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Cost Minimization

Varian, H. 2010. *Intermediate Microeconomics*, W.W. Norton.

Cost Minimization

- A firm is a cost-minimizer if it produces any given output level $y \geq 0$ at smallest possible total cost.

- $c(y)$ denotes the firm's smallest possible total cost for producing y units of output.

- $c(y)$ is the firm's total cost function.

Cost Minimization

- When the firm faces given input prices $w = (w_1,w_2,\ldots,w_n)$ the total cost function will be written as

$$c(w_1,\ldots,w_n,y).$$

The Cost-Minimization Problem

- Consider a firm using two inputs to make one output.
- The production function is
$$y = f(x_1,x_2).$$
- Take the output level $y \geq 0$ as given.
- Given the input prices w_1 and w_2, the cost of an input bundle (x_1,x_2) is $w_1x_1 + w_2x_2$.

The Cost-Minimization Problem

- For given w_1, w_2 and y, the firm's cost-minimization problem is to solve

$$\min_{x_1, x_2 \geq 0} w_1 x_1 + w_2 x_2$$

$$\text{s.t.} \quad f(x_1, x_2) = y.$$

The Cost-Minimization Problem

- The levels x_1*(w_1,w_2,y) and x_1*(w_1,w_2,y) in the least-costly input bundle are the firm's conditional demands for inputs 1 and 2.
- The (smallest possible) total cost for producing y output units is therefore

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y).$$

Conditional Input Demands

- Given w_1, w_2 and y, how is the least costly input bundle located?
- And how is the total cost function computed?

Iso-cost Lines

- A curve that contains all of the input bundles that cost the same amount is an iso-cost curve.

- E.g., given w_1 and w_2, the \$100 iso-cost line has the equation

$$w_1 x_1 + w_2 x_2 = 100.$$

Iso-cost Lines

- Generally, given w_1 and w_2, the equation of the \$c iso-cost line is

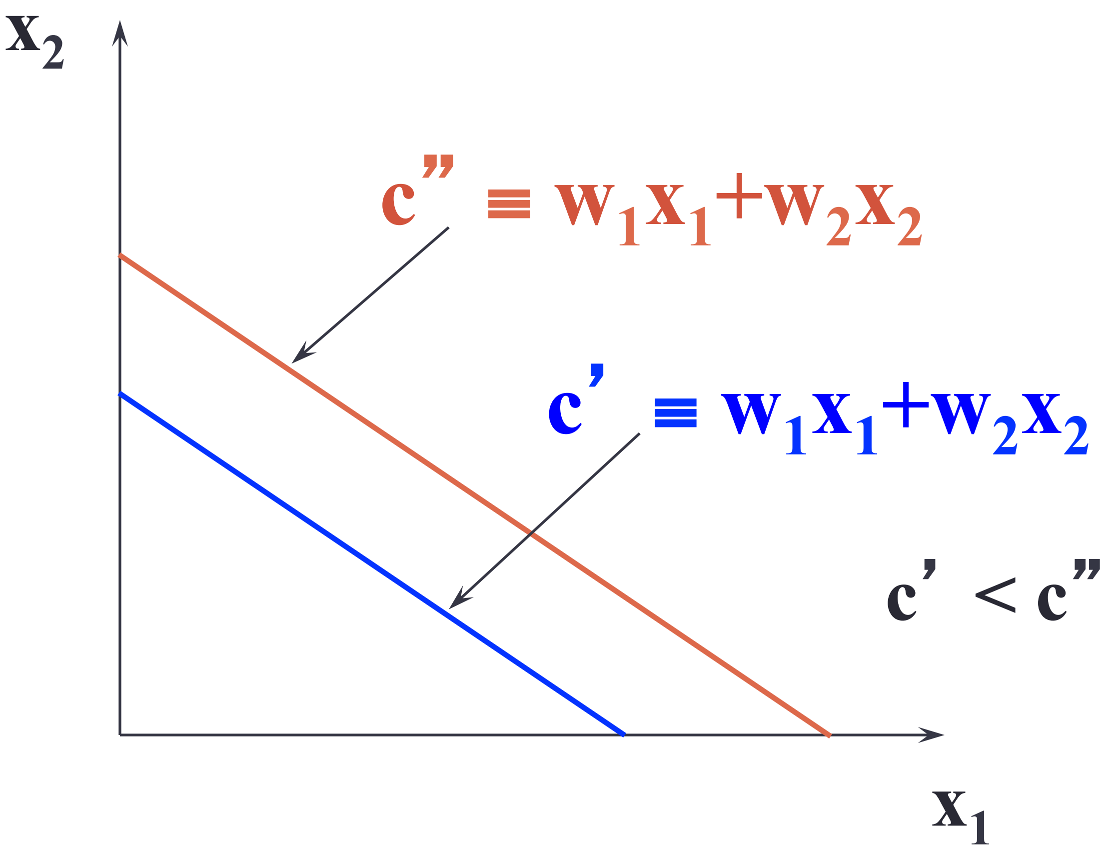
$$w_1x_1 + w_2x_2 = c$$

i.e.

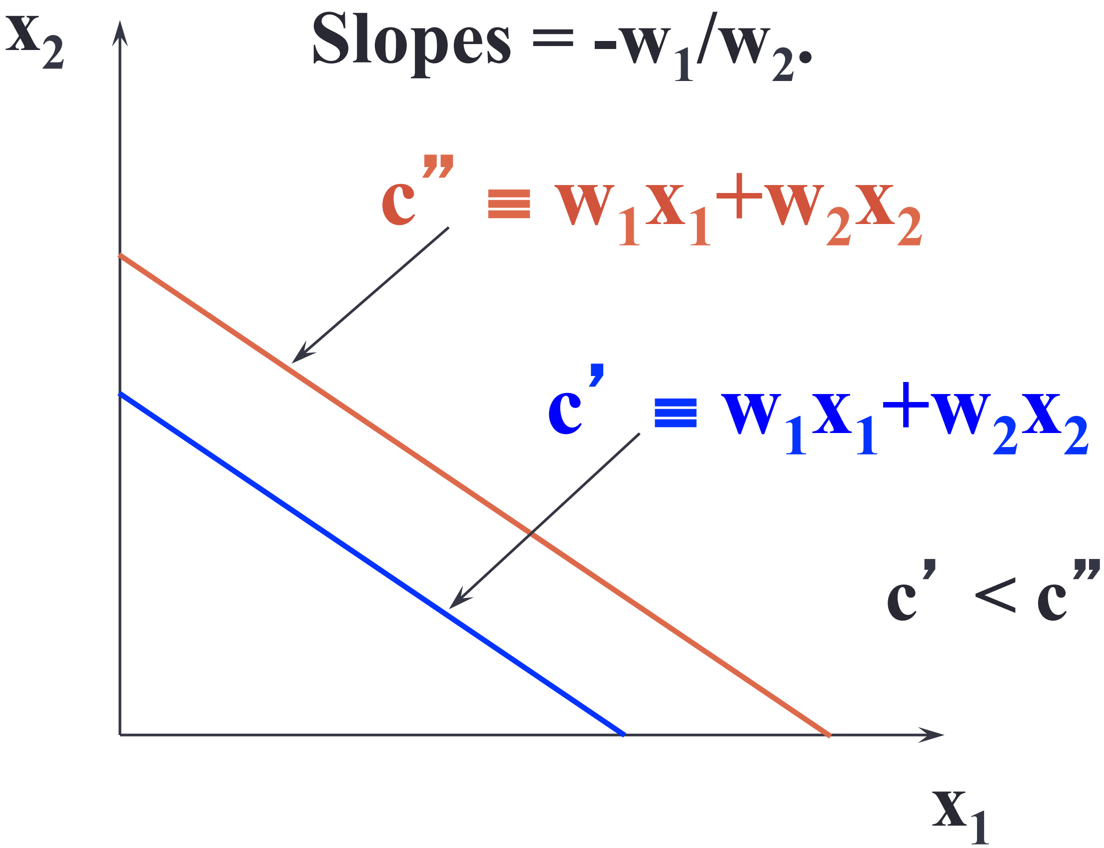
$$x_2 = -\frac{w_1}{w_2}x_1 + \frac{c}{w_2}.$$

- Slope is - w_1/w_2.

