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Utility Functions

Varian, H. 2010. *Intermediate Microeconomics*, W.W. Norton.

Representing preferences with a single function

The goal of this lecture is to develop our study of consumer preferences a step further.

We want to find a simple mathematical representation that takes preferences over bundles and assigns each bundle a single number, where higher numbers represent more preferred bundles (higher **indifference curves**).

We call this representation of preferences a **utility function**.

Formally, consider a utility function $u(X)$. This function will assign numbers to bundles such that:

$$(x_1, x_2) \succ (y_1, y_2) \text{ if and only if } u(x_1, x_2) > u(y_1, y_2)$$

Properties of the utility function

Note that the only important feature of the numbers a utility function assigns to bundles is that it preserves the *ranks*.

Suppose we have three bundles X, Y, and Z, where
$X > Y > Z$.

All of the following utility functions represent the same preferences:

$$u_1(X) = 5 \qquad u_1(Y) = 3 \qquad u_1(Z) = 1$$

$$u_2(X) = 1000 \qquad u_2(Y) = -10 \qquad u_2(Z) = -20$$

$$u_3(X) = 0.01 \qquad u_3(Y) = 0.001 \qquad u_3(Z) = 0.0001$$

Properties of the utility function

Since only the ranking of the bundles matters, if we can find one function to represent a particular set of preferences, we can find an infinite number of *other* functions that do the same thing.

Any transformation of a set of numbers that preserves the order of the numbers is called a **monotonic transformation**.

Typically, a transformation is achieved by applying some function $f(u(X))$ that transforms each utility number, u, into another number $f(u)$.

The transformation is monotonic if:

$$u_1 > u_2 \text{ implies } f(u_1) > f(u_2)$$

E.g. $f(u) = \frac{u}{2}$ or $f(u) = u + 5$

Monotonic transformations

Another way of saying that a transformation is (positive) monotonic is to say that it has a positive slope.

In other words, a function $f(u)$ will represent the same preferences as u as long as:

$$\frac{df(u)}{du} > 0$$

Monotonic transformations

Proposition: If $f(\cdot)$ is *any* monotonic transformation, and $u(x_1, x_2)$ is a utility function that represents a particular set of preferences, then $f(u(x_1, x_2))$ is another utility function that represent the same preferences.

Proof:

- If $u(x_1, x_2)$ represents a set of preferences, then by definition $u(x_1, x_2) > u(y_1, y_2)$ *iff* $(x_1, x_2) > (y_1, y_2)$
- And if $f(u)$ is a monotonic transformation, then (also by definition) $u(x_1, x_2) > u(y_1, y_2)$ *iff* $f(u(x_1, x_2)) > f(u(y_1, y_2))$
- Therefore, $f(u(x_1, x_2)) > f(u(y_1, y_2))$ *iff* $(x_1, x_2) > (y_1, y_2)$.

Thus, $f(u)$ represents the same preferences as u.

Utility and indifference curves

Question: What is the relationship between a utility function and indifference curves?

Answer: A utility function will assign the same number to *all bundles* that lie on a single indifference curve!

Higher indifference curves will be assigned a larger number.

And any monotonic transformation of a utility function just relabels indifference curves with different numbers while preserving the original ordering.

Utilitarian ethics

The concept of utility was originally conceived by the English philosopher Jeremy Bentham in the early 1800s as a way of evaluating how well a society was governed.

His vision of society was that it should produce "the greatest happiness for the greatest number".

His concept (and others' since him) of utility gave importance to the *magnitude* of utility, because his goal was to make *interpersonal comparisons* of utility.

Why might interpersonal utility comparisons (aka **cardinal utility**) be problematic?

Cardinal utility

Why might interpersonal utility comparisons (aka **cardinal utility**) be problematic?

But how do we know if person A likes ice cream twice as much as person B?

More importantly, how can *you* be sure that you like any bundle *twice as much as* any other?

This is where Bentham's philosophical project falters: we can't really compare happiness (or utility) across people. All we can say for sure is restricted to choices that one individual makes.

Utility theory in economics

Adam Smith, another famous philosopher (but Scottish, not English) was troubled by the following problem when he wrote his famous work "The Wealth of Nations" in 1776:

Why are diamonds so much more expensive than water? Water is necessary for survival, but diamonds are just pretty stones?

