemma 12$\Rightarrow c = O(\bar{\alpha}^2)$, so that $\epsilon_0 = |O(\bar{\alpha}^{1.5})|, \eta = \bar{\alpha}^{-.4} \Rightarrow \rho = 1 - |O(\bar{\alpha}^{1.1})| + |O(\bar{\alpha}^{1.2})| < 1$ for $\bar{\alpha}$ small. □

Lemma 14. *Under the assumptions of Theorem 3 $\forall \epsilon, \bar{\epsilon}$, $\exists K \in (0, \infty), \delta \in (0, 1)$ and $\bar{\alpha} \in (0, 1)$ such that $\forall \alpha \in (0, \bar{\alpha}]$,*

$$P\left[\frac{1}{t} \sum_{\tau=1}^{t} I_{d(P || p_\tau^{RN}) < \epsilon} \le 1 - \bar{\epsilon} \right] \le K \delta^t$$

Proof. We apply Hajek (1982) Theorem 3.1 putting in Eq. 16 the parameters
$\lambda = \frac{1}{\bar{\alpha}}, a = \bar{Y} + \bar{\alpha}^{.5}, \epsilon_0 = |O(\bar{\alpha}^{1.5})|, \eta = \bar{\alpha}^{-.4}, \rho = 1 - |O(\bar{\alpha}^{1.1})| + |O(\bar{\alpha}^{1.2})|$ and $b = \bar{Y} + \bar{\alpha}^{.2}$:

$$P\left[\frac{1}{t} \sum_{\tau=1}^{t} I_{Y_\tau < \bar{Y} + \bar{\alpha}^{.2}} \le \left(1 - \frac{1}{|O(\bar{\alpha}^{1.1})| + |O(\bar{\alpha}^{1.2})|} D e^{-\bar{\alpha}^{-.2}} e^{\bar{\alpha}^{.1}} \right) (1 - \epsilon') \right] \le K \delta^t;$$

so that for every $\epsilon, \bar{\epsilon}$, $\exists K \in (0,\infty), \delta \in (0,1)$ and $\bar{\alpha}$ such that $\forall \alpha \in (0,\bar{\alpha}]$:

$$P\left[\frac{1}{t}\sum_{\tau=1}^{t} I_{Y_\tau < \bar{Y}+\epsilon} \leq 1-\bar{\epsilon}\right] \leq K\delta^t$$

Repeating the same steps for the process $-(Y_t)_{t=1}^{\infty}$, we obtain the opposite bound.
For every $\epsilon, \bar{\epsilon}$, $\exists K \in (0,\infty), \delta \in (0,1)$ and $\bar{\alpha}$ such that $\forall \alpha \in (0,\bar{\alpha}]$:

$$P\left[\frac{1}{t}\sum_{\tau=1}^{t} I_{Y_\tau > \bar{Y}-\epsilon} \leq 1-\bar{\epsilon}\right] \leq K\delta^t$$

Therefore, $\forall \epsilon, \bar{\epsilon}$, $\exists K \in (0,\infty), \delta \in (0,1)$ and $\bar{\alpha} \in (0,1)$ such that $\forall \alpha \in (0,\bar{\alpha}]$,

$$P\left[\frac{1}{t}\sum_{\tau=1}^{t} I_{|Y_\tau - \bar{Y}|<\epsilon} \leq 1-\bar{\epsilon}\right] \leq K\delta^t$$

The following continuity argument proves the Lemma.
By continuity of $p^{RN}(Y), \forall \bar{\epsilon}' > 0, \exists \bar{\epsilon}'' > 0 : |Y_\tau - \bar{Y}| < \bar{\epsilon}'' \Rightarrow |p^{RN}(Y) - P| < \bar{\epsilon}'$.
By continuity of $d(P||p^{RN}), \forall \bar{\epsilon} > 0, \exists \bar{\epsilon}' : |p_t^{RN}(Y) - P| < \bar{\epsilon}' \Rightarrow d_t(P||p^{RN}) < \bar{\epsilon}$.
Thus $\forall \bar{\epsilon} > 0, \exists \bar{\epsilon}'' : |Y_t - \bar{Y}| < \bar{\epsilon}'' \Rightarrow d_t(P||p^{RN}) < \bar{\epsilon}$. □