Iso-cost Lines

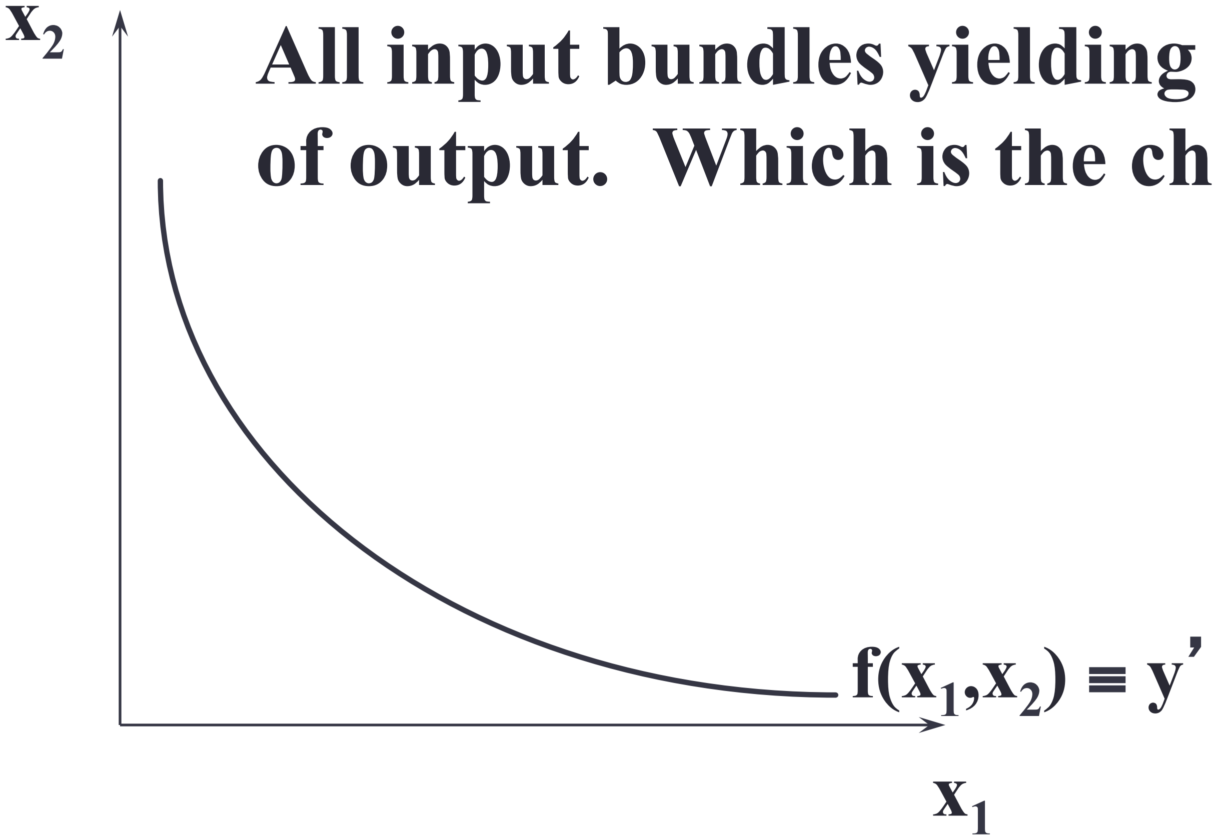


Iso-cost Lines

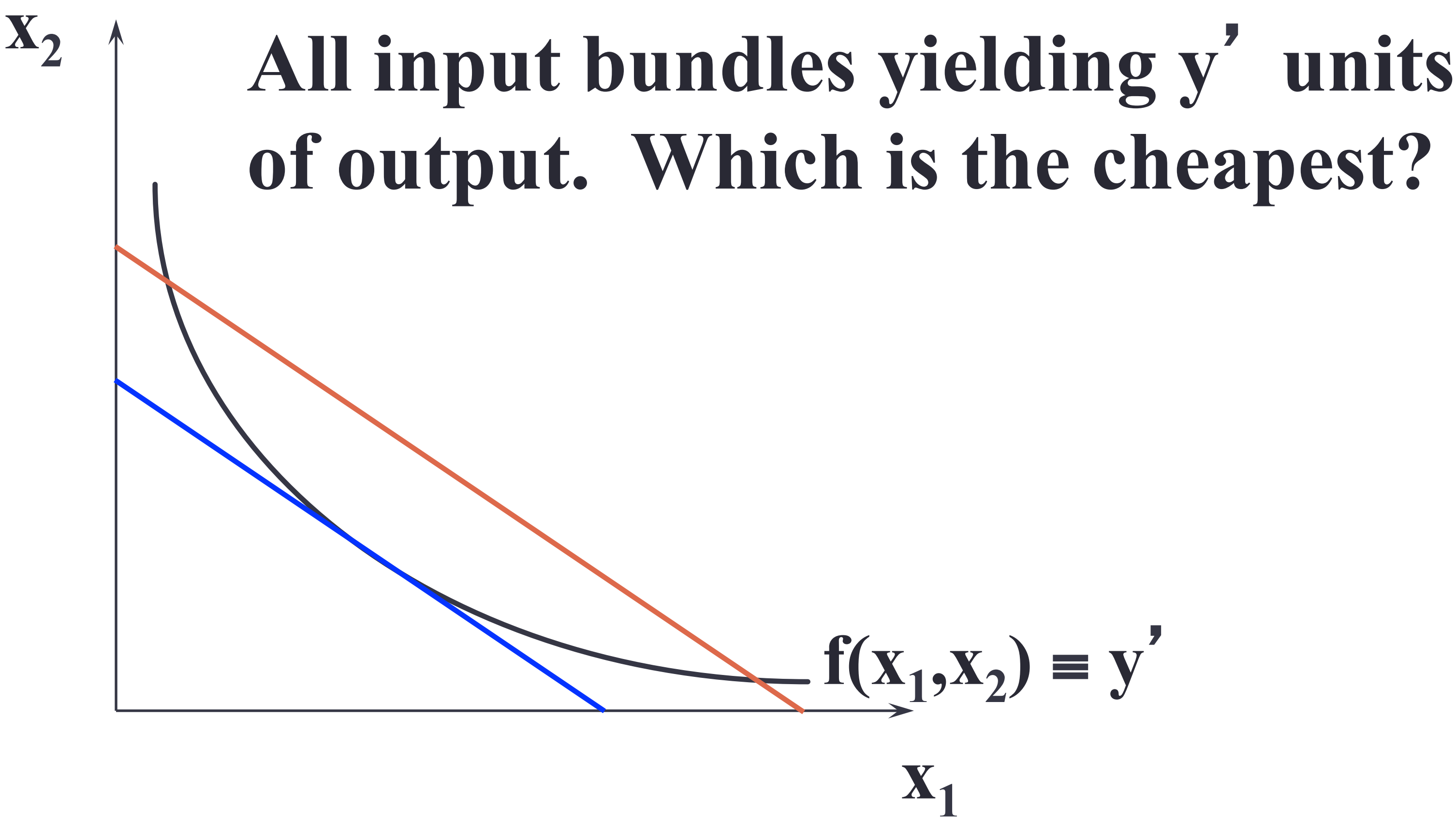


The *y'*-Output Unit Isoquant

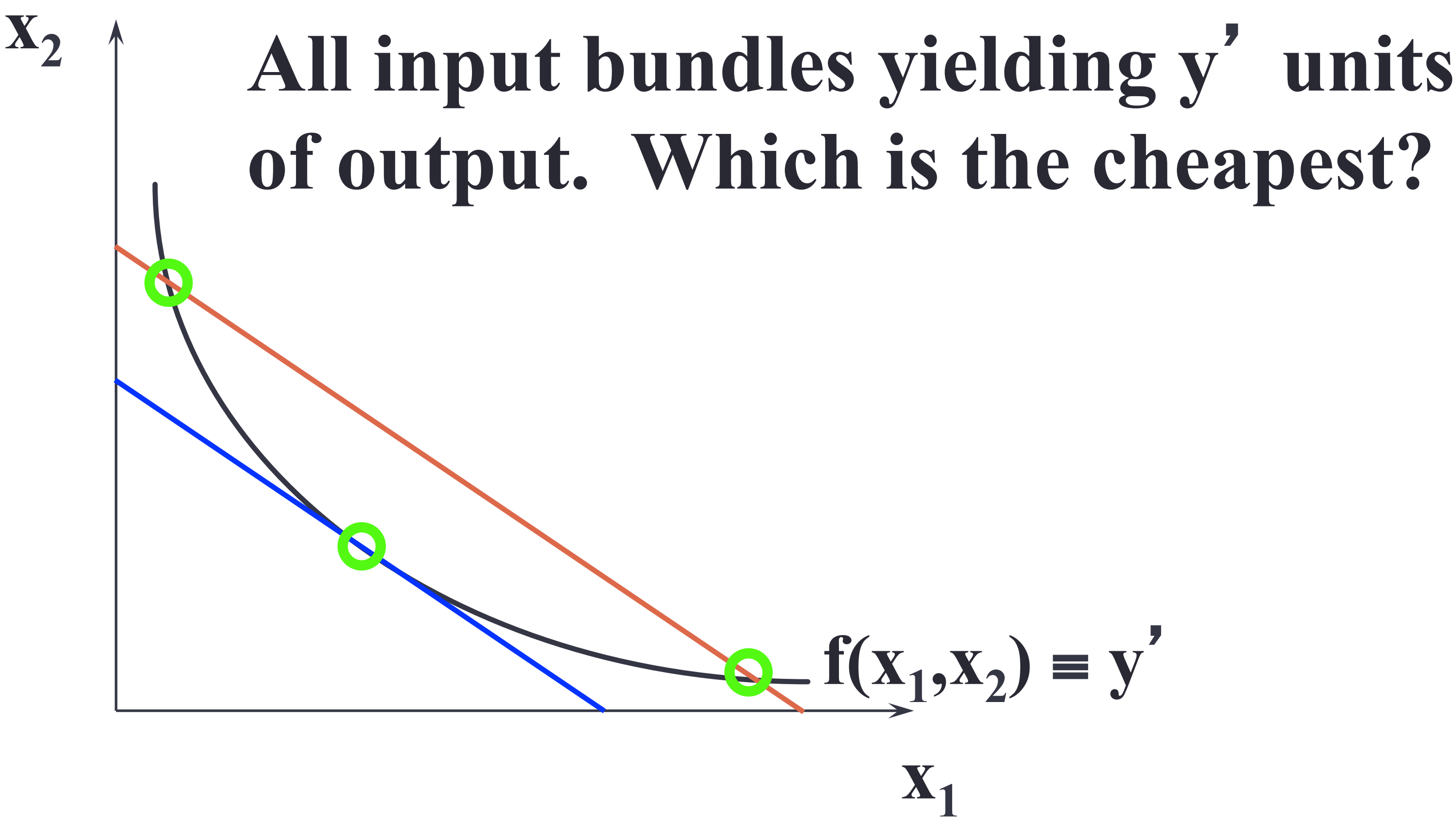
All input bundles yielding y' units of output. Which is the cheapest?



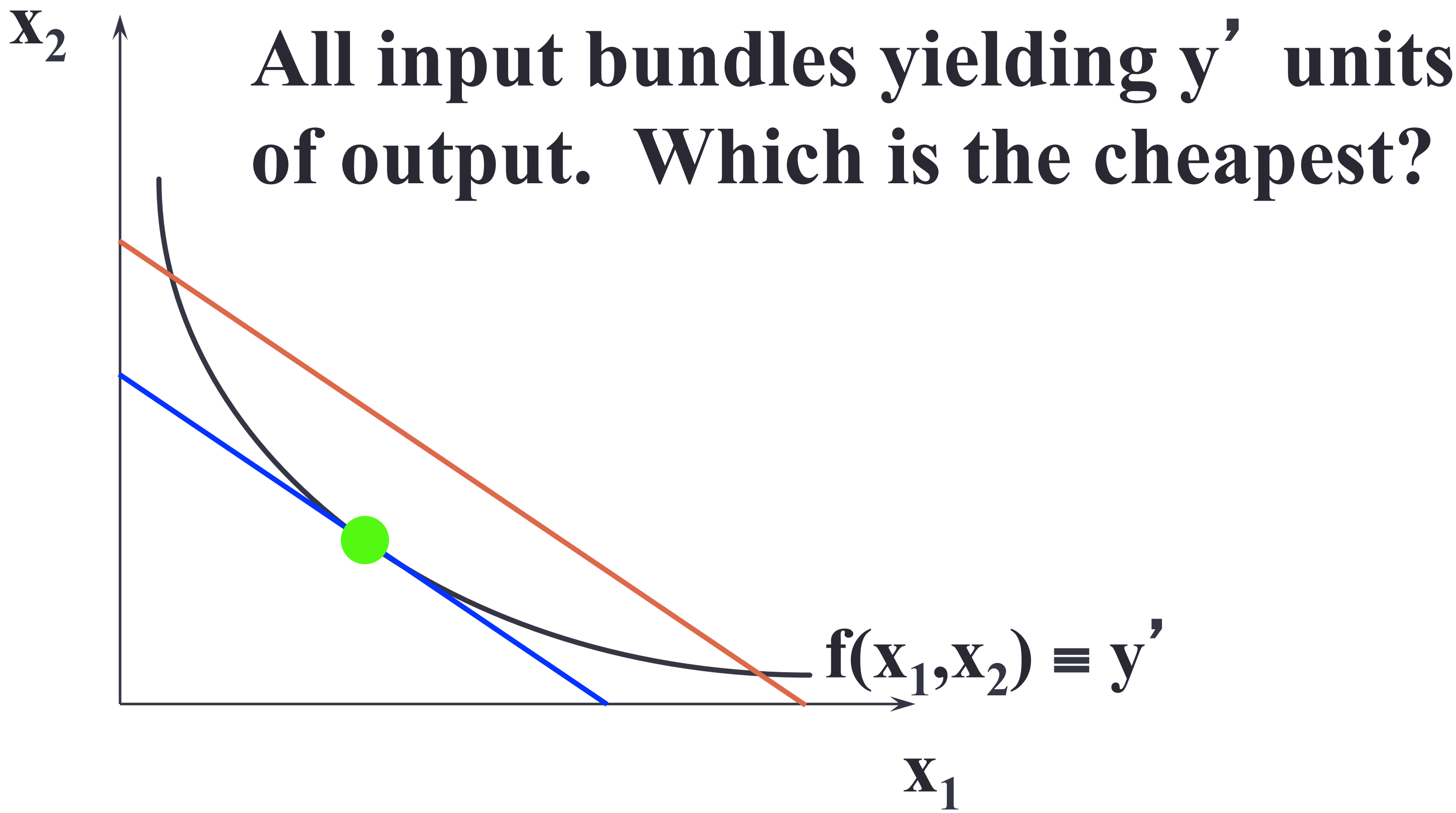
The Cost-Minimization Problem



The Cost-Minimization Problem

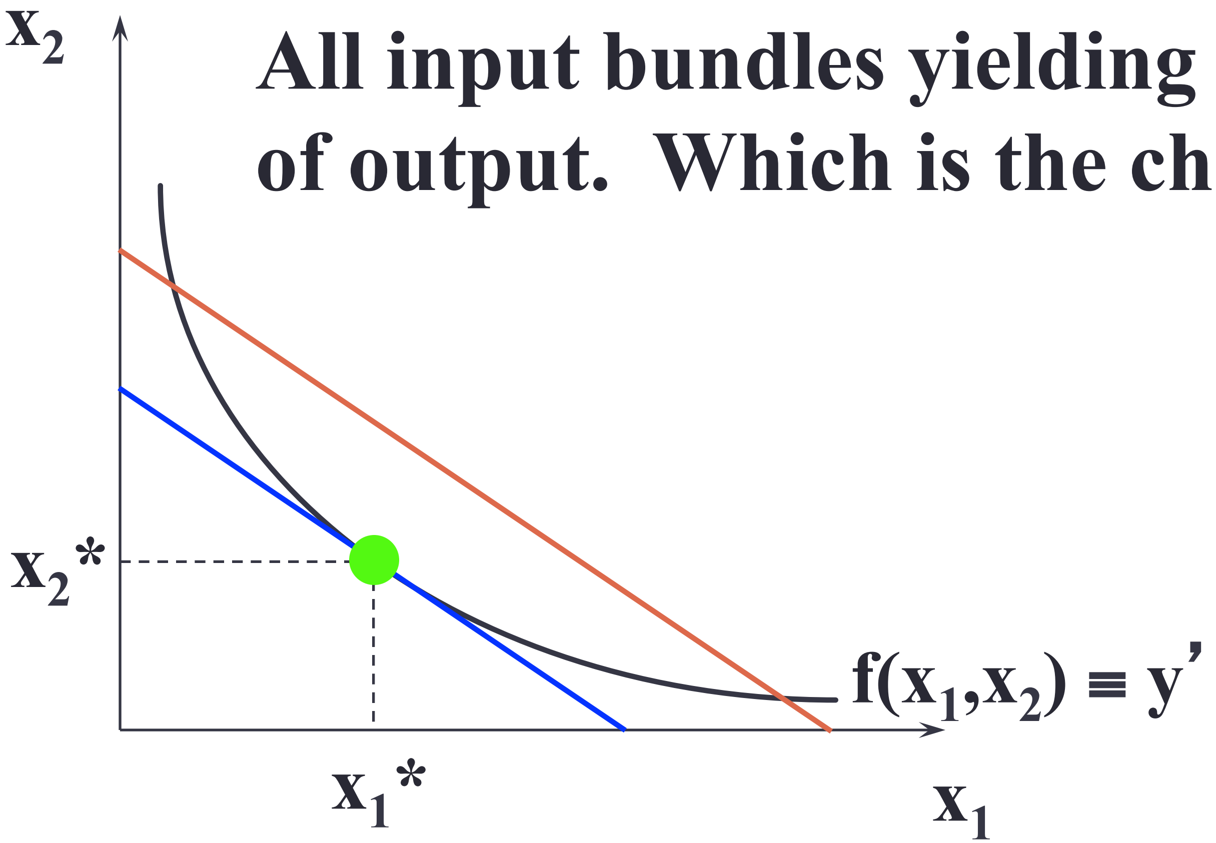


The Cost-Minimization Problem



The Cost-Minimization Problem

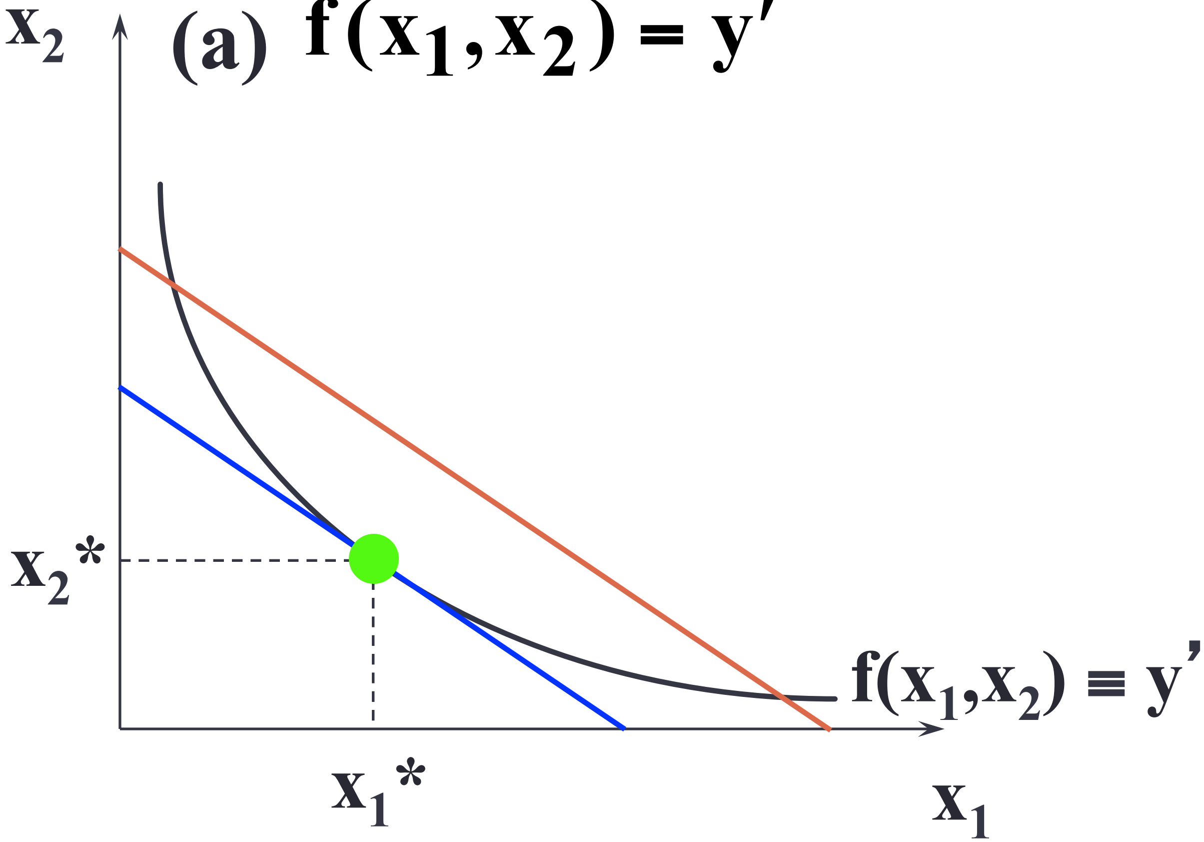
All input bundles yielding y' units of output. Which is the cheapest?