This is the famous Water-Diamond paradox, or the *paradox of value*. As we'll see, utility theory finally resolved this issue.
Keep it in mind as we go along.

Preview: The reason has to do with the notion of **marginal utility**; that is, the additional utility gained from another unit of either good. We'll come back to this point later!

What kinds of preferences lead to a utility function?

One kind of preferences that *cannot* be represented by a utility function is *intransitive* preferences.

Suppose someone has intransitive preferences so that
$X > Y > Z > X$.

Then, a utility function representing those preferences would have to output utility numbers such that
$u(X) > u(Y) > u(Z) > u(X)$. But this is impossible!

It turns out, that given as long as preferences satisfy our Axioms including the "Reasonable Restrictions" from Ch. 3, we can guarantee that a utility function exists to represent those preferences.

Axioms

A1 Completeness

For all (x_1, x_2) and (y_1, y_2) in X either $(x_1, x_2) < (y_1, y_2)$ or $(y_1, y_2) < (x_1, x_2)$, or both.

A2 Transitivity

For all (x_1, x_2), (y_1, y_2) and (z_1, z_2) in X if $(x_1, x_2) < (y_1, y_2)$ and $(y_1, y_2) < (z_1, z_2)$, then $(x_1, x_2) < (z_1, z_2)$.

A3 Reflexivity

For all (x_1, x_2) in X, $(x_1, x_2) < (x_1, x_2)$

The axioms continue

A4 Monotonicity

If $(x_1, x_2) \geq (y_1, y_2)$ and either $x_1 > y_1$ or $x_2 > y_2$, then $(x_1, x_2) > (y_1, y_2)$

A5 Strict Convexity

For some weight t between 0 and 1:

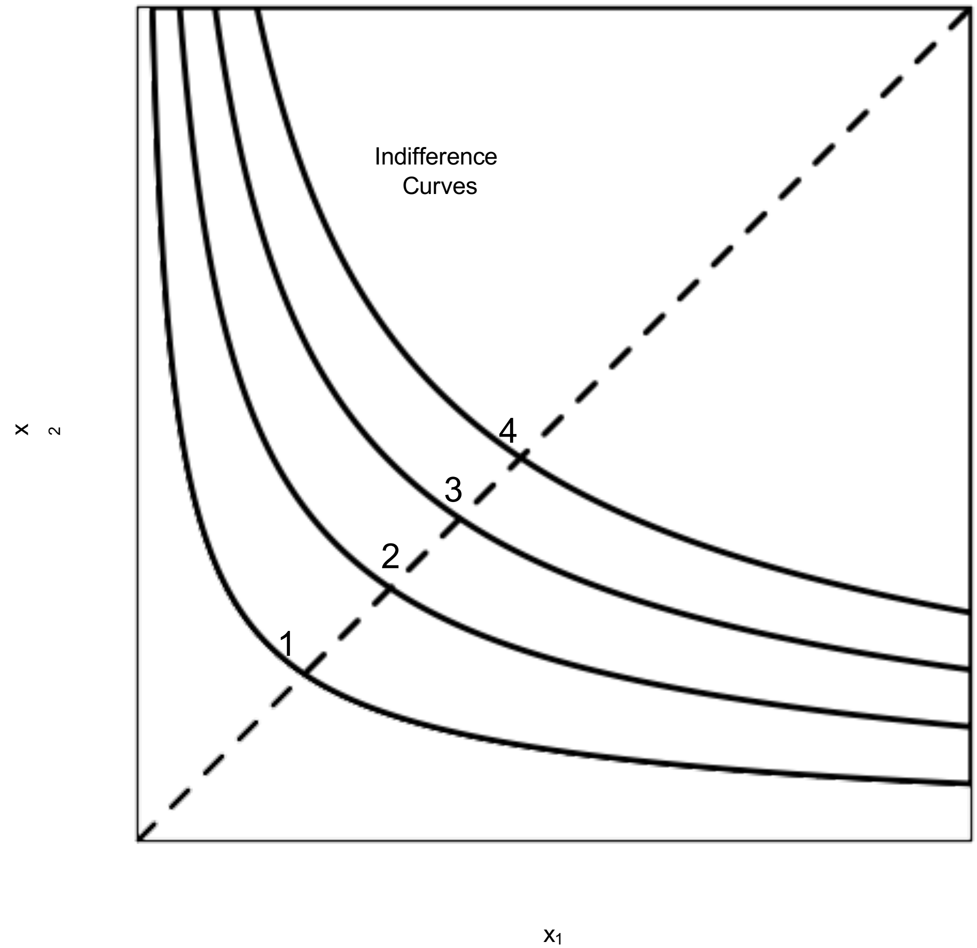
$(t\,x_1 + (1 - t)\,y_1, t\,x_2 + (1 - t)\,y_2) > (x_1, x_2) \sim (y_1, y_2)$