Proof of Theorem 3

Proof. By Lemma 14, $\forall \epsilon, \bar{\epsilon}$, $\exists K \in (0,\infty), \delta \in (0,1)$ and $\bar{\alpha} \in (0,1)$ such that $\forall \alpha \in (0,\bar{\alpha}]$,

$$P\left[\frac{1}{t}\sum_{\tau=1}^{t} I_{d(P||p_\tau^{RN})<\epsilon} \leq 1-\bar{\epsilon}\right] \leq K\delta^t. \tag{18}$$

What remains to show is that Eq. 18 $\Rightarrow \forall \epsilon > 0, \exists \bar{\alpha} : \alpha \in (0,\bar{\alpha}] \Rightarrow P\left\{\bar{d}(P||p^{RN}) < \epsilon\right\} = 1$.
For $\epsilon > 0$, let $F_t := \left\{\frac{1}{t}\sum_{\tau=1}^{t} d(P||p_\tau^{RN} < \epsilon\right\}$. We apply Borel-Cantelli Lemma to show that for every $\epsilon > 0$, $\bar{\alpha}$ can be chosen small enough to guaranty that the probability that F_t^C occurs infinitely often is zero, which it implies that $\forall \epsilon > 0, P\{\lim_{t\to\infty} F_t\} = P\left\{\bar{d}(P||p^{RN}) < \epsilon\right\} = 1$.

First, Lemma 14 implies that $\forall \bar{\epsilon} > 0$, $\bar{\alpha}$ can be chosen small enough to guaranty that

$$\begin{aligned}
P\left\{F_t < 2\bar{\epsilon}\right\} =& P\left\{\frac{1}{t}\sum_{\tau=1}^{t} d(P||p_\tau^{RN}\Big|_{\{d(P||p_\tau^{RN}<\bar{\epsilon}\}} + \frac{1}{t}\sum_{\tau=1}^{t} d(P||p_\tau^{RN}\Big|_{\{d(P||p_\tau^{RN}\geq\bar{\epsilon}\}} < 2\bar{\epsilon}\right\} \\
\geq& P\left\{\bar{\epsilon}\frac{1}{t}\sum_{\tau=1}^{t} d(P||p_\tau^{RN} < \bar{\epsilon}\} + \max(d(P||p_\tau^{RN})\frac{1}{t}\sum_{\tau=1}^{t} d(P||p_\tau^{RN} \geq \bar{\epsilon}\} < 2\bar{\epsilon}\right\} \\
\geq& P\left\{\max d(P||p_\tau^{RN}\frac{1}{t}\sum_{\tau=1}^{t} I_{\{d(P||p_\tau^{RN}\geq\bar{\epsilon}\}} < 2\bar{\epsilon}-\bar{\epsilon}\right\} \qquad ;\text{because } \frac{\sum_{\tau=1}^{t} I_{\{d(P||p_\tau^{RN}<\bar{\epsilon}\}}}{t} \leq 1 \\
=& P\left\{\frac{\sum_{\tau=1}^{t} I_{\{d(P||p_\tau^{RN}\geq\bar{\epsilon}\}}}{t} < \frac{\bar{\epsilon}}{\max(d(P||p_\tau^{RN})}\right\} \\
\geq& 1 - K\delta^t.
\end{aligned}$$

Next, we apply Borel-Cantelli Lemma to show that for all $\epsilon = 2\bar{\epsilon} > 0$ exists $\bar{\alpha} : \forall \alpha \in (0, \bar{\alpha})$:

$$\begin{aligned}
&P\left\{F_t^C\right\} = 1 - P\left\{\frac{1}{t}\sum_{\tau=1}^{t} |Y_\tau - \bar{Y}| < \epsilon\right\} = K\delta^t \\
&\Rightarrow \lim_{t\to\infty}\sum_{\tau=1}^{t} P\left\{F_t^C\right\} \leq \lim_{t\to\infty}\sum_{\tau=1}^{t} K\delta^t < \infty \\
&\Rightarrow^{\text{by Borel-Cantelli Lemma}} P\left\{\limsup_{t\to\infty} F_t^C\right\} = 0
\end{aligned}$$

□

Lemma 15. *On the solutions $p^{RN}(Y)$ of $F(Y, p^{RN}, \alpha) = 0$ it holds*

$$\frac{\partial}{\partial\alpha}\left(\frac{\partial F(Y,p^{RN},\alpha)}{\partial Y}\right)\Bigg|_{\alpha=0} = -\frac{1}{\gamma}(\pi^r - \pi^l)\phi(Y)\left(1-\phi(Y)\right) \quad \textit{and} \quad \frac{\partial}{\partial\alpha}\left(\frac{\partial F(Y,p^{RN},\alpha)}{\partial p^{RN}}\right)\Bigg|_{\alpha=0} = 1.$$