The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$

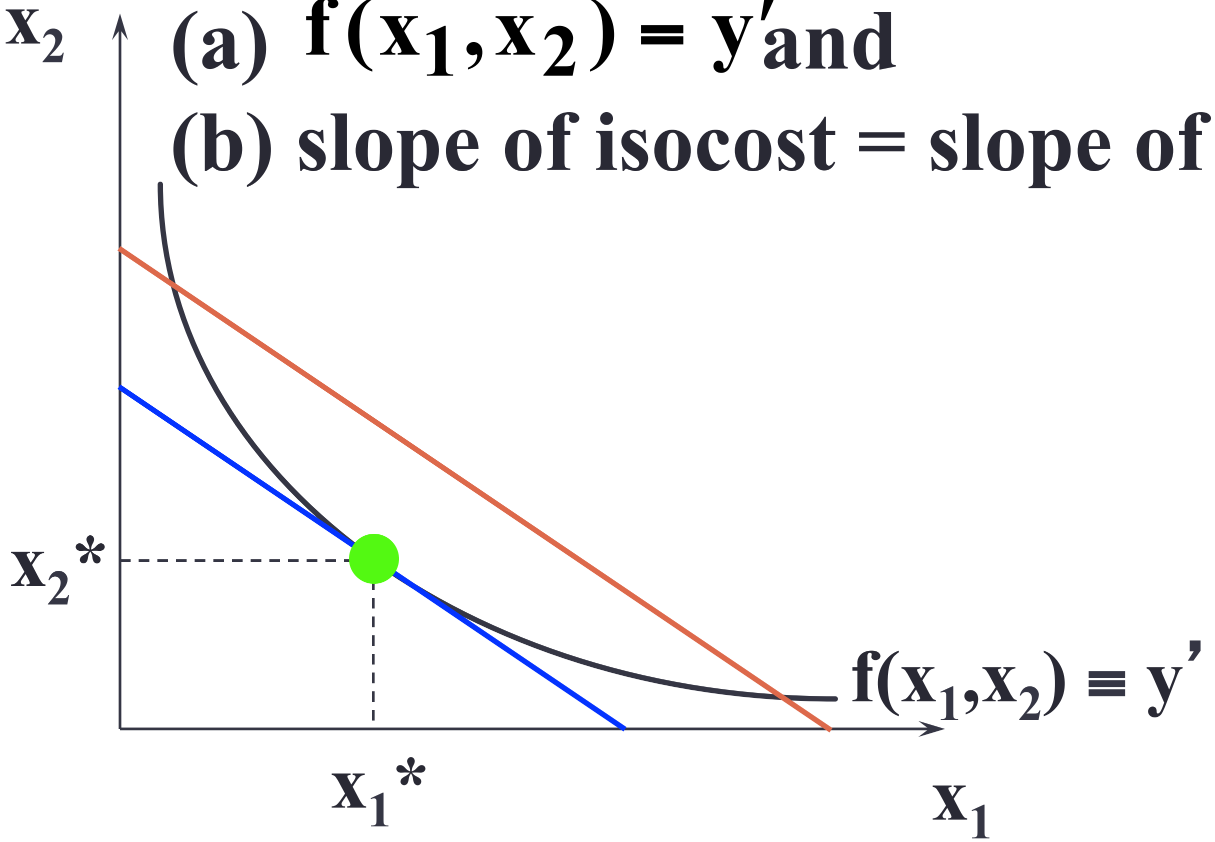


The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$ and

(b) slope of isocost = slope of isoquant



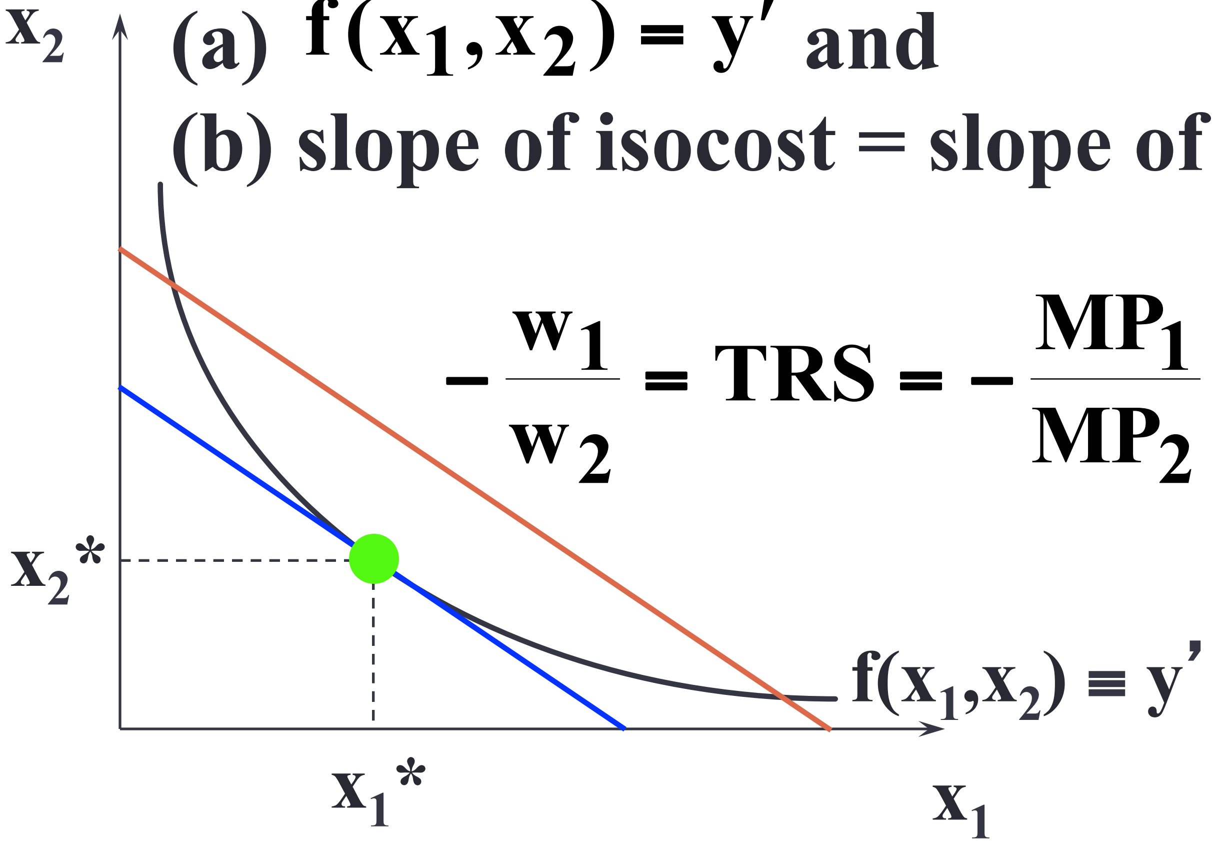
The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$ **and**

(b) slope of isocost = slope of isoquant; i.e.

$$-\frac{w_1}{w_2} = TRS = -\frac{MP_1}{MP_2} \text{ at } (x_1^*, x_2^*).$$



Cost Minimization with Calculus

We set up the problem:

$$\min_{x_1, x_2} w_1 x_1 + w_2 x_2$$
$$\text{s.t. } f(x_1, x_2) = y$$

Let's use one of the techniques for solving these problems that we covered in chapter 4, Lagrangians:

$$L = w_1 x_1 + w_2 x_2 - \lambda(f(x_1, x_2) - y)$$

Cost Minimization with Calculus

Next we need to differentiate w.r.t. x_1, x_2, and λ to get the F.O.Cs:

$$w_1 - \lambda \frac{\partial f(x_1, x_2)}{\partial x_1} = 0$$

$$w_1 2 - \lambda \frac{\partial f(x_1, x_2)}{\partial x_2} = 0$$

$$f(x_1, x_2) - y = 0$$

The last condition is just the constraint, and we can rearrange the first two equations and divide by the second to get:

$$\frac{w_1}{w_2} = \frac{\partial f(x_1, x_2)/\partial x_1}{\partial f(x_1, x_2)/\partial x_2}$$

Which is the same as the F.O.C. we showed earlier:
TRS = factor price ratio.

Cost Min with Calculus (CD Example)

If we have a Cobb-Douglas production function, the cost-minimization problem is:

$$\min_{x_1, x_2} w_1 x_1 + w_2 x_2$$

$$\text{s.t. } x_1^a x_2^b = y$$

To solve this, we could use the substitution method. First get x_2 as a function of x_1:

$$x_2 = (y x_1^{-a})^{1/b}$$

Then substitute this into the objective function to get an unconstrained problem:

$$\min_{x_1} w_1 x_1 + w_2 (y x_1^{-a})^{1/b}$$

Then we could solve this by differentiating w.r.t. x_1 and setting the result equal to 0. (Try it yourself!)

Cost Min with Calculus (CD Example)

We can also do it with Lagrangians. The three F.O.Cs are:

$$w_1 = \lambda a x_1^{a-1} x_2^b$$

$$w_2 = \lambda b x_1^a x_2^{b-1}$$

$$y = x_1^a x_2^b$$

Multiply the first equation x_1 and the second by x_2 to get

$$w_1 x_1 = \lambda a x_1^a x_2^b = \lambda a y$$

$$w_2 x_2 = \lambda a x_1^a x_2^b = \lambda b y$$

And thus

$$x_1 = \lambda \frac{ay}{w_1} \quad (1)$$

$$x_2 = \lambda \frac{by}{w_2} \quad (2)$$

Cost Min with Calculus (CD Example)

Now we can use the third F.O.C. to solve for λ by substituting (1) and (2):

$$\left(\frac{\lambda ay}{w_1}\right)^a \left(\frac{\lambda by}{w_2}\right)^b = y$$

With some messy algebra we can get:

$$\lambda = (a^{-b} b^{-b} w_1^a w_2^b y^{1-a-b})^{\frac{1}{a+b}}$$

Together with (1) and (2), this gives us our final expressions for x_1 and x_2:

$$x_1(w_1, w_2, y) = \left(\frac{a}{b}\right)^{\frac{b}{a+b}} w_1^{\frac{-b}{a+b}} w_2^{\frac{b}{a+b}} y^{\frac{1}{a+b}}$$

$$x_2(w_1, w_2, y) = \left(\frac{a}{b}\right)^{-\frac{a}{a+b}} w_1^{\frac{a}{a+b}} w_2^{\frac{-a}{a+b}} y^{\frac{1}{a+b}}$$

Cost Min with Calculus (CD Example)

Then we can get the cost function by writing down the costs incurred by the firm when making cost-minimizing choices:

$$c(w_1, w_2, y) = w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y)$$

With more tedious algebra we get:

$$c(w_1, w_2, y) = \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{a}{b}\right)^{\frac{-a}{a+b}} \right] w_1^{\frac{a}{a+b}} w_2^{\frac{b}{a+b}} y^{\frac{1}{a+b}}$$

Thus, costs will increase *more* than, *less* than, or *equal* to linearly with output as $a + b$ is *less* than, *more* than, or *equal* to 1.

This makes sense given that CD technologies exhibit different returns to scale depending on the value of $a + b$.

A Cobb-Douglas Example of Cost Minimization

- A firm's Cobb-Douglas production function is

$$y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}.$$

- Input prices are w_1 and w_2.
- What are the firm's conditional input demand functions?

A Cobb-Douglas Example of Cost Minimization

At the input bundle (x_1*,x_2*) which minimizes the cost of producing y output units:

(a)

$$y = (x_1^*)^{1/3}(x_2^*)^{2/3} \quad \text{and}$$

(b)

$$-\frac{w_1}{w_2} = -\frac{\partial y / \partial x_1}{\partial y / \partial x_2} = -\frac{(1/3)(x_1^*)^{-2/3}(x_2^*)^{2/3}}{(2/3)(x_1^*)^{1/3}(x_2^*)^{-1/3}}$$

$$= -\frac{x_2^*}{2x_1^*}.$$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ (b) $\dfrac{w_1}{w_2} = \dfrac{x_2^*}{2x_1^*}$.

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ (b) $\dfrac{w_1}{w_2} = \dfrac{x_2^*}{2x_1^*}$.

From (b), $x_2^* = \dfrac{2w_1}{w_2}x_1^*$.

A Cobb-Douglas Example of Cost Minimization

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From (b), $x_2^* = \dfrac{2w_1}{w_2}x_1^*$.

Now substitute into (a) to get

$$y = (x_1^*)^{1/3}\left(\frac{2w_1}{w_2}x_1^*\right)^{2/3}$$

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A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ (b) $\dfrac{w_1}{w_2} = \dfrac{x_2^*}{2x_1^*}$.

From (b), $x_2^* = \dfrac{2w_1}{w_2}x_1^*$.

Now substitute into (a) to get

$$y = (x_1^*)^{1/3}\left(\frac{2w_1}{w_2}x_1^*\right)^{2/3} = \left(\frac{2w_1}{w_2}\right)^{2/3}x_1^*.$$

So $x_1^* = \left(\dfrac{w_2}{2w_1}\right)^{2/3} y$ is the firm's conditional demand for input 1.

A Cobb-Douglas Example of Cost Minimization

Since $x_2^* = \frac{2w_1}{w_2} x_1^*$ and $x_1^* = \left(\frac{w_2}{2w_1}\right)^{2/3} y$

$$x_2^* = \frac{2w_1}{w_2}\left(\frac{w_2}{2w_1}\right)^{2/3} y = \left(\frac{2w_1}{w_2}\right)^{1/3} y$$

is the firm's conditional demand for input 2.

A Cobb-Douglas Example of Cost Minimization

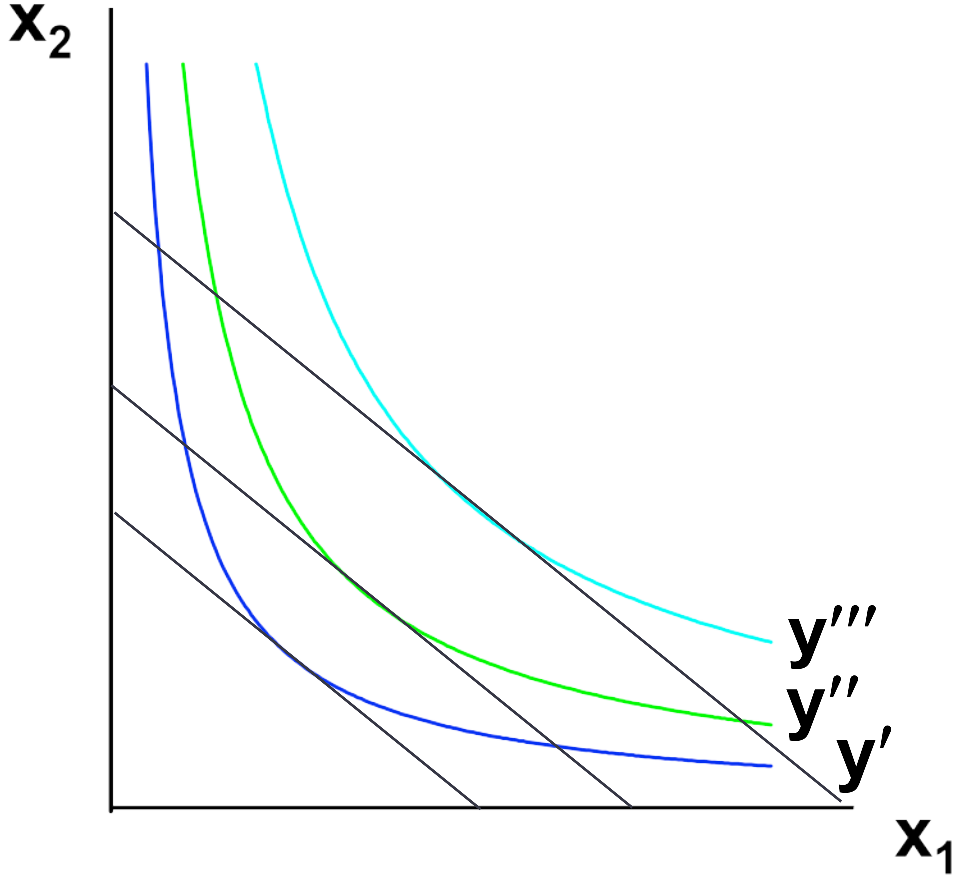
So the cheapest input bundle yielding y output units is

$$\left(x_1^*(w_1, w_2, y), x_2^*(w_1, w_2, y)\right)$$

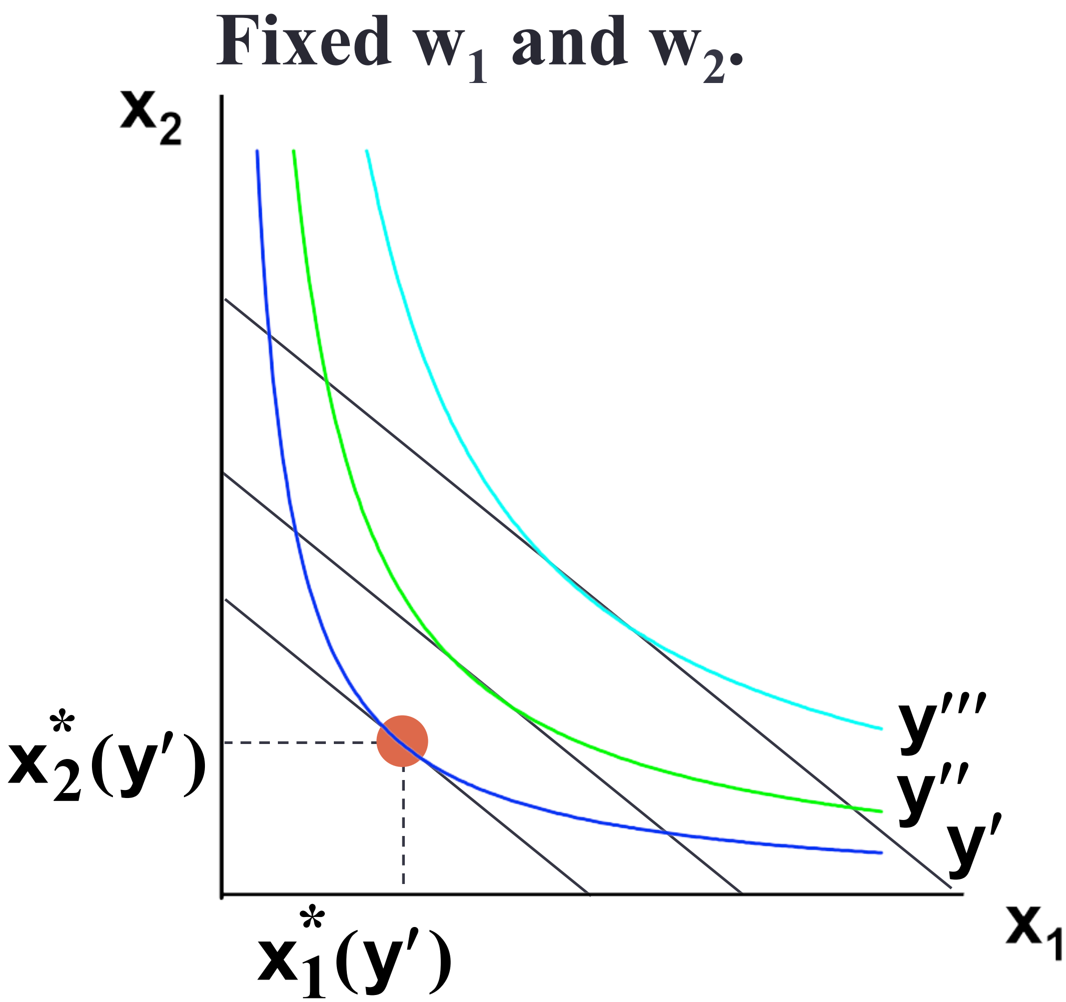
$$= \left(\left(\frac{w_2}{2w_1}\right)^{2/3} y, \left(\frac{2w_1}{w_2}\right)^{1/3} y\right).$$

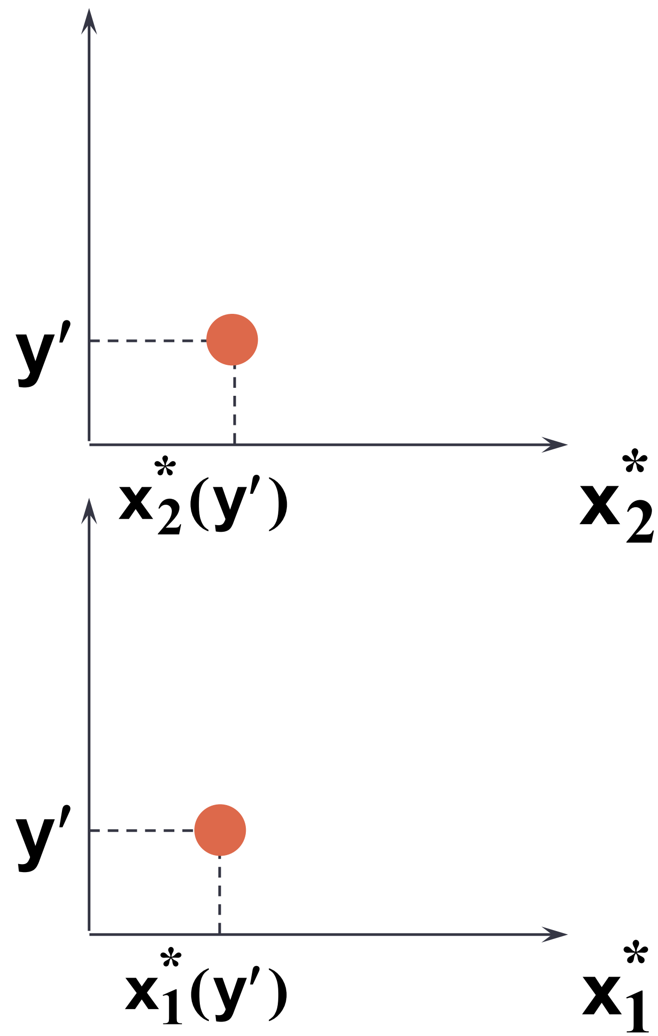
Conditional Input Demand Curves

Fixed w_1 and w_2.