A6 Continuity

If $(x_1, x_2) > (y_1, y_1)$ then all points "close to" $(x_1, x_2) >$ all points "close to" (y_1, y_2).
No jumps in preferences.

Proposition: For any set of preferences that satisfy A1-A6, there exists a continuous utility function that represents those preferences.
We won't be going over the proof. This is just another reason that we imposed all those restrictions on preferences in Ch. 3.

Constructing a utility function



Just assign any "reasonable" preferences numeric labels in increasing order as you move away from the origin.

Graphic representation

If you have a utility function $u(x_1, x_2)$ and you want to create a graphical representation, it's easy to draw indifference curves.

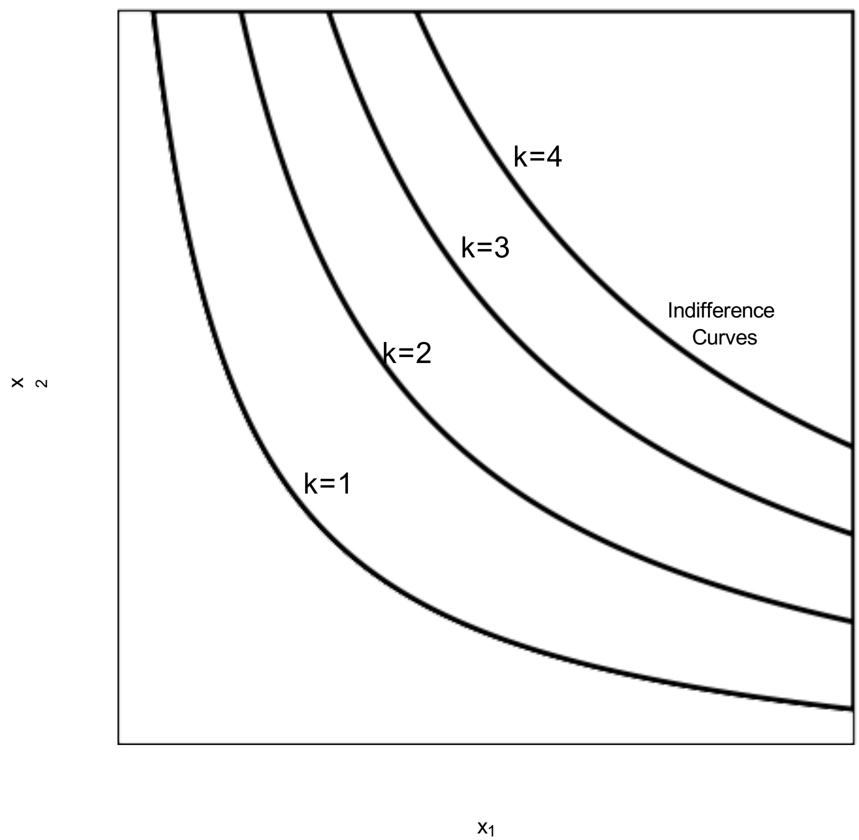
Just find all the bundles (x_1, x_2) such that $u(x_1, x_2)$ equals some constant, k.

Then do it again. This set of numbers, that is, all (x_1, x_2) such that $u(x_1, x_2) = k$ is called a **level set**.

Drawing indifference curves

Suppose we have a utility function $u(x_1, x_2) = x_1 x_2$

Each indifference curve is defined by the formula $x_2 = \frac{k}{x_1}$



A second example

Now suppose we have another utility function $\mathrm{v}\left(x_1, x_2\right) = x_1^2 x_2^2$

Then we know that:

$$v(x_1, x_2) = x_1^2 x_2^2 = (x_1 x_2)^2 = u(x_1, x_2)^2$$

Since we must have at least 0 of each good, $u(x_1, x_2)$ cannot be negative.