Proof. As previously shown,

$$\begin{aligned}
\frac{\partial F(Y,p^{RN},\alpha)}{\partial Y} =& -(1-p^{RN})\gamma\left((p^r)^{\frac{1}{\gamma}}\phi(Y) + (p^l)^{\frac{1}{\gamma}}(1-\phi(Y))\right)^{\gamma-1}\left((p^r)^{\frac{1}{\gamma}} - (p^l)^{\frac{1}{\gamma}}\right)\phi'(Y) \\
&+ p^{RN}\gamma\left((1-p^r)^{\frac{1}{\gamma}}\phi(Y) + (1-p^l)^{\frac{1}{\gamma}}(1-\phi(Y))\right)^{\gamma-1}\left((1-p^r)^{\frac{1}{\gamma}} - (1-p^l)^{\frac{1}{\gamma}}\right)\phi'(Y),
\end{aligned}$$

where $\phi'(Y) = \frac{1}{\gamma}\phi(Y)\left(1-\phi(Y)\right)$. For $i = r, l$,

$$p^i = \alpha\pi^i + (1-\alpha)p^{RN} \Rightarrow \begin{array}{rl} \frac{\partial (p^i)^{\frac{1}{\gamma}}}{\partial\alpha}\Big|_{\alpha=0} = & (p^i)^{\frac{1}{\gamma}-1}\frac{\pi^i - p^{RN}}{\gamma}\Big|_{\alpha=0} = (p^{RN})^{\frac{1}{\gamma}-1}\frac{\pi^i - p^{RN}}{\gamma} \\ \frac{\partial (1-p^i)^{\frac{1}{\gamma}}}{\partial\alpha}\Big|_{\alpha=0} = & (1-p^i)^{\frac{1}{\gamma}-1}\frac{\pi^i - p^{RN}}{\gamma}\Big|_{\alpha=0} = (1-p^{RN})^{\frac{1}{\gamma}-1}\frac{p^{RN} - \pi^i}{\gamma} \end{array} \tag{19}$$

Using the above to evaluate $\frac{\partial}{\partial\alpha}\frac{\partial F(Y,p^{RN},\alpha)}{\partial Y}$ in $\alpha = 0$, leads to

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\frac{\partial F(Y,p^{RN},\alpha)}{\partial Y}\bigg|_{\alpha=0} &= -(1-p^{RN})\gamma\left((p^{RN})^{\frac{1}{\gamma}}\phi(Y)+(p^{RN})^{\frac{1}{\gamma}}(1-\phi(Y))\right)^{\gamma-1}(p^{RN})^{\frac{1}{\gamma}-1}\frac{\pi^r-\pi^l}{\gamma}\phi'(Y)\\
&\quad + p^{RN}\gamma\left((1-p^{RN})^{\frac{1}{\gamma}}\phi(Y)+(1-p^{RN})^{\frac{1}{\gamma}}(1-\phi(Y))\right)^{\gamma-1}(1-p^{RN})^{\frac{1}{\gamma}-1}\frac{\pi^l-\pi^r}{\gamma}\phi'(Y)\\
&= -(1-p^{RN})(\pi^r-\pi^l)\phi'(Y)+p^{RN}(\pi^l-\pi^r)\gamma\phi'(Y)\\
&= -(\pi^r-\pi^l)\phi'(Y)\\
&= -\frac{1}{\gamma}(\pi^r-\pi^l)\phi(Y)\,(1-\phi(Y))
\end{aligned}$$

Turning to $\Delta(\alpha)=\frac{\partial}{\partial\alpha}\left(\frac{\partial F(Y,p^{RN},\alpha)}{\partial p^{RN}}\right)$. As previously shown,

$$\begin{aligned}
\frac{\partial F(Y,p^{RN},\alpha)}{\partial p^{RN}} &= \left((p^r)^{\frac{1}{\gamma}}\phi+(p^l)^{\frac{1}{\gamma}}(1-\phi)\right)^{\gamma}\\
&\quad + \left((1-p^r)^{\frac{1}{\gamma}}\phi+(1-p^l)^{\frac{1}{\gamma}}(1-\phi)\right)^{\gamma}\\
&\quad - (1-p^{RN})(1-\alpha)\left((p^r)^{\frac{1}{\gamma}}\phi+(p^l)^{\frac{1}{\gamma}}(1-\phi)\right)^{\gamma-1}\left((p^r)^{\frac{1}{\gamma}-1}\phi+(p^l)^{\frac{1}{\gamma}-1}(1-\phi)\right)\\
&\quad - p^{RN}(1-\alpha)\left((1-p^r)^{\frac{1}{\gamma}}\phi+(1-p^l)^{\frac{1}{\gamma}}(1-\phi)\right)^{\gamma-1}\left((1-p^r)^{\frac{1}{\gamma}-1}\phi+(1-p^l)^{\frac{1}{\gamma}-1}(1-\phi)\right).
\end{aligned}$$