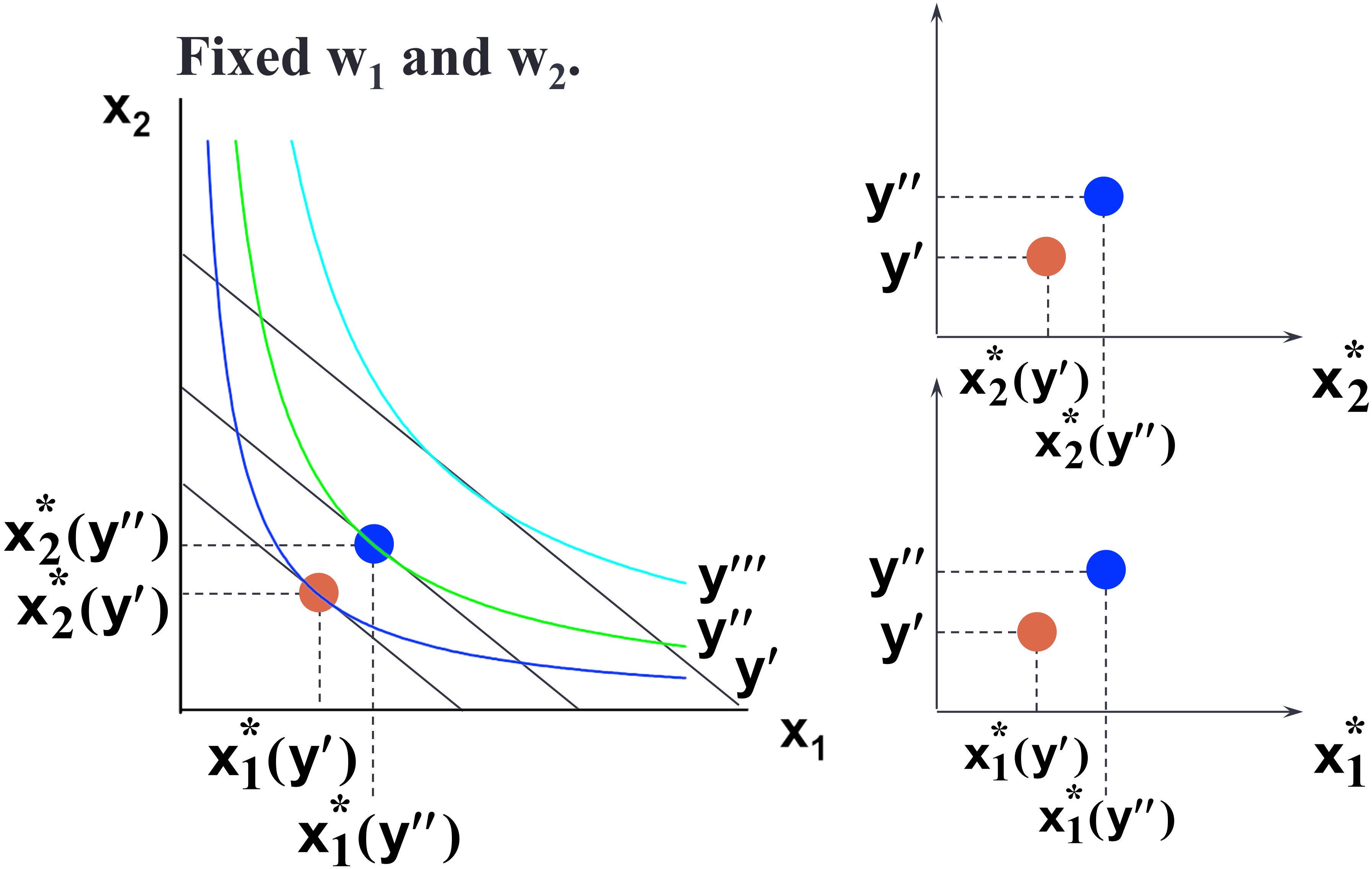


Conditional Input Demand Curves



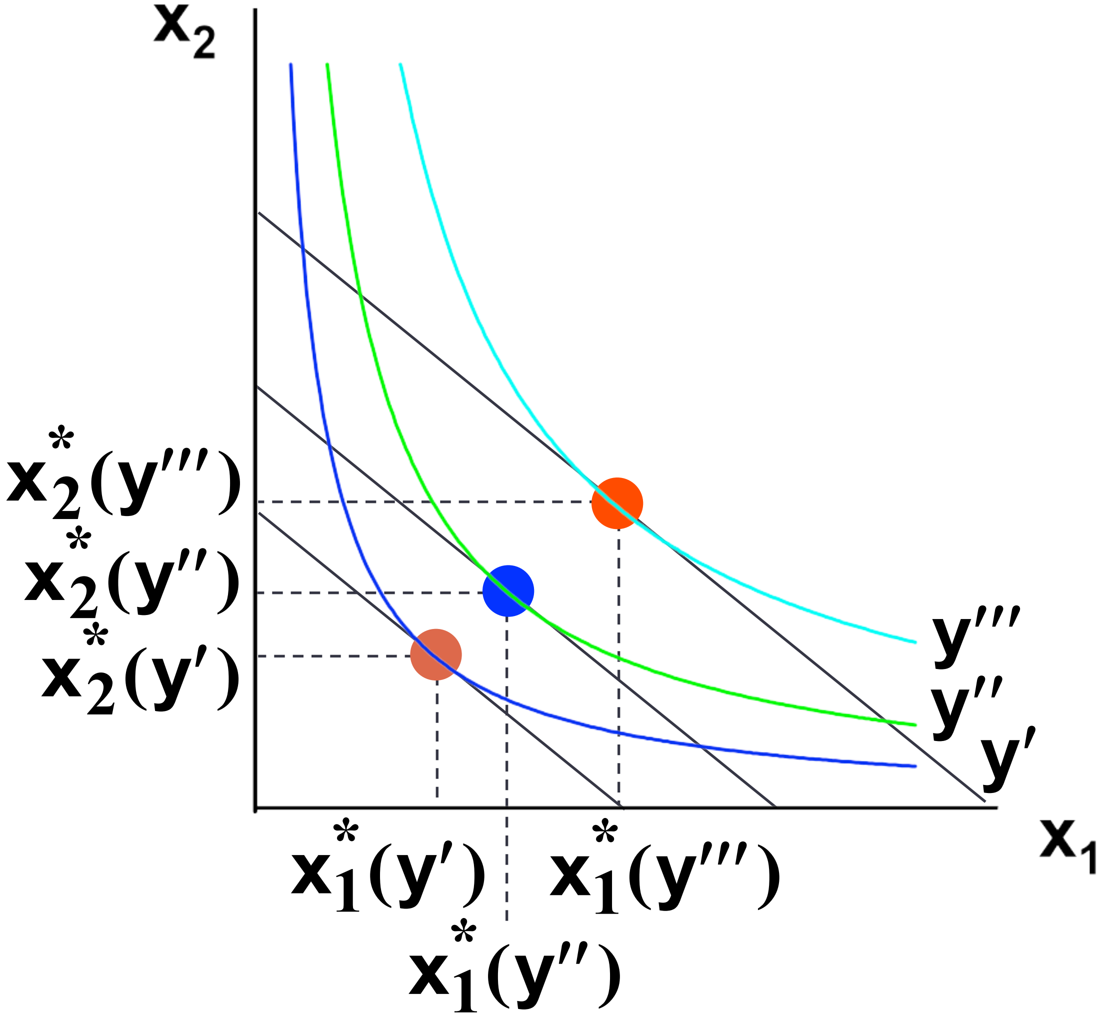


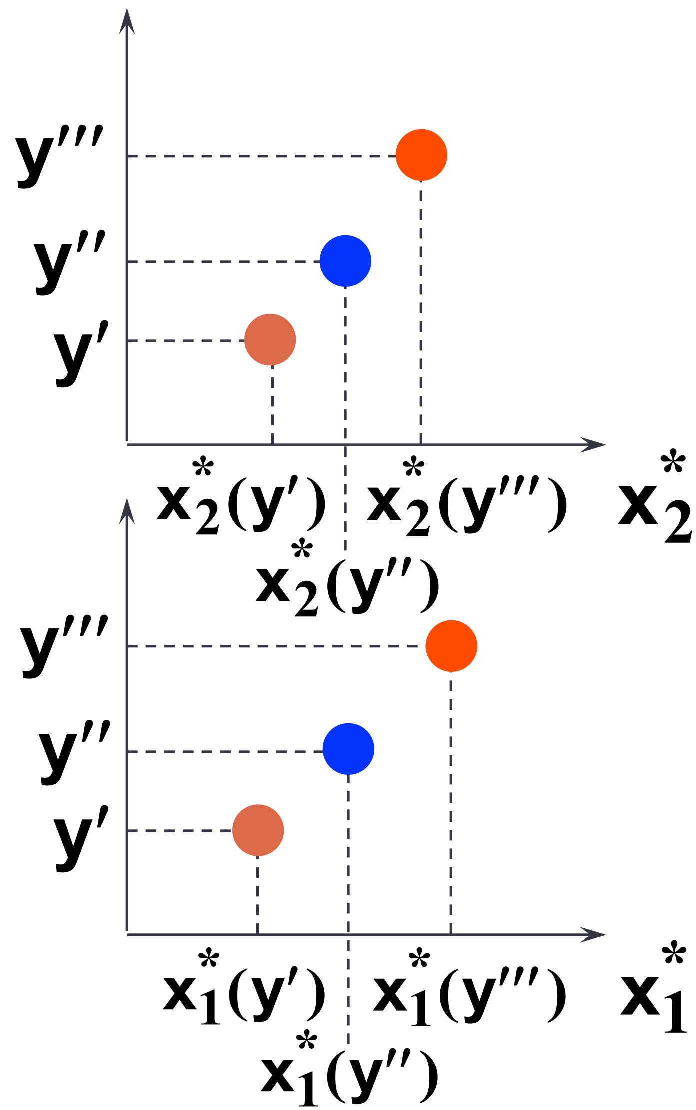
Conditional Input Demand Curves



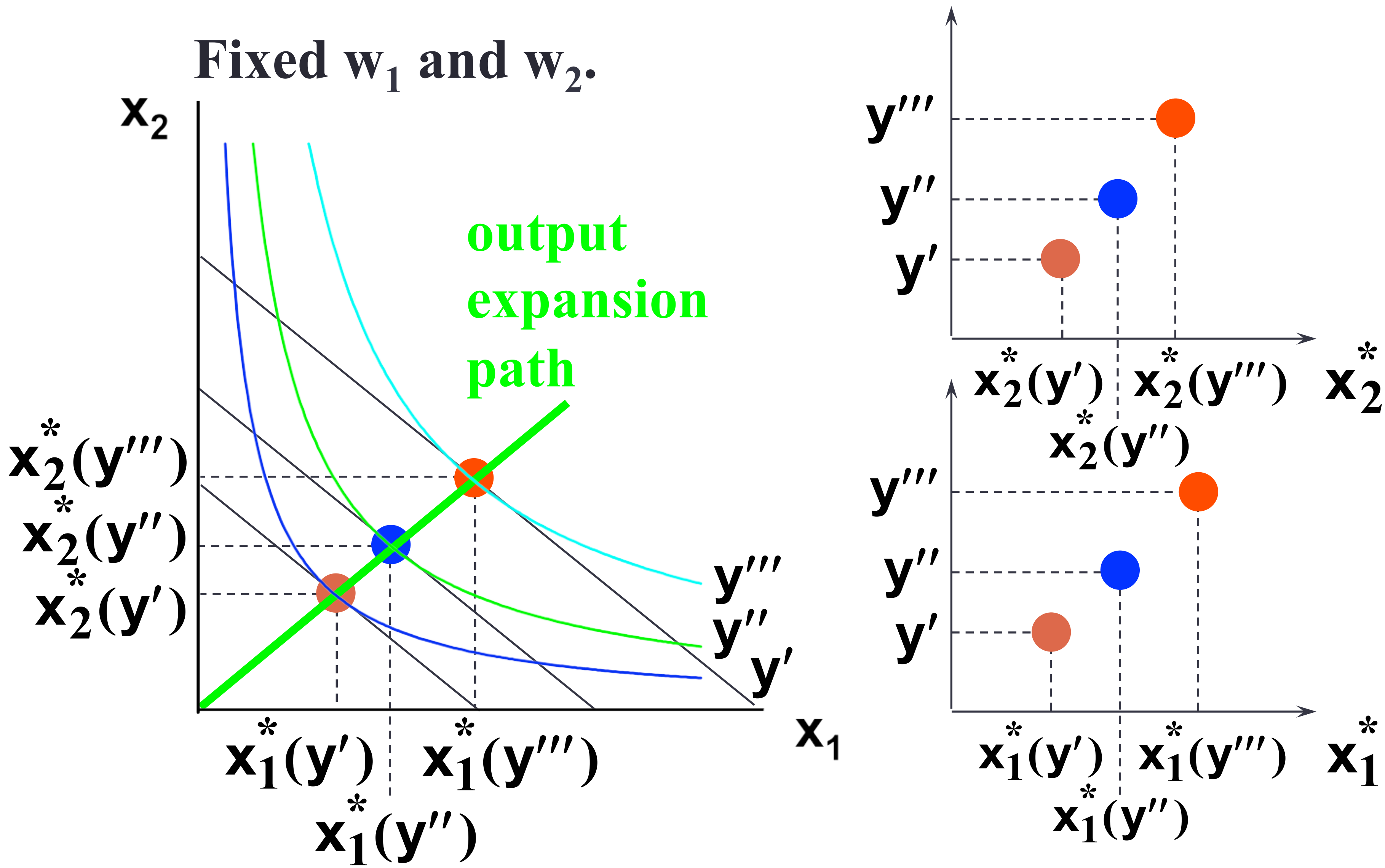
Conditional Input Demand Curves

Fixed w_1 and w_2.