Thus, $v(\cdot)$ is just a monotonic transformation of $u(\cdot)$, and *both functions have the same shaped indifference curves*!

Only the labels change.

From Indifference Curves to a Utility Representation

Going this direction is more complicated. There are two methods.

1. Mathematically, we can try to find a function that is constant along the given indifference curves and assigns a higher value to higher indifference curves.

2. Or, given choice behavior of an individual, we can try to figure out what the consumer is trying to maximize, i.e. what combinations of goods describes the choice behavior.

Perfect substitutes

Remember the example comparing Coca-Cola and Pepsi (or red and blue peeps)?

Since all that matters is the total number of sodas (or peeps) we can use a very simple utility function to represent these preferences:

$$u(x_1, x_2) = x_1 + x_2$$

How can we be sure that this utility function represents preferences for **perfect substitutes**?

1. The function is constant along each indifference curve.
2. The function assigns a higher number to more preferred bundles.

Perfect substitutes

We can apply any monotonic transformation (squaring, adding 5), and it will represent the same preferences.

What happens if the rate of substitution is greater than or less than 1 to 1?
Suppose the consumer requires 2 units of x_2 to make up for losing 1 unit of x_1.

$$u(x_1, x_2) = 2x_1 + x_2$$

equivalently

$$u(x_1, x_2) = x_1 + \frac{\&}{(}x_2$$

What will be the slope of this indifference curve?
Answer: -2

Perfect substitutes

More generally, preferences for **perfect substitutes** can be represented by any utility function of the form:

$$u(x_1, x_2) = \mathrm{a}x_1 + \mathrm{b}x_2$$

Where a and b are positive and measure the "value" of goods 1 and 2 to the consumer.

The slope of the indifference curve is $-a/b$ and indicates the rate at which a is traded off against b.

Perfect complements

Peanut butter and jelly. In this case, all you care is how many sandwiches you can make.

This is based on the *minimum* of the units of peanut butter and the units of jelly that you have.

$$u(x_1, x_2) = \min\ \{x_1, x_2\}$$

Think about it this way: how many sandwiches can you make with 3 units of peanut butter and 3 units of jelly?

Answer: $\min\{3, 3\} = 3$

Now suppose you add one more unit of jelly, does that move you to a new indifference curve?

Answer: No, $\min\{3, 3\} = \min\{3, 4\} = 3$.

Perfect complements

What happens if you like to have twice as much peanut butter as jelly on each sandwich?

$$u(x_1, x_2) = \min\left\{\frac{1}{2}x_1, x_2\right\}$$

Think about this, if we have more than twice as much peanut butter than jelly, then we will have some peanut butter left over, so we'll still only get $\frac{1}{2}x$ sandwiches.

And remember, we can always impose monotonic transformations, so $\min\{x_1, 2x_2\}$ represents the same preferences.

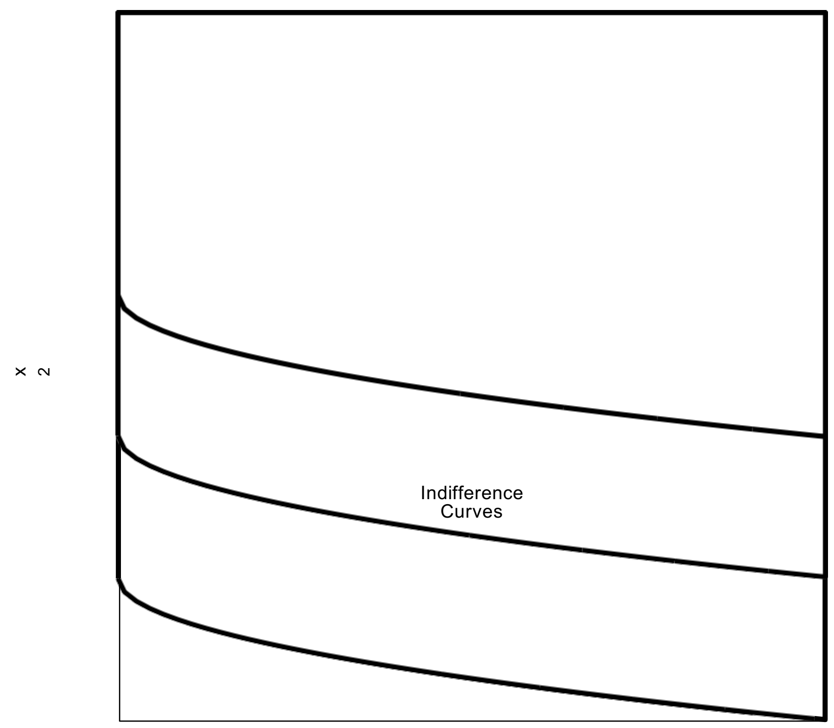
More generally, **perfect complements** can be represented by:

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

With $a, b > 0$ representing the proportions of each good.

Quasilinear preferences

Suppose someone has indifference curves that are vertical translations of one another.



We can write this form of utility function as $u(x_1, x_2) = v(x_1) + x_2$

Quasilinear preferences

To draw an indifference curve, set $u(x_1, x_2) = v(x_1) + x_2 = k$

Then, $x_2 = k - v(x_1)$

Thus, utility is linear in good 2, but it doesn't have to be linear in good 1.

And as a result, **perfect substitutes** are a special case of **quasilinear preferences**.

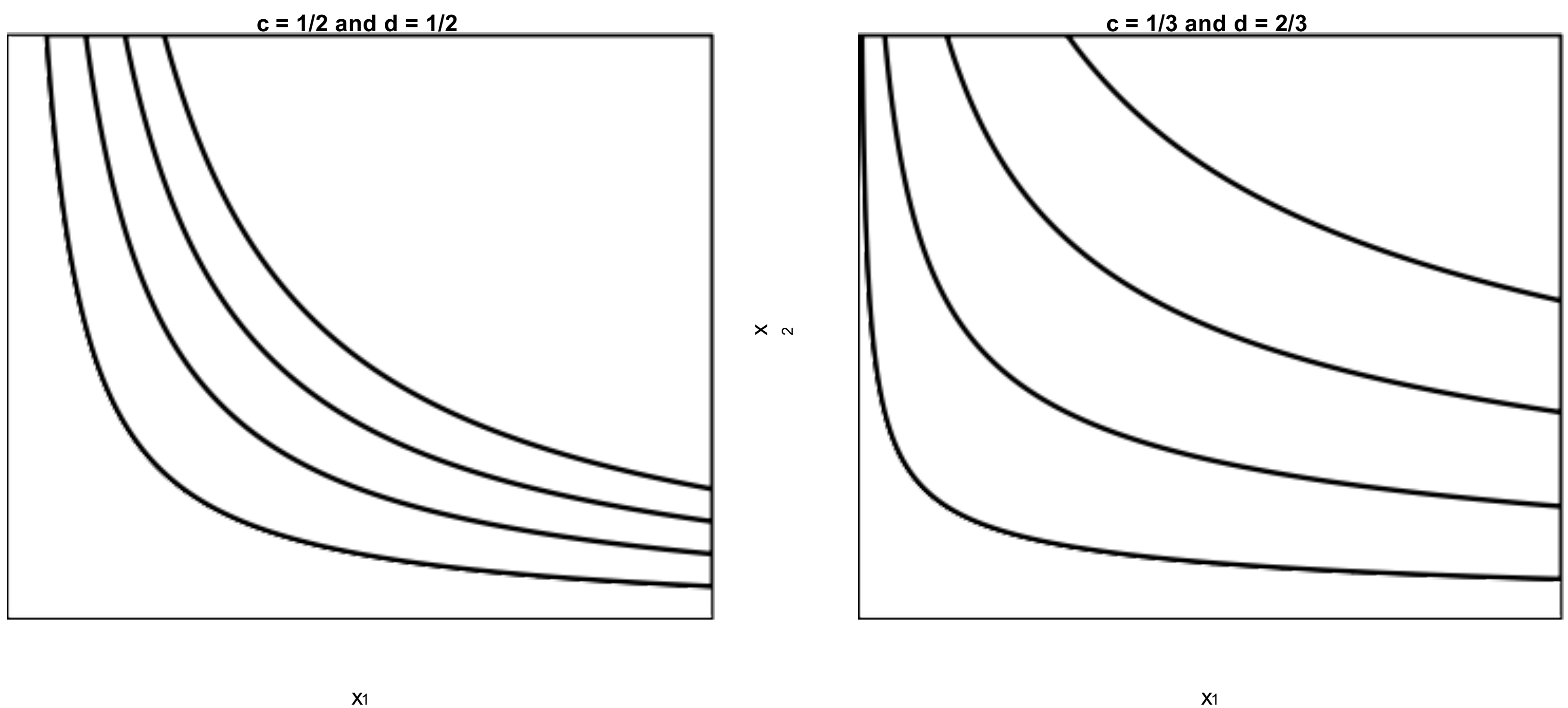
Economists like quasilinear preferences because they make a lot of the math simpler.