Taking the derivate w.r.t. α and using (19) we get

$$\begin{aligned}
\Delta|_{\alpha=0} &= \gamma\left((p^{RN})^{\frac{1}{\gamma}}\right)^{\gamma-1}(p^{RN})^{\frac{1}{\gamma}-1}\left(\frac{\pi^r-p^{RN}}{\gamma}\phi(Y)+\frac{\pi^l-p^{RN}}{\gamma}(1-\phi(Y))\right)\\
&\quad + \gamma\left((1-p^{RN})^{\frac{1}{\gamma}}\right)^{\gamma-1}(1-p^{RN})^{\frac{1}{\gamma}-1}\left(\frac{p^{RN}-\pi^r}{\gamma}\phi(Y)+\frac{p^{RN}-\pi^l}{\gamma}(1-\phi(Y))\right)\\
&\quad + (1-p^{RN})\left((p^{RN})^{\frac{1}{\gamma}}\right)^{\gamma-1}\left((p^{RN})^{\frac{1}{\gamma}-1}\right)\\
&\quad - (1-p^{RN})(\gamma-1)\left((p^{RN})^{\frac{1}{\gamma}}\right)^{\gamma-2}\left((p^{RN})^{\frac{1}{\gamma}-1}\right)\left(\frac{\pi^r-p^{RN}}{\gamma}\phi(Y)+\frac{\pi^l-p^{RN}}{\gamma}(1-\phi(Y))\right)(p^{RN})^{\frac{1}{\gamma}-1}\\
&\quad - (1-p^{RN})\left((p^{RN})^{\frac{1}{\gamma}}\right)^{\gamma-1}\left((p^{RN})^{\frac{1}{\gamma}-2}\right)(1-\gamma)\left(\frac{\pi^r-p^{RN}}{\gamma}\phi(Y)+\frac{\pi^l-p^{RN}}{\gamma}(1-\phi(Y))\right)\\
&\quad + p^{RN}\left((1-p^{RN})^{\frac{1}{\gamma}}\right)^{\gamma-1}\left((1-p^{RN})^{\frac{1}{\gamma}-1}\right).\\
&\quad - p^{RN}(\gamma-1)\left((1-p^{RN})^{\frac{1}{\gamma}}\right)^{\gamma-2}\left((1-p^{RN})^{\frac{1}{\gamma}-1}\right)\left(\frac{p^{RN}-\pi^r}{\gamma}\phi(Y)+\frac{p^{RN}-\pi^l}{\gamma}(1-\phi(Y))\right)(1-p^{RN})^{\frac{1}{\gamma}-1}\\
&\quad - p^{RN}\left((1-p^{RN})^{\frac{1}{\gamma}}\right)^{\gamma-1}\left((1-p^{RN})^{\frac{1}{\gamma}-2}\right)(1-\gamma)\left(\frac{p^{RN}-\pi^r}{\gamma}\phi(Y)+\frac{p^{RN}-\pi^l}{\gamma}(1-\phi(Y))\right).
\end{aligned}$$