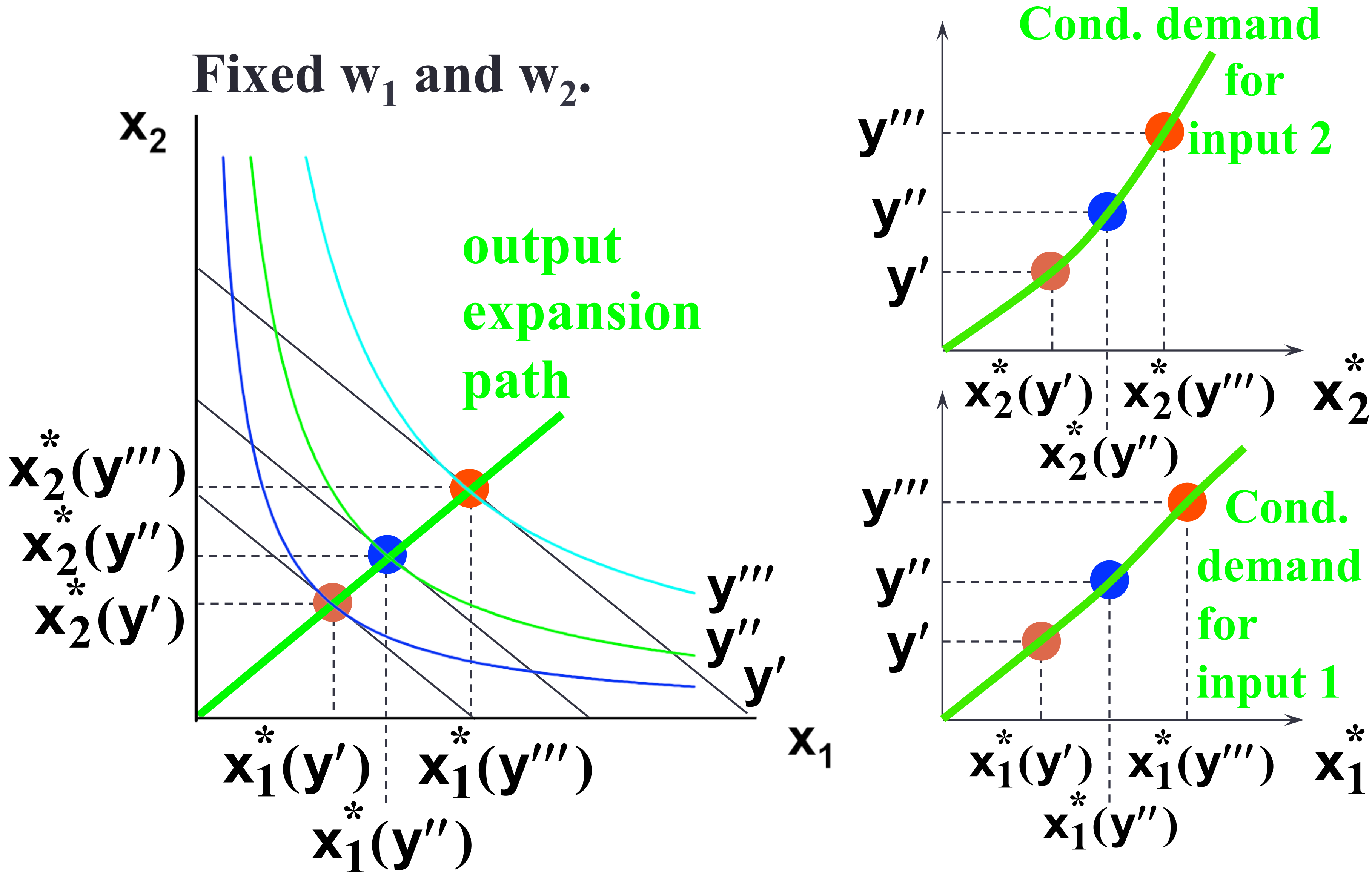




Conditional Input Demand Curves



Conditional Input Demand Curves



A Cobb-Douglas Example of Cost Minimization

For the production function

$$y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$$

the cheapest input bundle yielding y output units is

$$\left(x_1^*(w_1, w_2, y), x_2^*(w_1, w_2, y)\right)$$

$$= \left(\left(\frac{w_2}{2w_1} \right)^{2/3} y, \left(\frac{2w_1}{w_2} \right)^{1/3} y \right).$$

A Cobb-Douglas Example of Cost Minimization

So the firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

A Cobb-Douglas Example of Cost Minimization

So the firm's total cost function is

$$c(w_1,w_2,y) = w_1x_1^*(w_1,w_2,y) + w_2x_2^*(w_1,w_2,y)$$

$$= w_1\left(\frac{w_2}{2w_1}\right)^{2/3} y + w_2\left(\frac{2w_1}{w_2}\right)^{1/3} y$$

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So the firm's total cost function is

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$$= w_1 \left(\frac{w_2}{2w_1}\right)^{2/3} y + w_2 \left(\frac{2w_1}{w_2}\right)^{1/3} y$$

$$= \left(\frac{1}{2}\right)^{2/3} w_1^{1/3} w_2^{2/3} y + 2^{1/3} w_1^{1/3} w_2^{2/3} y$$

A Cobb-Douglas Example of Cost Minimization

So the firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

$$= w_1 \left(\frac{w_2}{2w_1}\right)^{2/3} y + w_2 \left(\frac{2w_1}{w_2}\right)^{1/3} y$$

$$= \left(\frac{1}{2}\right)^{2/3} w_1^{1/3} w_2^{2/3} y + 2^{1/3} w_1^{1/3} w_2^{2/3} y$$

$$= 3\left(\frac{w_1 w_2^2}{4}\right)^{1/3} y.$$

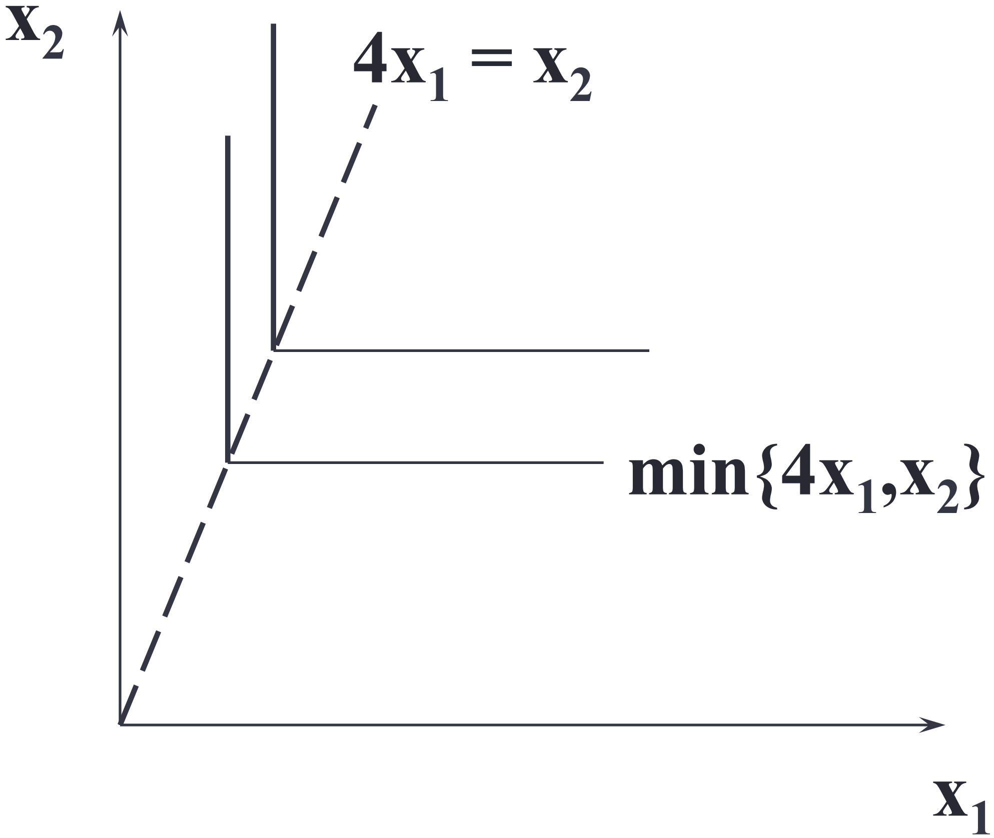
A Perfect Complements Example of Cost Minimization

- The firm's production function is

$$y = \min\{4x_1, x_2\}.$$

- Input prices w_1 and w_2 are given.
- What are the firm's conditional demands for inputs 1 and 2?
- What is the firm's total cost function?

A Perfect Complements Example of Cost Minimization



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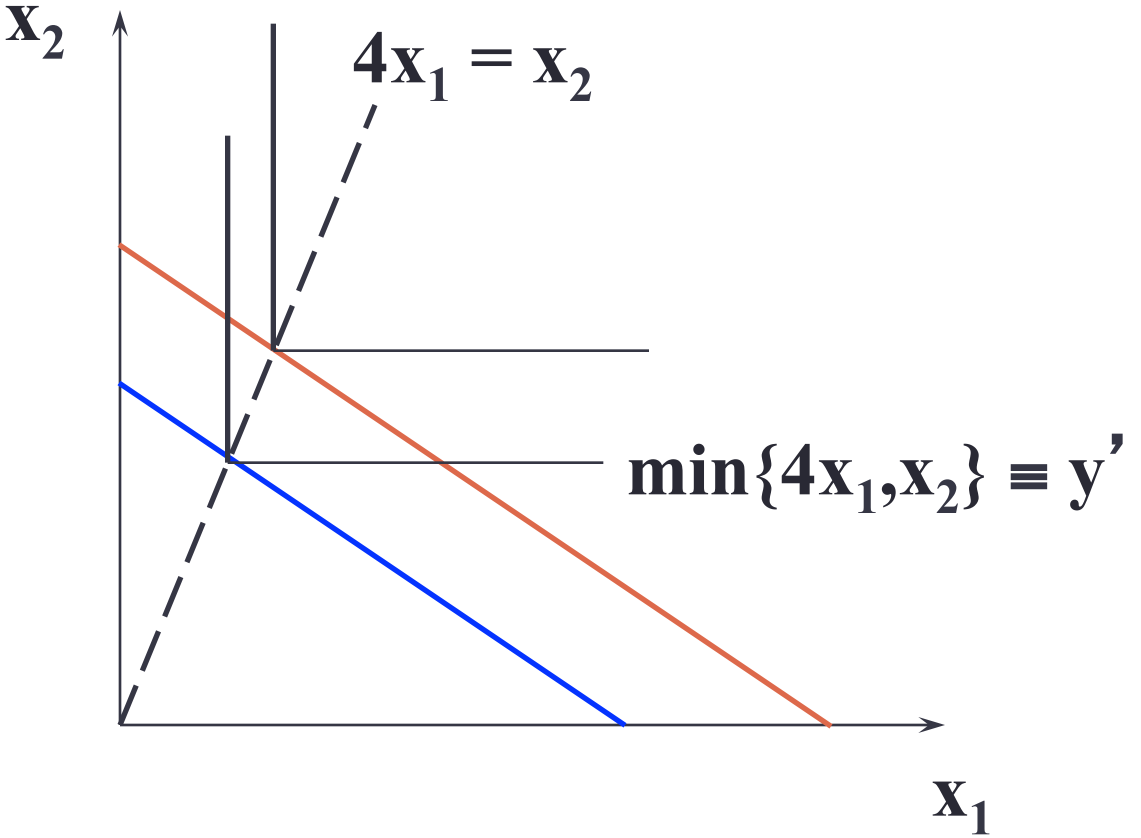
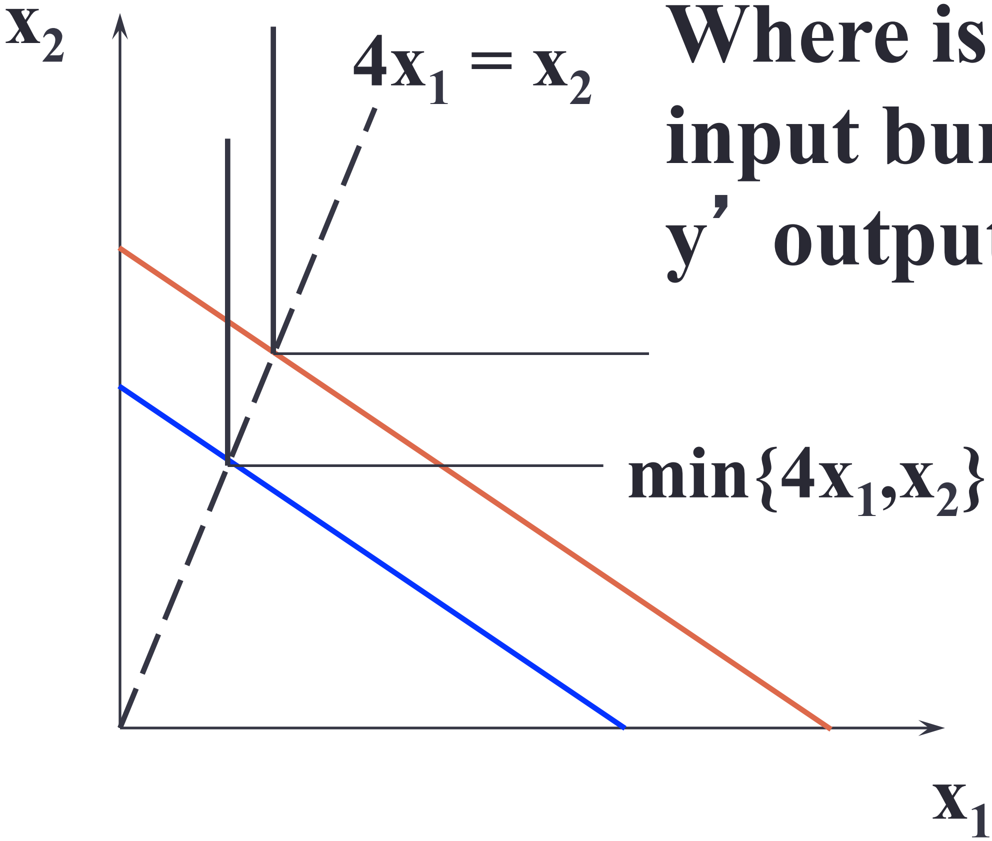
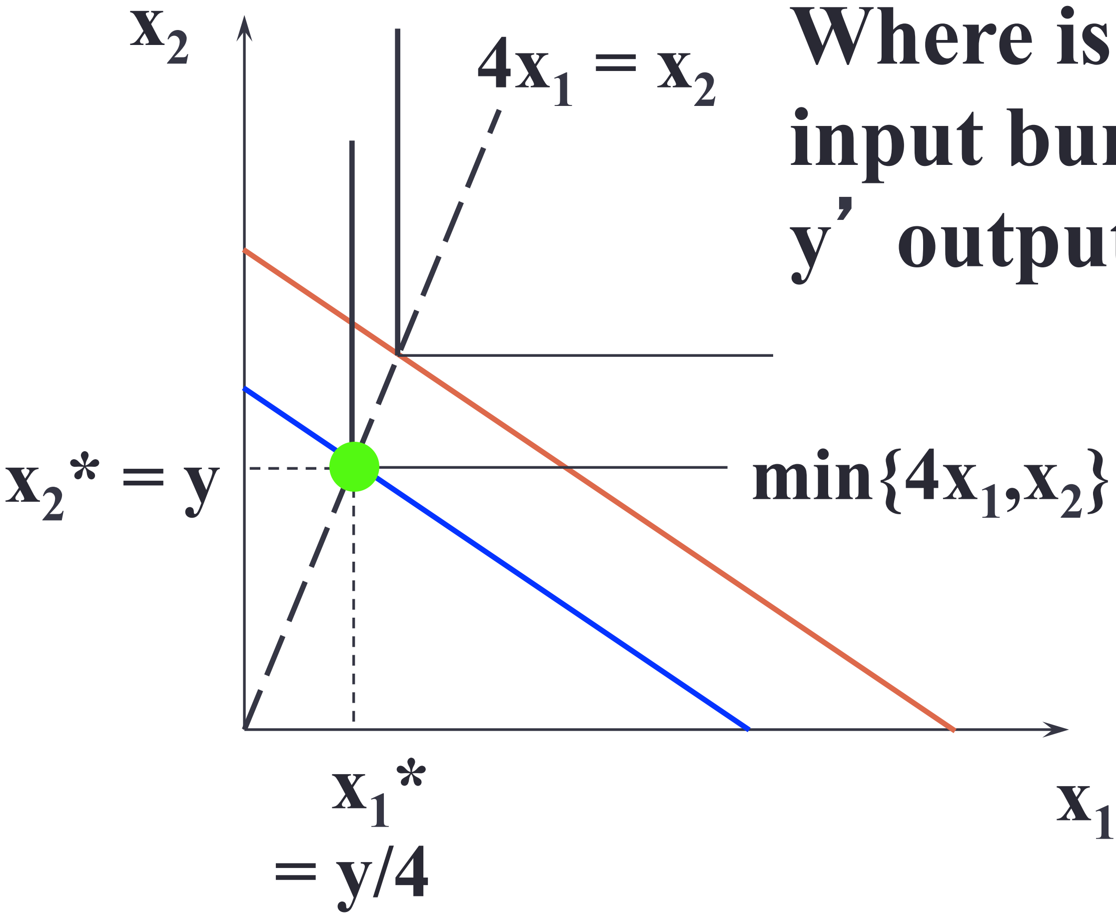


A Perfect Complements Example of Cost Minimization



Where is the least costly input bundle yielding y' output units?