Examples: $u(x_1, x_2) = \ln(x_1) + x_2$

Cobb-Douglas preferences

This is another commonly used utility function, because it is also relatively easy to manipulate mathematically.

$$u(x_1, x_2) = x_1 x_2$$



Cobb-Douglas preferences

These are basically the simplest of the "well-behaved" preferences we discussed in chapter 3.

For that reason, we're going to be using this kind of utility function *a lot*.

As with other utility functions, we can represent the exact same preferences with any monotonic transformation of the Cobb-Douglas utility function.

Cobb-Douglas preferences

Here are two useful examples:

1. $v(u(x_1, x_2)) = \ln(x^c x^d) = c \ln x_1 + d \ln x_2$

   - These indifference curves will look exactly the same as the Cobb-Douglas indifference curves!

2. $v(u(x_1, x_2)) = \left(x_\$^c, x_2^d\right)^{\frac{1}{c\#d}} = x_\$^{\frac{c}{c\#d}} \; x_2^{\frac{d}{c\#d}}$

Then we define $a = \frac{c}{c\,'\,d}$ (so $\frac{c\,'\,d}{c\,'\,d} = 1$) and we can rewrite:

$v(u(x_1, x_2)) = x_\$^a \; x_2^{\$)\,a}$

This means we can always use a monotonic transformation to ensure that the exponents in a Cobb-Douglas utility function sum to 1!

Marginal utility

How does a consumer's utility as we increase the quantity of good 1 in her bundle?

The rate of change of the utility function as we add a little more of good 1 is called the **marginal utility** with respect to good 1.

$$MU_1 = \frac{\Delta U}{\Delta x_1} = \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1}$$

This shows how utility changes as you change the quantity of good 1, *holding the quantity of good 2 fixed.*

Marginal utility

Let's make this definition a little more precise.

When we talk about *marginal*, what we're actually referring to is a derivative.

In this case a *partial* derivative, of the utility function with respect to good 1.

$$MU_{\&} = \frac{\Delta U}{\Delta x_1} = \lim_{\Delta x_1 \to -} \frac{u(x_1 . \Delta x_1, x_2)/u(x_1, x_2)}{\Delta x_1} = \frac{0u(x_1, x_2)}{0x_1}$$

This shows how utility changes as you change the quantity of good 1, *holding the quantity of good 2 fixed*.

Marginal utility

We can do the same thing for the **marginal utility** w.r.t. good 2.

$$MU_(= \frac{\Delta U}{\Delta x_2} = \lim_{\Delta x_2 \rightarrow -} \frac{u(x_1.x_2,.\Delta x_2)/u(x_1,x_2)}{\Delta x_2} = \frac{0u(x_1,x_2)}{0x_2}$$

Here we're holding good 1 constant.

Taking monotonic transformations will change the magnitude of the marginal utility, which means that we can't calculate marginal utilities by looking at choice.

But remember, all we care about is the *ordering* of bundles implied by any utility function!

Marginal utility and the MRS

Remember, the MRS (**marginal rate of substitution**) measures the slope of an indifference curve at any particular bundle of goods.

It tells you the rate at which a consumer will trade off good 1 and good 2 *at current consumption levels*.

Note that this implies an easy method of computing the MRS.

Marginal utility and the MRS

Consider a change in consumption that keeps utility constant.

$$MU_{\&}\Delta x_{\&} + MU_{(}\Delta x_{(} = \Delta U = 0$$

Then, we can solve for the slope of the indifference curve.

$$MRS = \frac{\Delta x_2}{\Delta x_1} = -\frac{MU_1}{MU_2}$$

As before, the MRS is negative, because to stay on the same indifference curve, an increase in one good implies a decrease in the other!