Finally, evaluating the powers of p^{RN} and simplifying gives

$$
\begin{aligned}
\Delta|_{\alpha=0} &= \gamma\left(\frac{\pi^r - p^{RN}}{\gamma}\phi(Y) + \frac{\pi^l - p^{RN}}{\gamma}(1-\phi(Y))\right) \\
&+ \gamma\left(\frac{p^{RN} - \pi^r}{\gamma}\phi(Y) + \frac{p^{RN} - \pi^l}{\gamma}(1-\phi(Y))\right) \\
&+ (1 - p^{RN}) \\
&- \frac{1-p^{RN}}{p^{RN}}(\gamma - 1)\left(\frac{\pi^r - p^{RN}}{\gamma}\phi(Y) + \frac{\pi^l - p^{RN}}{\gamma}(1-\phi(Y))\right) \\
&- \frac{1-p^{RN}}{p^{RN}}(1 - \gamma)\left(\frac{\pi^r - p^{RN}}{\gamma}\phi(Y) + \frac{\pi^l - p^{RN}}{\gamma}(1-\phi(Y))\right) \\
&+ p^{RN} \\
&- \frac{p^{RN}}{1-p^{RN}}(\gamma - 1)\left(\frac{p^{RN} - \pi^r}{\gamma}\phi(Y) + \frac{p^{RN} - \pi^l}{\gamma}(1-\phi(Y))\right) \\
&- \frac{p^{RN}}{1-p^{RN}}(1 - \gamma)\left(\frac{p^{RN} - \pi^r}{\gamma}\phi(Y) + \frac{p^{RN} - \pi^l}{\gamma}(1-\phi(Y))\right) \\
&= 1\,.
\end{aligned}
$$

□

C Proof of competitive equilibrium existence

We define a *competitive equilibrium with consensus* as a $2I+2$-tuple of sequences of consumption allocations $(c_t^i(\sigma))_{t=0}^{\infty}$, beliefs $p^i(\sigma_t|)_{t=0}^{\infty}$, consensus beliefs $p^C(\sigma_t|)_{t=0}^{\infty}$ and prices $(q(\sigma^t))_{t=0}^{\infty}$, one for each $\sigma \in \Sigma$, such that

1. Each agent $i \in \mathcal{I}$ consumption solves the utility maximization given endogenous beliefs p^i and prices $(q(\sigma^t))_{(t,\sigma^t)}$

$$
\max_{(c_t^i(\sigma))_{t=0}^{\infty}} \mathrm{E}_{p^i}\left[\sum_{t=0}^{\infty} \beta^t u^i(c_t^i(\sigma))\right] \quad s.t. \quad \sum_{t\geq 0}\sum_{\sigma^t\in\Sigma^t} q(\sigma^t)\left(c_t^i(\sigma) - e_t^i(\sigma)\right) \leq 0. \tag{20}
$$

2. All good markets clear:

$$
\sum_{i\in\mathcal{I}} c_t^i(\sigma) = \sum_{i\in\mathcal{I}} e_t^i(\sigma) \quad \text{for all } (t,\sigma). \tag{21}
$$

3. Each agent $i \in \mathcal{I}$ beliefs p^i are as in Definition 4 for a given choice of consensus belief p^C and idiosyncratic belief π^i.
4. The consensus belief p^C is p^M as in Definition 7 or p^{RN} as in Definition 8 or p_γ^{RN} as in Definition 9.

The *competitive equilibrium with consensus* differs from the standard one in that agent beliefs are endogenously determined.

In what follows we prove that under **A1-A4 (A5)** there exists a *competitive equilibrium with consensus*. In the **first step**, we shall assign an initial consumption share distribution ϕ_0 and derive sequences of consumption, prices, individual beliefs, and consensus beliefs consistent with the First Order Conditions (FOC) of agents utility maximization problem, with market clearing, and with the definition of individual and consensus beliefs. This step is similar to the computation of a Pareto optimal allocation given a set of Pareto weights but, due to the endogeneity of beliefs, involves an additional fixed point argument for each iteration. The Brouwer fixed point theorem, together with the smoothness of our maps, guarantees the existence of such fixed point for each iteration. The details of this step are different for p^M and the other consensuses because of their different analytical form.

In the **second step**, we show that there exists an initial distribution of consumption shares such that each agent's budget constraint is satisfied. The main difference between this step and the standard proof of the existence of the competitive equilibrium with exogenous beliefs is that in our case the initial consumption-share distribution affects prices also via its effect on beliefs. This further complication does not change the typical argument. Even in this case, Brouwer's fixed point theorem guarantees the existence of a fixed point.

Remark Our proof ensures existence, not uniqueness. Multiplicity of equilibria is not problematic because our results hold in all the equilibria that exist.

Let us start from the system of FOCs. Having defined $\bar{c}_t^i(\sigma) = \frac{1}{u^i(c_t^i(\sigma))'}$, the system of agent i FOC and his budget constraint is

$$\begin{cases} \bar{c}_0^i = \frac{1}{\lambda^i}, \\ \bar{c}_t^i(\sigma) = \frac{\beta p^i(\sigma_t|)}{q(\sigma_t|)} \bar{c}_{t-1}^i(\sigma) \quad \text{for all} \quad (t,\sigma), \\ \sum\limits_{t\geq 0} \sum\limits_{\sigma^t \in \Sigma^t} q(\sigma^t) \left(c_t^i(\sigma) - e_t^i(\sigma) \right) = 0, \end{cases} \tag{22}$$

where λ^i is the multiplier associated with agent i's budget constraint.