A Perfect Complements Example of Cost Minimization



Where is the least costly input bundle yielding y' output units?

A Perfect Complements Example of Cost Minimization

The firm's production function is

$$y = \min\{4x_1, x_2\}$$

and the conditional input demands are

$$x_1^*(w_1, w_2, y) = \frac{y}{4} \quad \text{and} \quad x_2^*(w_1, w_2, y) = y.$$

A Perfect Complements Example of Cost Minimization

The firm's production function is

$$y = \min\{4x_1, x_2\}$$

and the conditional input demands are

$$x_1^*(w_1, w_2, y) = \frac{y}{4} \quad \text{and} \quad x_2^*(w_1, w_2, y) = y.$$

So the firm's total cost function is

$$\begin{aligned} c(w_1, w_2, y) &= w_1 x_1^*(w_1, w_2, y) \\ &\quad + w_2 x_2^*(w_1, w_2, y) \end{aligned}$$

A Perfect Complements Example of Cost Minimization

The firm's production function is

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$$\begin{aligned} c(w_1, w_2, y) &= w_1 x_1^*(w_1, w_2, y) \\ &\quad + w_2 x_2^*(w_1, w_2, y) \\ &= w_1 \frac{y}{4} + w_2 y = \left(\frac{w_1}{4} + w_2\right) y. \end{aligned}$$

Average Total Production Costs

- For positive output levels y, a firm's average total cost of producing y units is

$$AC(w_1, w_2, y) = \frac{c(w_1, w_2, y)}{y}.$$

Returns-to-Scale and Av. Total Costs

- The returns-to-scale properties of a firm's technology determine how average production costs change with output level.
- Our firm is presently producing y' output units.
- How does the firm's average production cost change if it instead produces 2y' units of output?

Constant Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits constant returns-to-scale then doubling its output level from y' to 2y' requires doubling all input levels.

Constant Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits constant returns-to-scale then doubling its output level from y' to 2y' requires doubling all input levels.

- Total production cost doubles.

Constant Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits constant returns-to-scale then doubling its output level from y' to 2y' requires doubling all input levels.
- Total production cost doubles.
- Average production cost does not change.

Decreasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits decreasing returns-to-scale then doubling its output level from y' to 2y' requires more than doubling all input levels.

Decreasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits decreasing returns-to-scale then doubling its output level from y' to 2y' requires more than doubling all input levels.

- Total production cost more than doubles.

Decreasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits decreasing returns-to-scale then doubling its output level from y' to 2y' requires more than doubling all input levels.
- Total production cost more than doubles.
- Average production cost increases.

Increasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits increasing returns-to-scale then doubling its output level from y' to 2y' requires less than doubling all input levels.

Increasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits increasing returns-to-scale then doubling its output level from y' to 2y' requires less than doubling all input levels.

- Total production cost less than doubles.

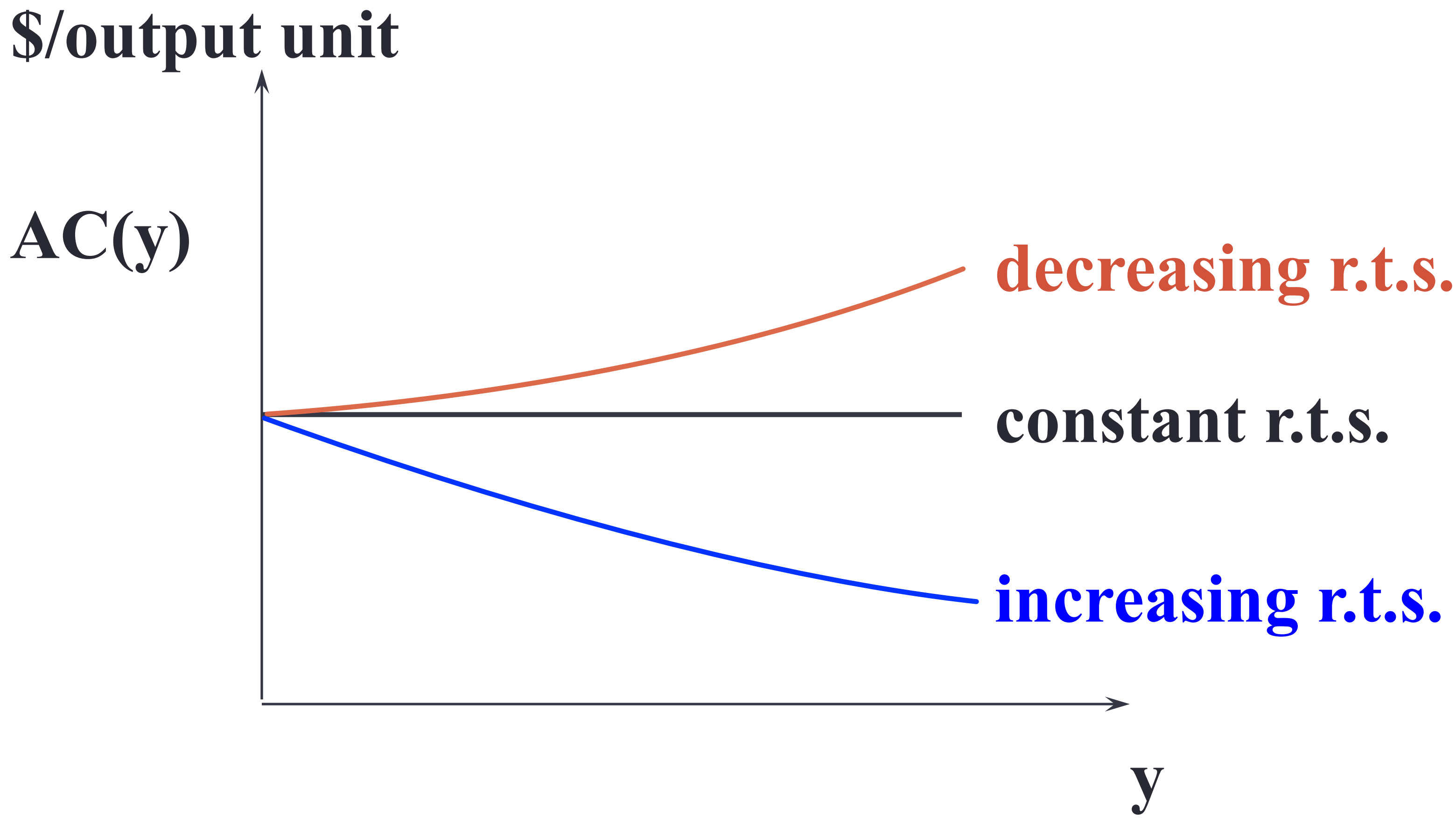
Increasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits increasing returns-to-scale then doubling its output level from y' to 2y' requires less than doubling all input levels.

- Total production cost less than doubles.

- Average production cost decreases.

Returns-to-Scale and Av. Total Costs

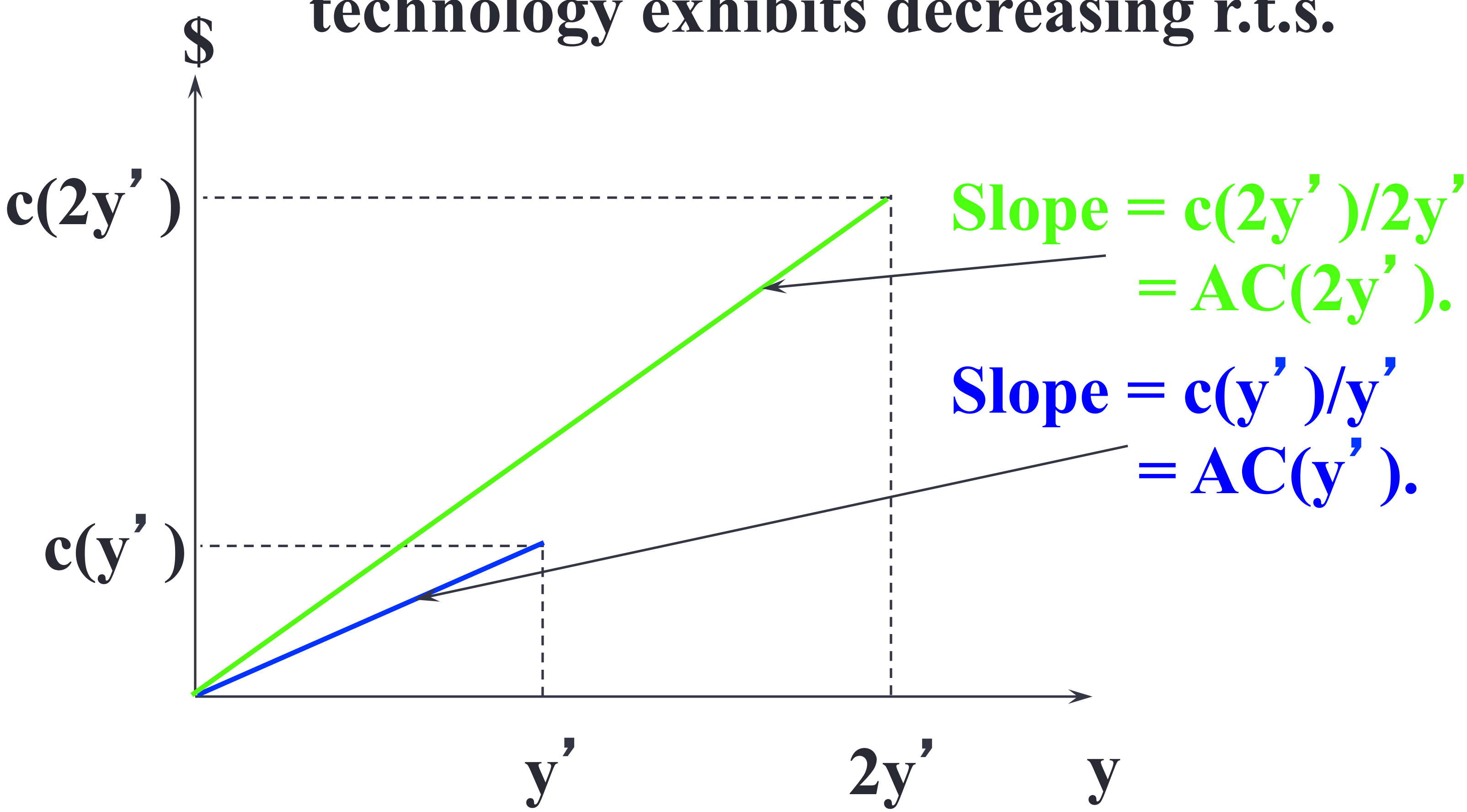


Returns-to-Scale and Total Costs

- What does this imply for the shapes of total cost functions?

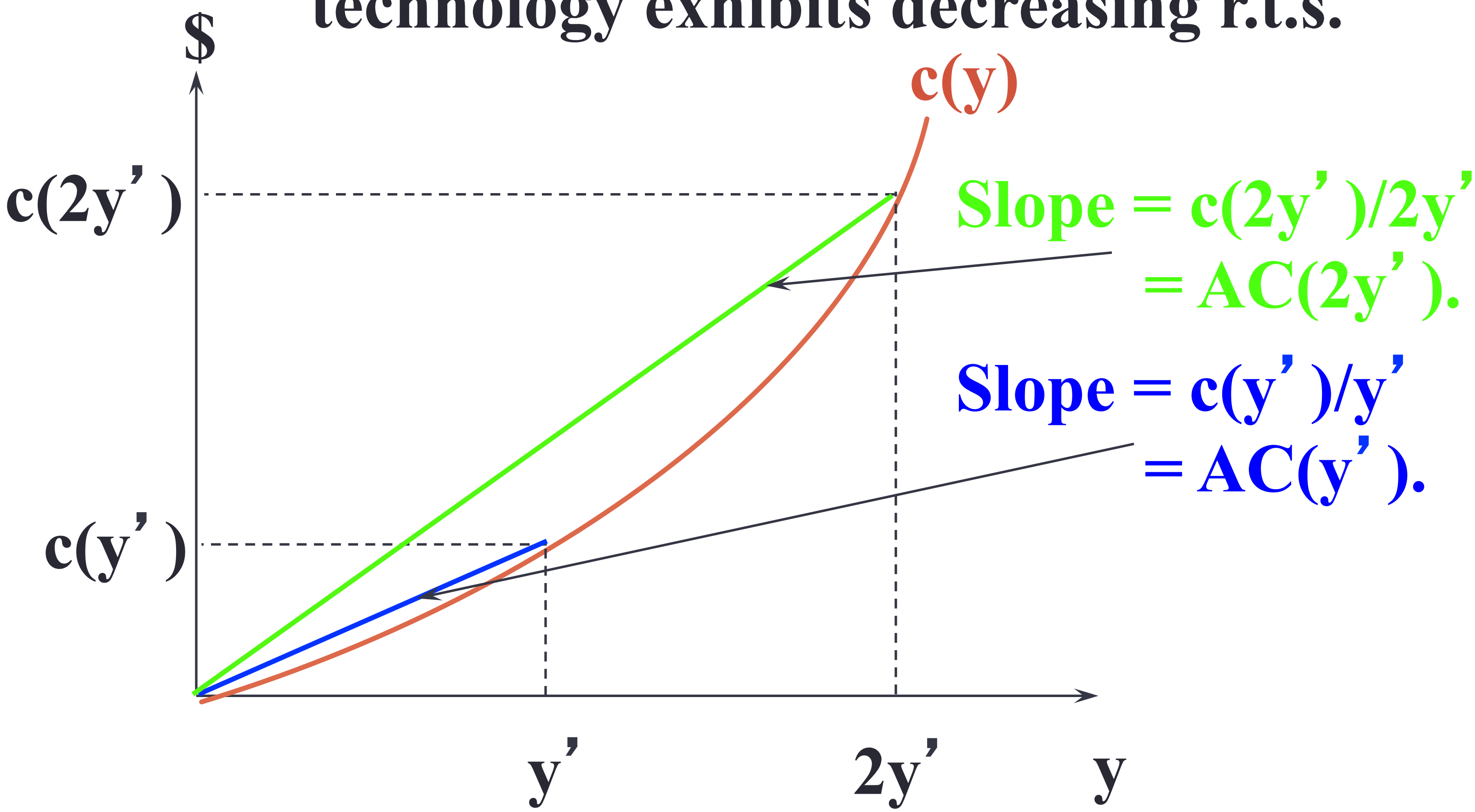
Returns-to-Scale and Total Costs

Av. cost increases with y if the firm's technology exhibits decreasing r.t.s.