Marginal utility and the MRS

With calculus:

$$du = \frac{\partial u(x_1, x_2)}{\partial x_1}\, dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2}\, dx_2$$

The first term measures the change in utility as we change x_1, and the second term measures the change in utility as we change x_2.

Remember, we're interested in changes such that utility is held constant (i.e. we remain on a single indifference curve, $du = 0$).

Marginal utility and the MRS

Now we can solve for $\frac{dx_2}{dx_1}$, which is the MRS.

This is just:

$$\frac{dx_2}{dx_1} = \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}$$

Which is directly analogous to our algebraic derivation a few slides back.

Monotonic Transformations

Now suppose we take a monotonic transformation,
$v(x_1, x_2) = f(u(x_1, x_2))$

Using the *chain rule*, we can calculate the MRS for $v(\cdot)$:

$$\text{MRS} = -\frac{\partial v/\partial x_1}{\partial v/\partial x_2} = -\frac{\partial f/\partial u \, \partial u/\partial x_1}{\partial f/\partial u \, \partial u/\partial x_2} = \frac{\partial u/\partial x_1}{\partial u/\partial x_2}$$

This reiterates that the MRS is independent of the utility representation.

If two utility functions have the same MRS, then they represent the same preferences.

Cobb-Douglas Preferences

Recall that Cobb-Douglas preferences are of the form:

$$u(x_1, x_2) = x_{\&}^{c} x_{(}^{d}$$

Then, the MRS is:

$$MRS = -\frac{\partial u/\delta\partial}{\partial u/\partial x_2}$$

$$= -\frac{c x_1^{c\%1} x_{\&}^{d}}{d x_1^{c} x_{\&}^{d\%1}}$$

$$= -\frac{c x_{\&}}{d x_1}$$

Cobb-Douglas Preferences

We can take also compute the MRS by taking the log transformation of the Cobb-Douglas utility function (assured that this will not alter the underlying preferences) to get:

$$\mathrm{v}(x_1, x_2) = \mathrm{c} \ln x_{\&} + \mathrm{d} \ln x_{(}$$

Here, the MRS is:

$$MRS = -\frac{\partial u/\partial x_{\&}}{\partial u/\partial x_{(}}$$

$$= -\frac{c/x_1}{d/x_2}$$

$$= -\frac{cx_2}{dx_1}$$

Which is just the same as before.
In both cases, the MRS depends only on the *c, d* and the quantities of the two goods currently in thparameterse bundle!

Perfect substitutes

Suppose we have the utility function $u\left(x_{\&,} x_{(}\right) = \mathrm{a}x_{\&} + bx_{(}$.

What is the MRS?

$$MRS = -\frac{\partial u/\partial x_{\&}}{\partial u/\partial x_{(}} = -\frac{a}{b}$$

Estimating utility functions

Preferences for commuting may depend on, among other things, travel time, waiting time, explicit cost, etc...

Suppose:

$$\mathrm{U}\left(x_{\&}, \ldots, x_n\right) = \beta_{\&} x_{\&} + \cdots + \beta_n x_n$$

By observing commuter behavior, economists can infer the average utility function in the population.

Then we can ask questions like, what happens to commuting decisions when the price of gasoline goes up? or when the price of transit passes goes up?

Estimating utility functions

A famous 1975 study by Thomas Domenich and Daniel McFadden estimated just such a utility function (using powerful statistical techniques that can find the β associated with each term in the utility function) based on commuter data from 1967.

They got the following utility function, which fits their data very well:

$$U(TW, TT, C) = -0.147TW - 0.0411TT - 2.24C$$

TW = Time Walking, TT = Travel Time, and C = $ Cost.

So walking time is viewed as 3 times as costly as time in transit.

Summary

1. Utility functions provide a convenient mathematical summary of a preference ordering, so long as the preferences satisfy certain axioms.
2. Utility numbers have no inherent meaning, so utility functions that are positive monotonic transformations of one another will all represent the same underlying preferences.
3. The marginal rate of substitution can be computed by taking partial derivatives of the utility function:

$$MRS = dx_2/dx_1 = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2}$$