First step - p^M is the consensus used by all $i \in \mathcal{I}$

By Lemma 5 for all (t, σ)

$$p_t^M = \sum_{i\in\mathcal{I}} \pi^i \frac{\alpha^i \bar{c}_{t-1}^i}{\sum_{j\in\mathcal{I}} \alpha^j \bar{c}_{t-1}^j}, \tag{23}$$

so that, using Definition 4,

$$p_t^i = (1-\alpha^i) \sum_{j\in\mathcal{I}} \pi^j \frac{\alpha^j \bar{c}_{t-1}^j}{\sum_{k\in\mathcal{I}} \alpha^k \bar{c}_{t-1}^k} + \alpha^i \pi^i. \tag{24}$$

Thus, for each given initial consumption distribution $(\phi_0^i)_{i=1}^I$ we can compute initial marginal utilities $(\bar{c}_0^i)_{i=1}^I$, consensus beliefs p_1^M, and individual beliefs $(p_1^i)_{i=1}^I$.

Having determined beliefs, we can proceed to compute equilibrium consumption in date $t = 1$ as usual. From the second equation of (22), the ratio of agent $i = 1$ to agent j FOC between $t = 0$ and $(t = 1, \sigma_1)$ gives

$$((u^j)')^{-1} \left(\frac{\bar{c}_0^1}{\bar{c}_0^j} \frac{p^1(\sigma_1|)}{p^j(\sigma_1|)} u^1(\phi_1^1(\sigma_1)) e_1(\sigma_1))' \right) = \phi_1^j(\sigma_1) e_1(\sigma_1).$$

Aggregating over agents we find

$$\sum_{j\in\mathcal{I}} ((u^j)')^{-1} \left(\frac{\bar{c}_0^1}{\bar{c}_0^j} \frac{p^1(\sigma_1|)}{p^j(\sigma_1|)} u^1(\phi_1^1(\sigma_1) e_1(\sigma_1))' \right) = e_1(\sigma_1). \tag{25}$$

Agent $i=1$ consumption share $\phi_1^1(\sigma_1)$ can be derived from the above. A solution $\phi_1^1(\sigma_1)$ of (25) always exists in $(0,1)$ because, by **A1, A3**, the l.h.s. is continuous in $\phi_1^1(\sigma_1)=x$, goes to $0<e_1(\sigma_1)$ for $x\to 0$, and is larger than $e_1(\sigma_1)$ in $x=1$.[14] Repeating the same argument for all the agents we find $(\phi_1^i(\sigma_1))_{i=1}^I$ and, repeating for all $\sigma_1\in\mathcal{S}$, we find $(\phi_1^i)_{i=1}^I$. Iterating these steps for all t and all σ^t gives the stream of individual consumptions, individual beliefs, and consensus beliefs for each choice of path $\sigma\in\Sigma$ and for each choice of $(\phi_0^i)_{i=1}^I$.

First step - p^{RN} is the consensus used by all $i\in\mathcal{I}$

By Lemma 2, in $t=0$ the consensus p^{RN} in $(t=1,\sigma_1)$ defined in (8) can be written as

$$p^{RN}(\sigma_1|)=\frac{\sum_{i\in\mathcal{I}}p^i(\sigma_1|)\frac{\bar{c}_0^i}{\sum_{j\in\mathcal{I}}\bar{c}_1^j(\sigma_1)}}{\sum_{\tilde{\sigma}_1\in\mathcal{S}}\sum_{i\in\mathcal{I}}p^i(\tilde{\sigma}_1|)\frac{\bar{c}_0^i}{\sum_{j\in\mathcal{I}}\bar{c}_1^j(\tilde{\sigma}_1)}}\quad\text{for all }\sigma_1\in\mathcal{S},\tag{26}$$

where $p^{RN}(\sigma_1|)$ is also on the r.h.s. in each individual belief $p^i(\sigma_1|)$ for all $i\in\mathcal{I}$. The above for all σ_1 together with (25) for all σ_1 define a map from Δ^S to Δ^S as follows. For each given $\rho\in\Delta^S$, (25) for all σ_1 and all i allows to compute $(c_1^i(\rho))_{i=1}^I$ when individual beliefs $(p_1^i)_{i=1}^I$ are built using ρ as the consensus, $(p_1^i(\rho))_{i=1}^I$. Then, having consumption $(c_1^i(\rho))_{i=1}^I$ and beliefs $(p_1^i(\rho))_{i=1}^I$, (26) gives the consensus beliefs $p_1^{RN}(\rho)$. We have an equilibrium consensus when $p_1^{RN}(\rho)=\rho$.