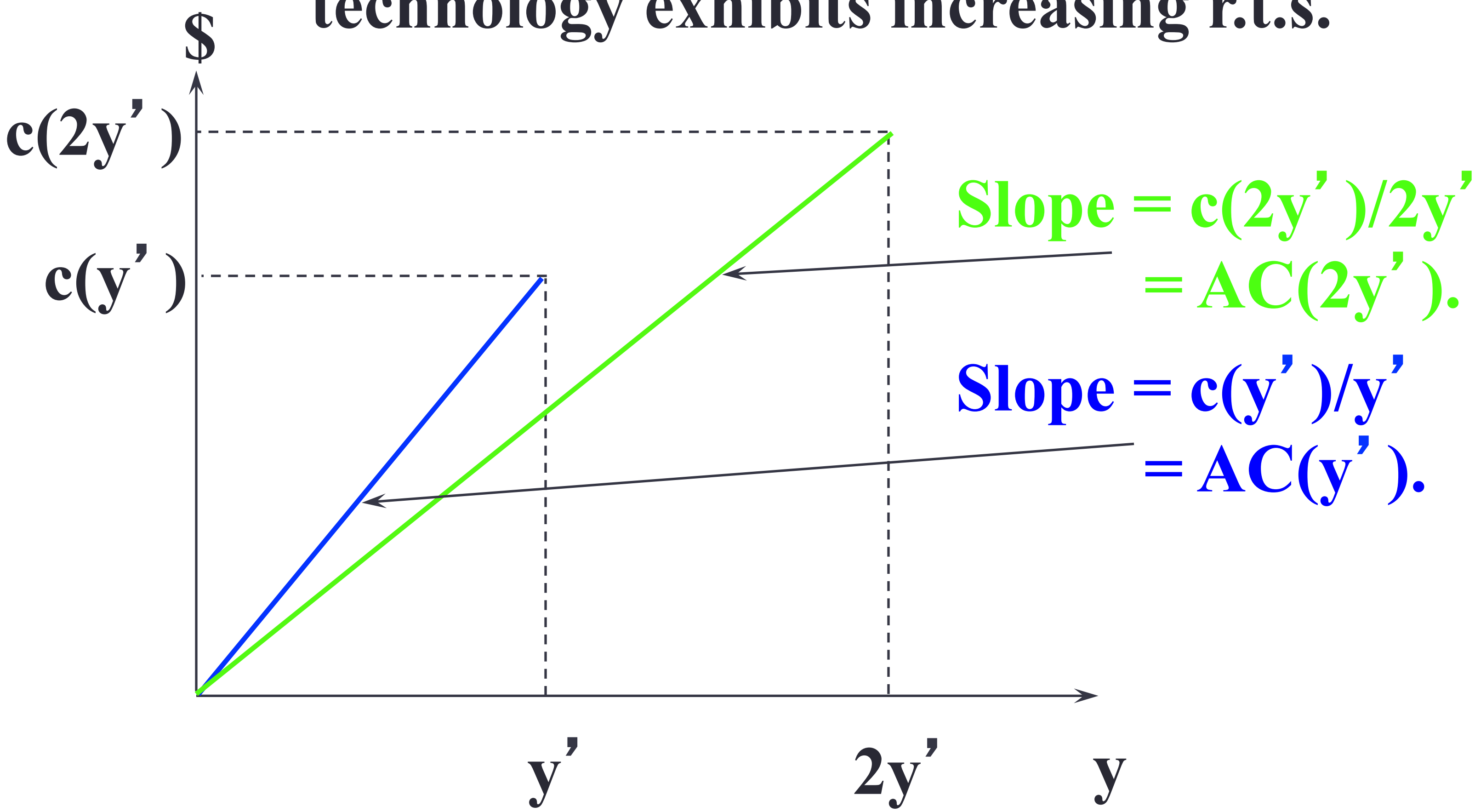
Returns-to-Scale and Total Costs

Av. cost increases with y if the firm's technology exhibits decreasing r.t.s.



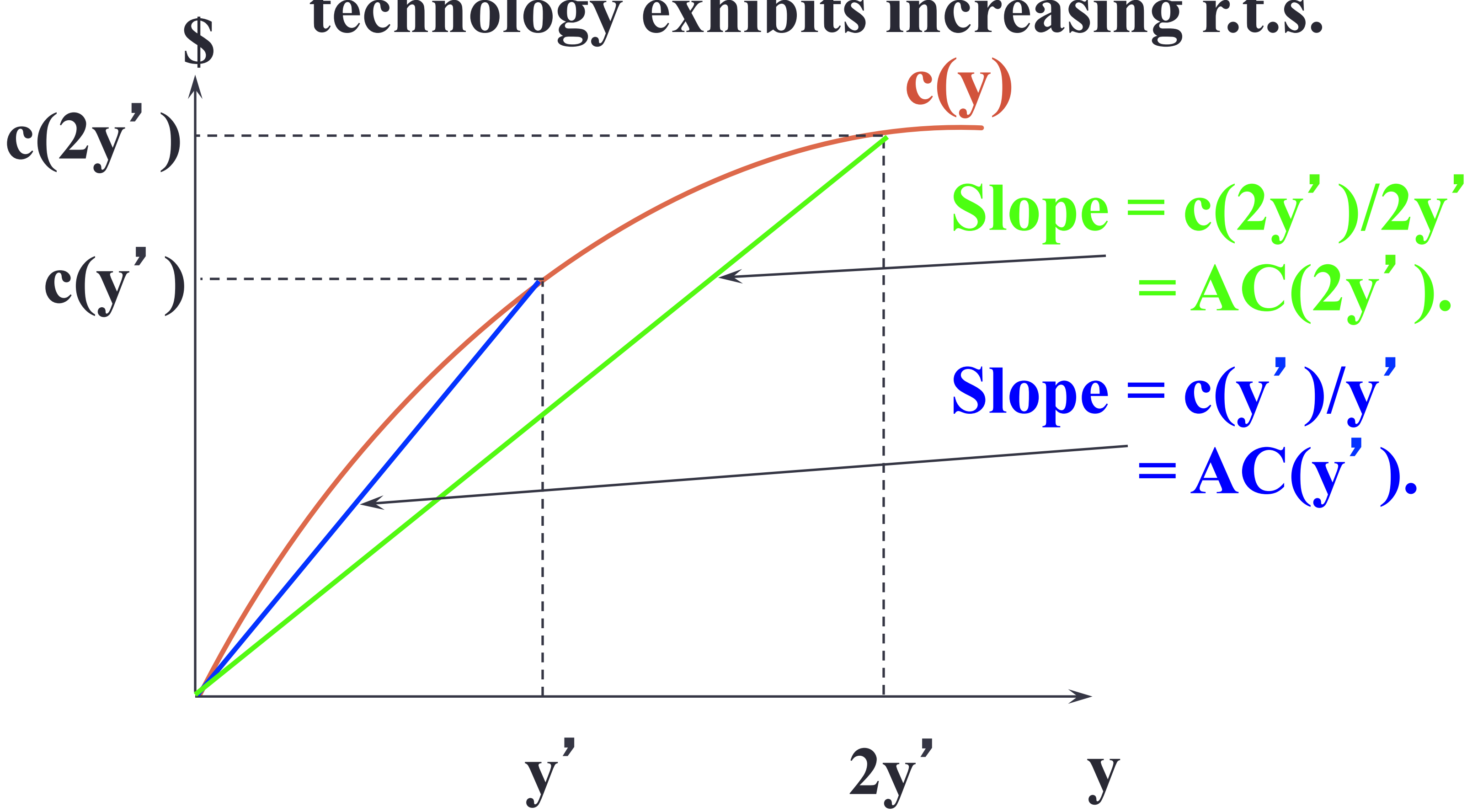
Returns-to-Scale and Total Costs

Av. cost decreases with y if the firm's technology exhibits increasing r.t.s.



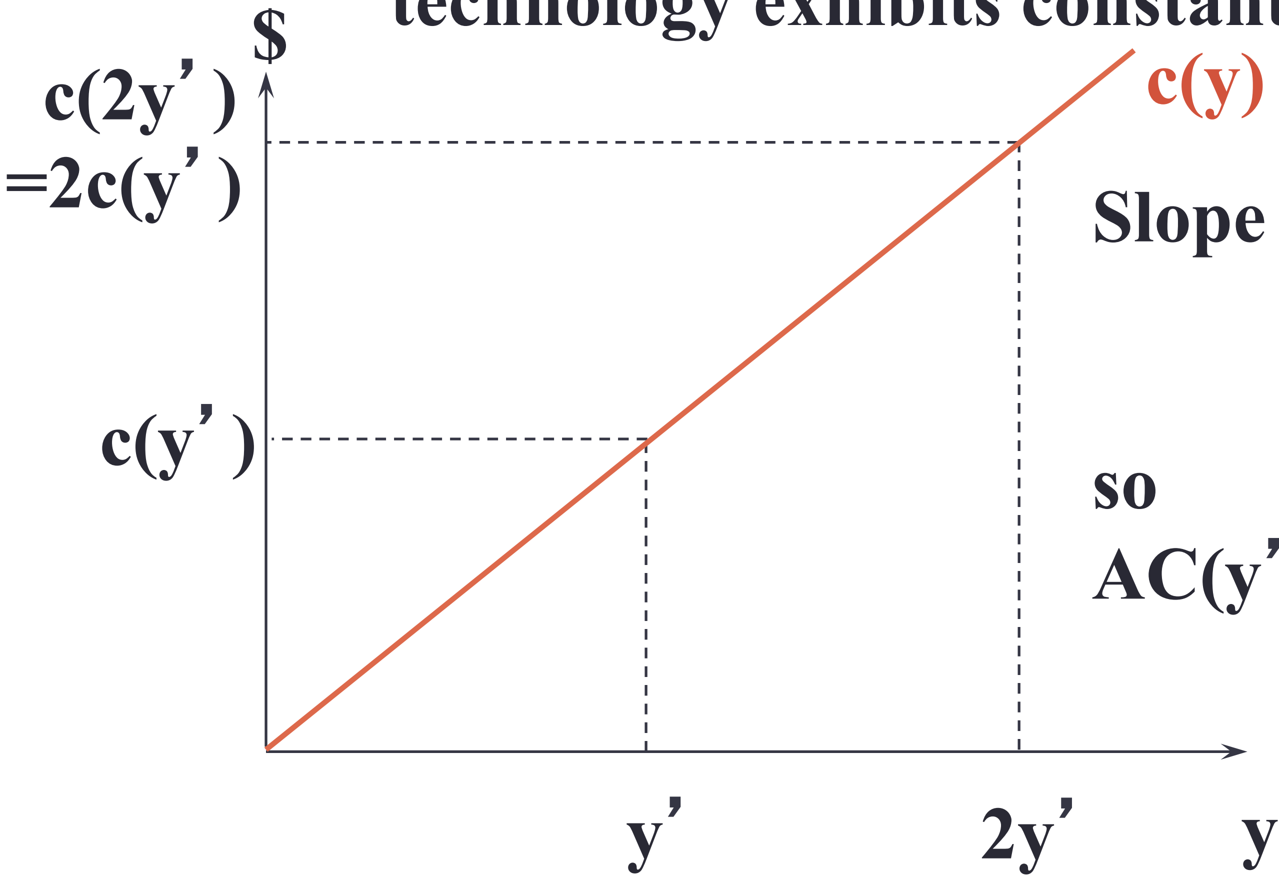
Returns-to-Scale and Total Costs

Av. cost decreases with y if the firm's technology exhibits increasing r.t.s.



Returns-to-Scale and Total Costs

Av. cost is constant when the firm's technology exhibits constant r.t.s.



Slope = c(2y')/2y'
= 2c(y')/2y'
= c(y')/y'
so
AC(y') = AC(2y').

Short-Run & Long-Run Total Costs

- In the long-run a firm can vary all of its input levels.
- Consider a firm that cannot change its input 2 level from x_2' units.
- How does the short-run total cost of producing y output units compare to the long-run total cost of producing y units of output?

Short-Run & Long-Run Total Costs

- The long-run cost-minimization problem is

$$\min_{x_1, x_2 \geq 0} w_1 x_1 + w_2 x_2$$

$$\textbf{s.t.} \;\; f(x_1, x_2) = y.$$

- The short-run cost-minimization problem is

$$\min_{x_1 \geq 0} w_1 x_1 + w_2 x_2'$$

$$\textbf{s.t.} \;\; f(x_1, x_2') = y.$$

Short-Run & Long-Run Total Costs

- The short-run cost-min. problem is the long-run problem subject to the extra constraint that $x_2 = x_2'$.
- If the long-run choice for x_2 was x_2' then the extra constraint $x_2 = x_2'$ is not really a constraint at all and so the long-run and short-run total costs of producing y output units are the same.

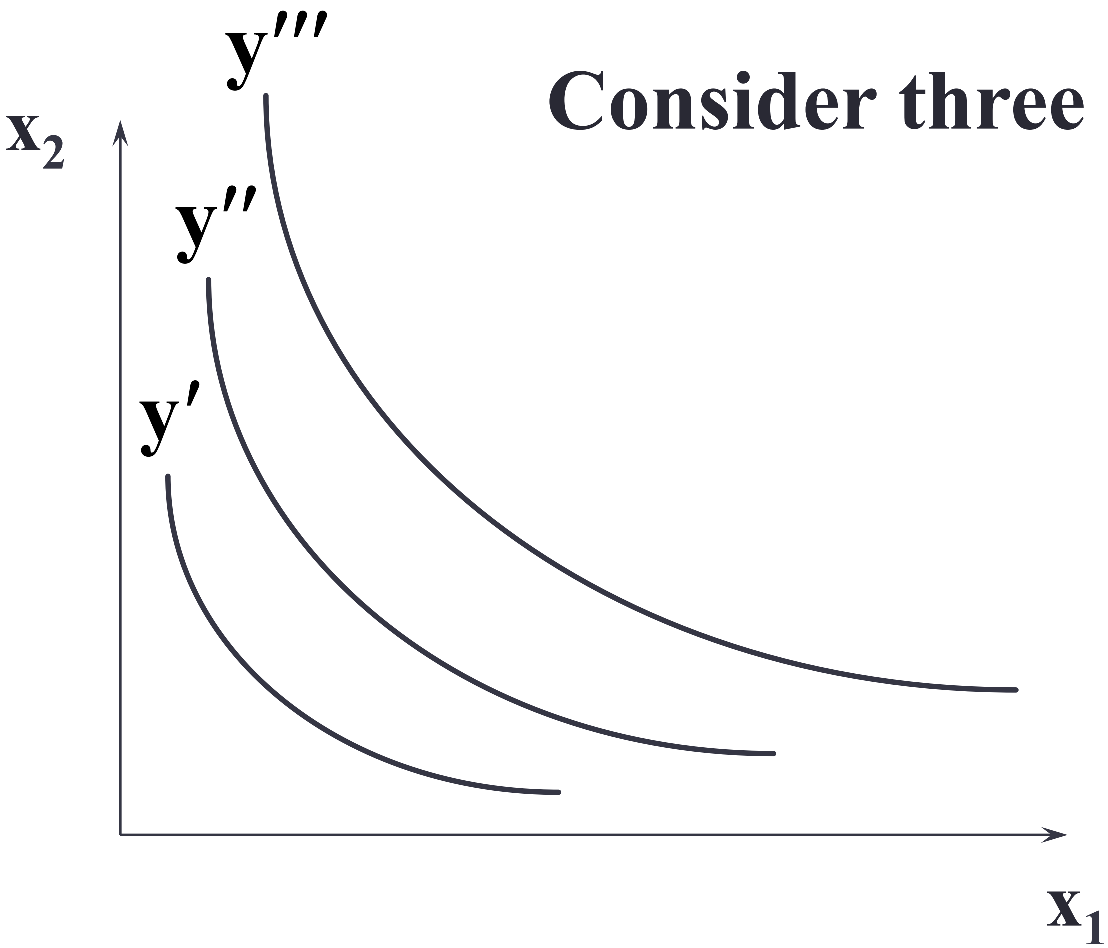
Short-Run & Long-Run Total Costs

- The short-run cost-min. problem is therefore the long-run problem subject to the extra constraint that $x_2 = x_2''$.