The existence of the latter follows from Brouwer's fixed point theorem because the map that we have built composing (25) for all i and σ_1 and (26) for all σ_1 goes from the simplex Δ^S to the simplex Δ^S and is continuous. To prove continuity note that, given ρ, for each i and σ_1 (25) defines a function $F^i(\rho,\phi_1^i(\sigma_1))$ such that the solution of

$$F_i(\rho,\phi_1^i(\sigma_1))=0\quad\text{determines}\quad c_1^i(\sigma_1)(\rho)=e_1(\sigma_1)\phi_1^i(\sigma_1)\,.$$

Continuity of $c_1^i(\sigma_1)(\rho)$ in ρ follows from the Implicit Function Theorem because, by $\mathbf{A1},\mathbf{A3}$, $F_i(\rho,x)$ is the sum of compositions of monotone functions, and thus monotone, implying that the derivative $\partial F_i/\partial x$ is different from zero in the solution $\phi_1^i(\sigma_1)$ of $F_i(\rho,x)=0$. Continuity of the composed map follows from continuity of $c_1^i(\sigma_1)(\rho)$ for all i and σ_1 and from continuity of (26).

Having found the date $t=0$ consensus beliefs p_1^{RN}, the corresponding date $t=1$ consumption distribution and individual beliefs are $(c_1^i(p_1^{RN}))_{i=1}^I$ and $(p_1^i(p_1^{RN}))_{i=1}^I$, respectively.

Iterating these steps for all t and all σ^t gives a sequence of consumptions, individual and consensus beliefs as a function of the initial consumption distribution ϕ_0.

First step - p_γ^{RN} is the consensus used by all $i\in\mathcal{I}$

Note that when the consensus beliefs is p_γ^{RN} defined as in 9, this first step of the proof is the same provided that map (26) is replaced by the corresponding expression of p_γ^{RN} as a function of equilibrium consumption derived in Lemma 3.

[14]For the latter note that

$$\sum_{j\in\mathcal{I}}((u^j)')^{-1}\left(\frac{\bar{c}_0^1}{\bar{c}_0^j}\frac{p^1(\sigma_1|)}{p^j(\sigma_1|)}u^1(\phi_1^1(\sigma_1)e_1(\sigma_1))'\right)\Bigg|_{\phi_1^1(\sigma_1)=1}=\sum_{j\neq 1}((u^j)')^{-1}\left(\frac{\bar{c}_0^1}{\bar{c}_0^j}\frac{p^1(\sigma_1|)}{p^j(\sigma_1|)}u^1(e_1(\sigma_1))'\right)+e_1(\sigma_1)$$

.

First step - different agents use different consensuses

The computation of streams of consumption, individual beliefs, and consensus beliefs given an initial consumption distribution ϕ_0 can be performed also when different agents use different consensuses. We consider two cases: i) agents use either p^M or p^{RN}; ii) agents use either p^M or p^{RN}_γ.[15]

When agents use either p^M or p^{RN} the proof proceeds similarly to when all agents use only p^{RN}. In $t = 0$, given a candidate consensus beliefs $\rho \in \Delta^S$, initial individual beliefs of those mixing with p^{RN} are computed directly from ρ while individual beliefs of those mixing with p^M are computed using (24) with $\pi^j = p^j$ if j chooses p^{RN} as consensus. Having all agents individual beliefs for a given ρ, the combination of (25) and (26) for all $s \in \mathcal{S}$ determines the fixed point ρ such that $p^{RN}_1(\rho) = \rho$. From here we proceed as above.

The case when agents use either p^M or p^{RM}_γ proceeds along the same way provided that the map (26) is replaced by the corresponding expression of p^{RN}_γ as a function of equilibrium consumption as in Lemma 3.

Second step

With the first step we have found individual consumption and beliefs for each given consumption distribution ϕ_0. Using the FOC, to such consumption streams there corresponds a sequence of state- contingent prices $(q(\sigma^t))_{(t,\sigma^t)}$. We have an equilibrium when ϕ_0 is chosen such that all agents budget constraints, third equation in (22), are satisfied.

More formally, define

$$\begin{aligned} f_i(\phi_0) &= \sum_{t\geq 0}\sum_{\sigma^t\in\Sigma^t} q(\sigma^t)e^i_t(\sigma^t) - \sum_{t\geq 0}\sum_{\sigma^t\in\Sigma^t} q(\sigma^t)e_t(\sigma^t)\phi^i_t(\sigma^t), \\ \vdots \;\; &= \qquad\qquad \vdots \qquad\qquad = \qquad\qquad \vdots \\ f_I(\phi_0) &= \sum_{t\geq 0}\sum_{\sigma^t\in\Sigma^t} q(\sigma^t)e^I_t(\sigma^t) - \sum_{t\geq 0}\sum_{\sigma^t\in\Sigma^t} q(\sigma^t)e_t(\sigma^t)\phi^I_t(\sigma^t). \end{aligned}$$

We have a competitive equilibrium with consensus if we can find $\phi \in \Delta^I$ such that $f(\phi) = 0$. The existence of (at least) one of these points follows from Brouwer's fixed point Theorem, as follows.

First note that each function is well defined and continuous. Well defined because the aggregate endowment is bounded (**A2**) and state prices go to zero as fast as β^t (FOC and **A1-A4**). Continuous because, as shown in the proof of the first step for p^{RN}, the equilibrium consumption that solves (25) for all i, t, and σ_t is continuous in its parameters (we have proved continuity with respect to ρ but the argument is the same for continuity in ϕ_0, monotonicity in the unknown consumption allows to use the Implicit Function Theorem).

Define the function $f^+ : \Delta^I \to [0,\infty)^I$ as

$$f^+_i(\phi) = \max\{f_i(\phi), 0\} \quad \text{for all } i \in \mathcal{I}.$$

Denote

$$\alpha(\phi) = 1 + \sum_{i\in\mathcal{I}} f^+_i(\phi)\,.$$

[15]In each case the definition of *competitive equilibrium with consensus* should be changed accordingly. In 3. each agent should be allowed to use the consensus he chooses. In 4. both consensuses, p^M or p^{RN} in i) and p^M or p^{RN}_γ in ii), should be determined in equilibrium.

By construction $\alpha(\phi) \geq 1$ for all $\phi \in \Delta^I$. Define the function $F : \Delta^I \to \Delta^I$ as

$$F(\phi) = \frac{\phi + f^+(\phi)}{\alpha(\phi)} .$$

Continuity of f_i for all $i \in \mathfrak{I}$ imply that the function F is continuous on the compact convex set Δ^I and thus has a fixed point $\bar{\phi}$ by the Brouwer Fixed Point Theorem. Showing that $f(\bar{\phi}) = 0$ ends the proof.

$F(\bar{\phi}) = \bar{\phi}$ implies

$$\frac{\bar{\phi} + f^+(\bar{\phi})}{1 + \sum_{i \in \mathfrak{I}} f_i^+(\bar{\phi})} = \bar{\phi} \quad \Rightarrow \quad f^+(\bar{\phi}) = \bar{\phi} \left(\sum_{i \in \mathfrak{I}} f_i^+(\bar{\phi}) \right) . \tag{27}$$

Assume first that $\sum\limits_{i \in \mathfrak{I}} f_i^+(\bar{\phi}) > 0$. If $\bar{\phi}_i = 0$, then, by construction, the budget constraint does not hold for i and $f_i(\bar{\phi}) > 0$, so that $f_i^+(\bar{\phi}) > 0$ too, leading to a contradiction with (27). Then it must be $\bar{\phi}_i > 0$ for all i, implying $f_i^+(\bar{\phi}) > 0$ for all i, by (27), and leading to a contradiction with $\sum\limits_{i \in \mathfrak{I}} f_i(\phi) = 0$ for all ϕ (Walras Law). It follows that $\sum\limits_{i \in \mathfrak{I}} f_i^+(\bar{\phi}) = 0$ and thus, being the sum of non-negative functions, $f_i^+(\bar{\phi}) = 0$ for all i, implying $f_i(\bar{\phi}) \leq 0$ for all i. The latter together with $\sum\limits_{i \in \mathfrak{I}} f_i(\bar{\phi}) = 0$ (Walras Law) implies $f_i(\bar{\phi}) = 0$ for all $i \in \mathfrak{I}$.

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