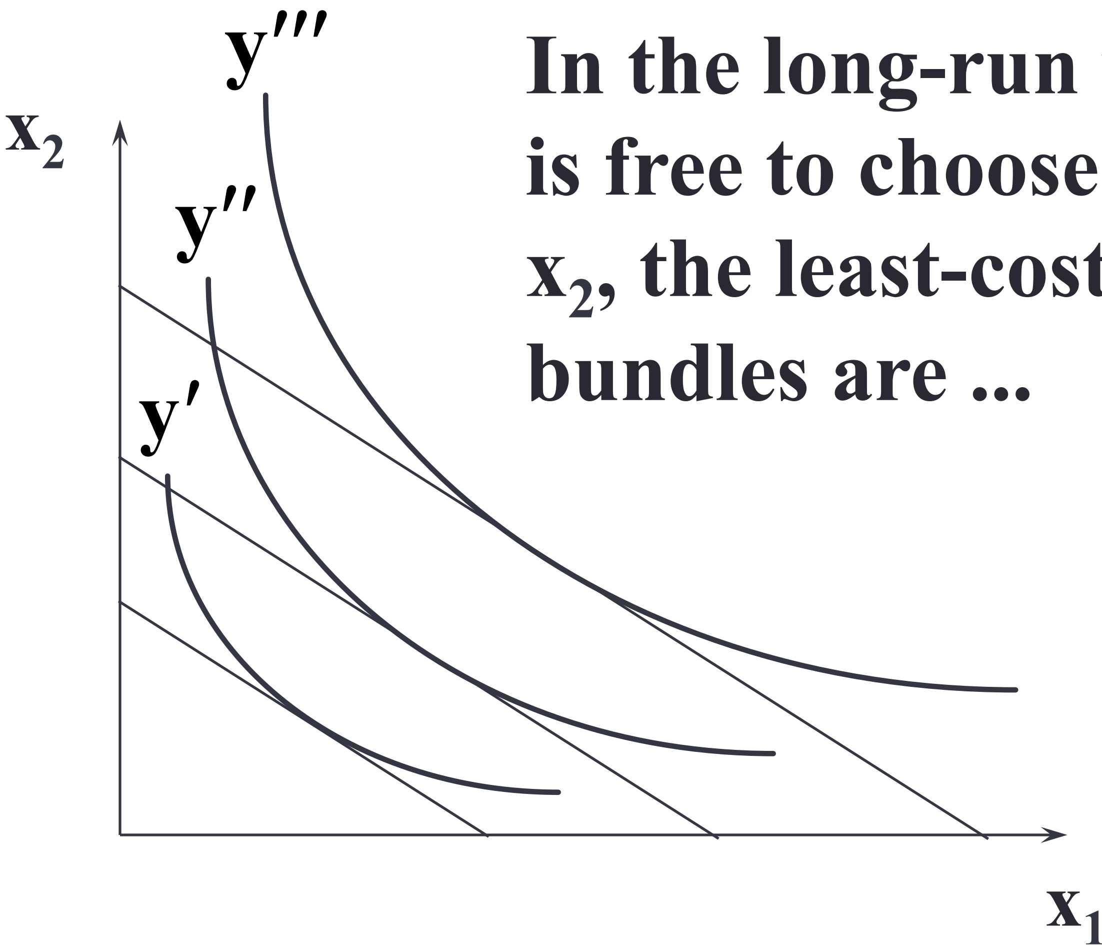
- But, if the long-run choice for $x_2 \neq x_2''$ then the extra constraint $x_2 = x_2''$ prevents the firm in this short-run from achieving its long-run production cost, causing the short-run total cost to exceed the long-run total cost of producing y output units.

Short-Run & Long-Run Total Costs

Consider three output levels.

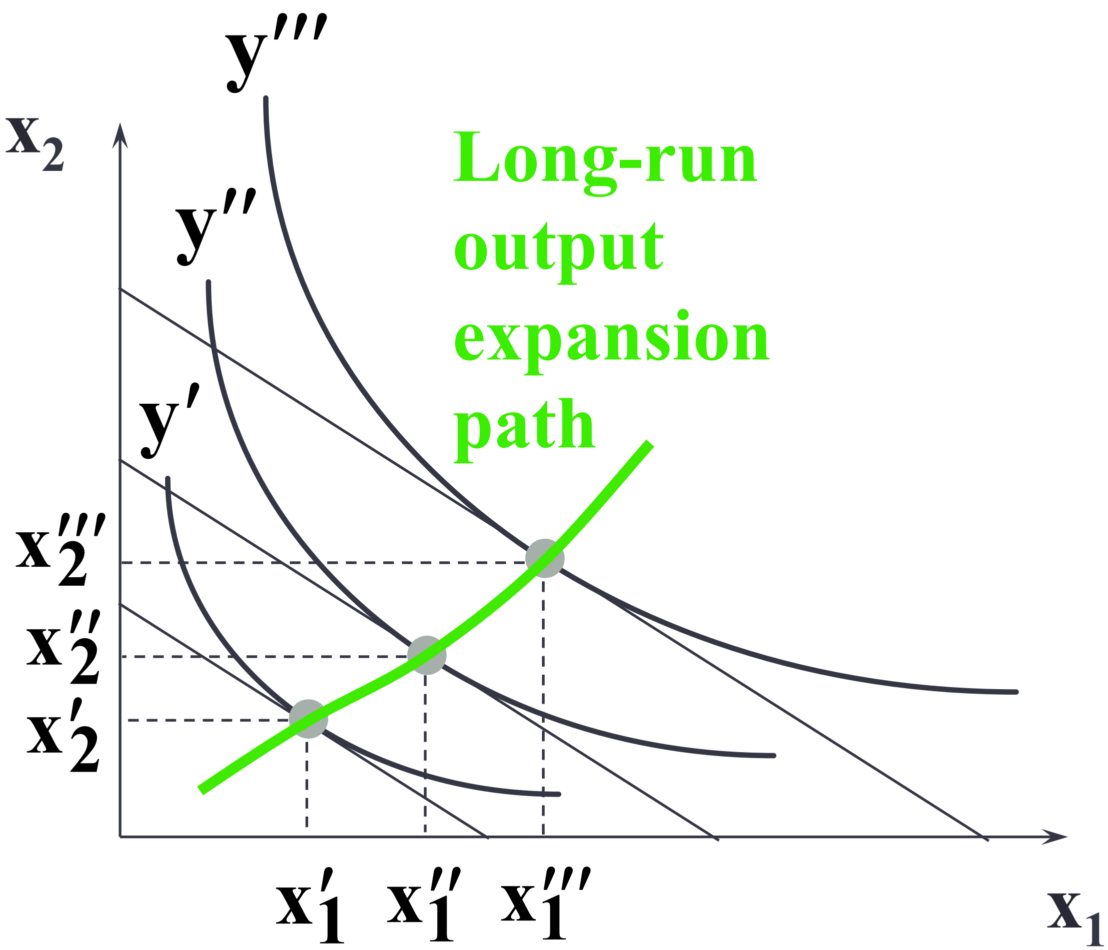


Short-Run & Long-Run Total Costs

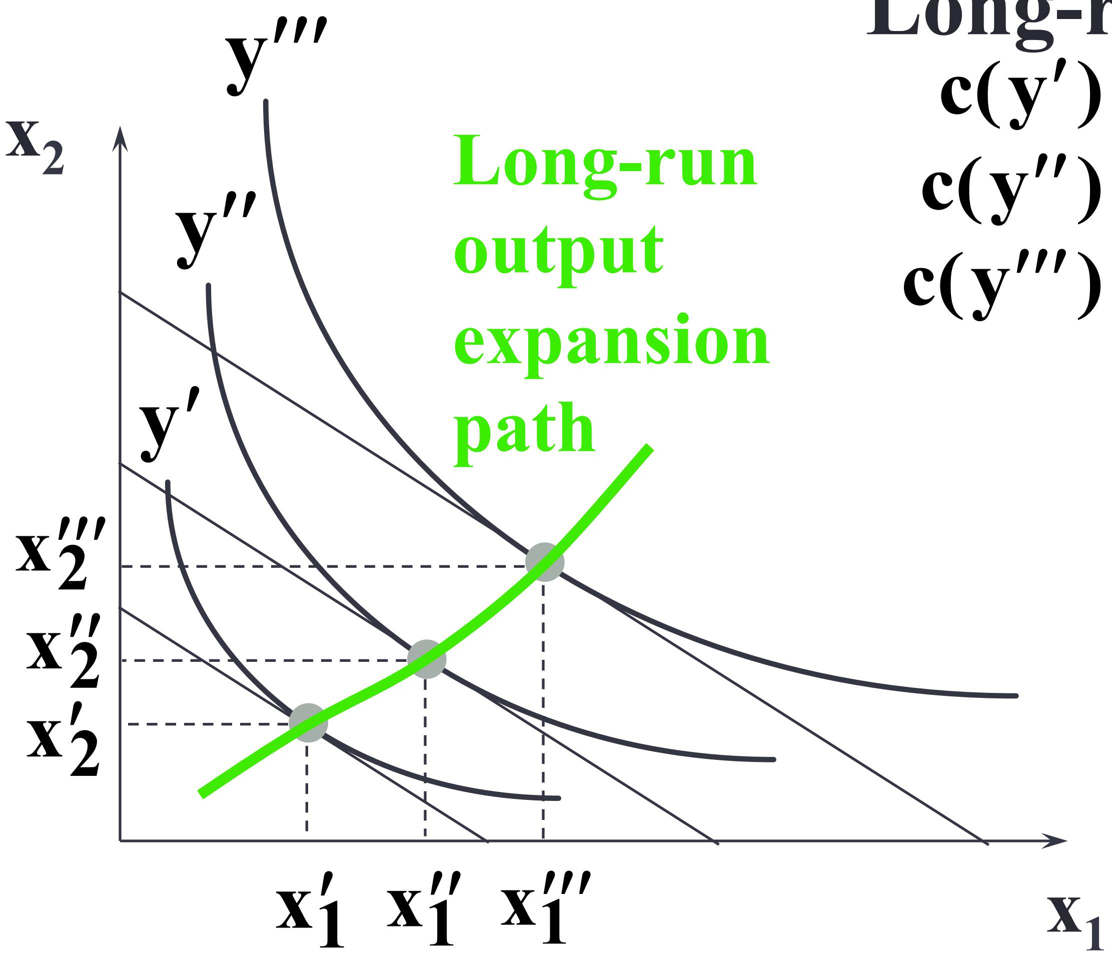


In the long-run when the firm is free to choose both x_1 and x_2, the least-costly input bundles are ...

Short-Run & Long-Run Total Costs



Short-Run & Long-Run Total Costs



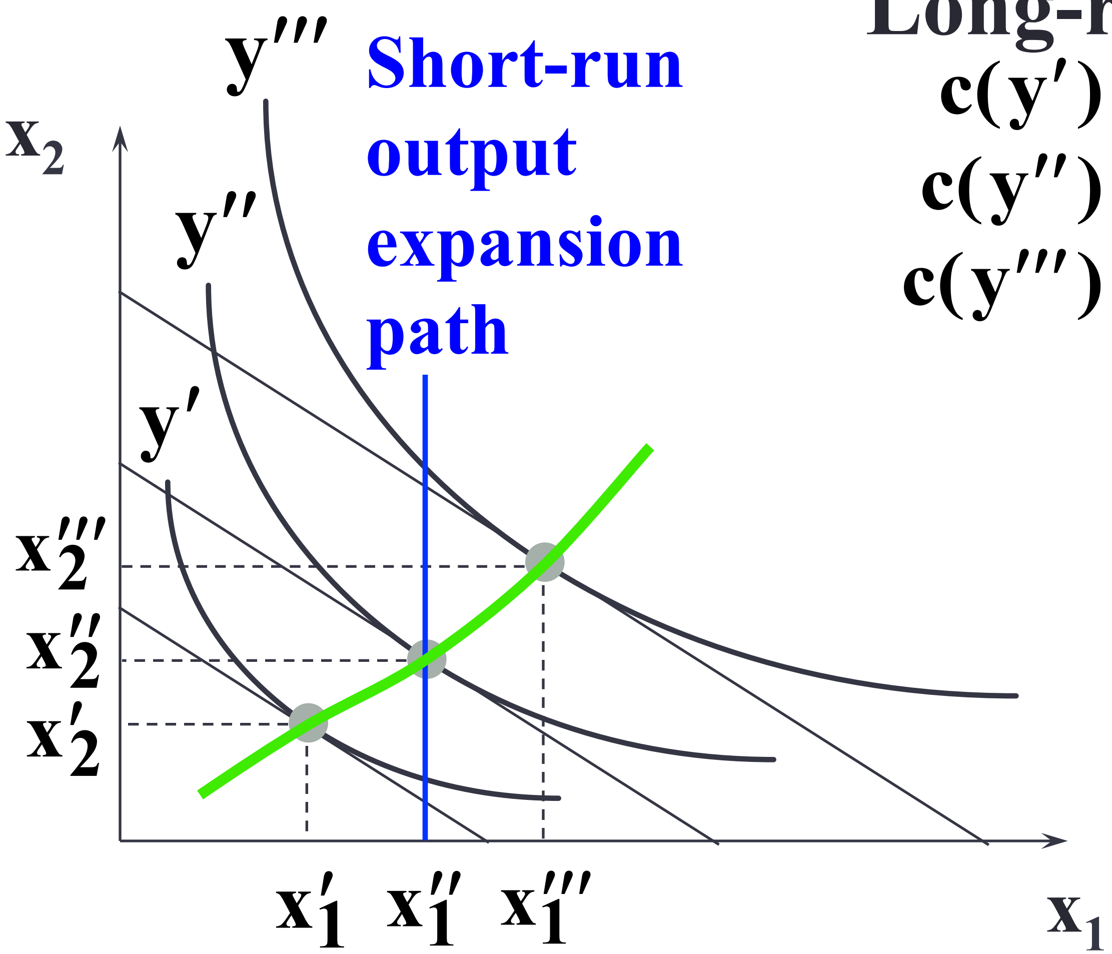
Long-run costs are:

$$c(y') = w_1 x_1' + w_2 x_2'$$
$$c(y'') = w_1 x_1'' + w_2 x_2''$$
$$c(y''') = w_1 x_1''' + w_2 x_2'''$$

Short-Run & Long-Run Total Costs

- Now suppose the firm becomes subject to the short-run constraint that $x_1 = x_1''$.

Short-Run & Long-Run Total Costs



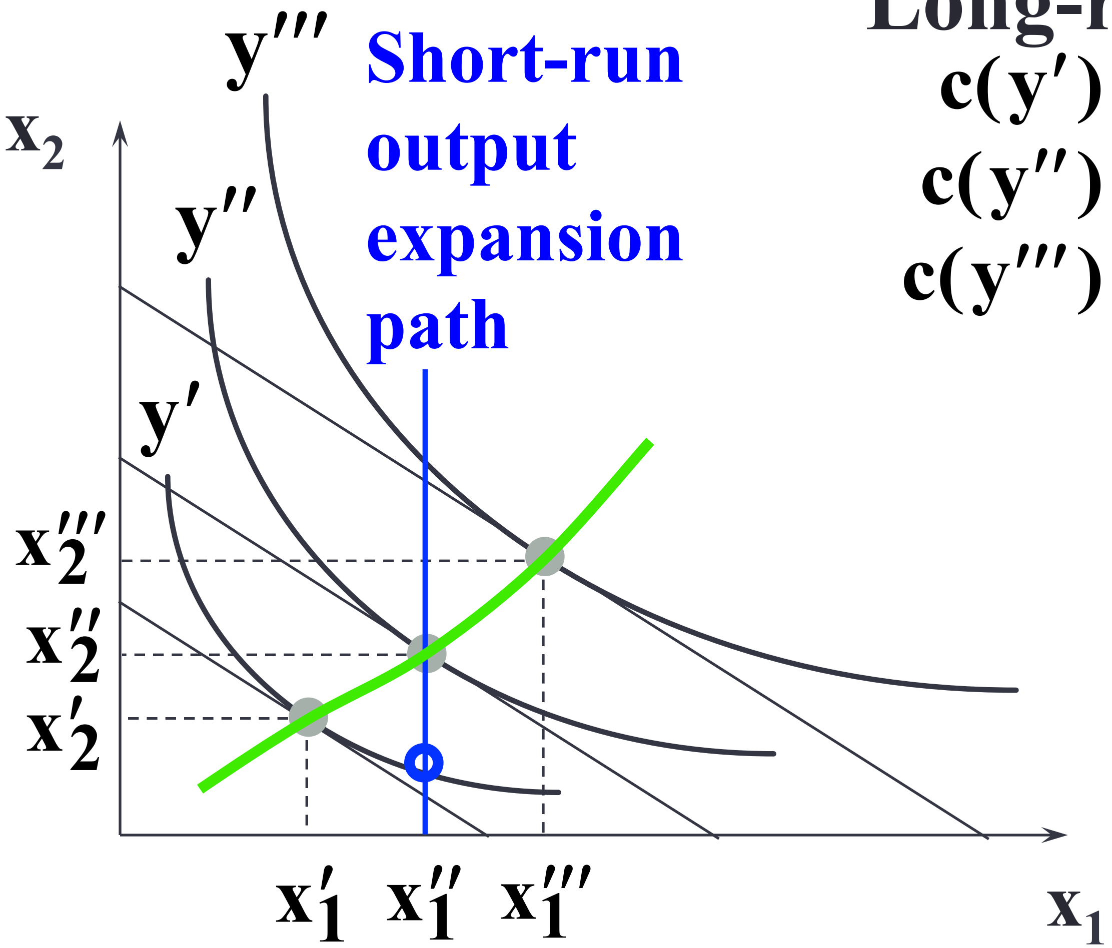
Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$

$$c(y'') = w_1x_1'' + w_2x_2'''$$

$$c(y''') = w_1x_1''' + w_2x_2'''$$

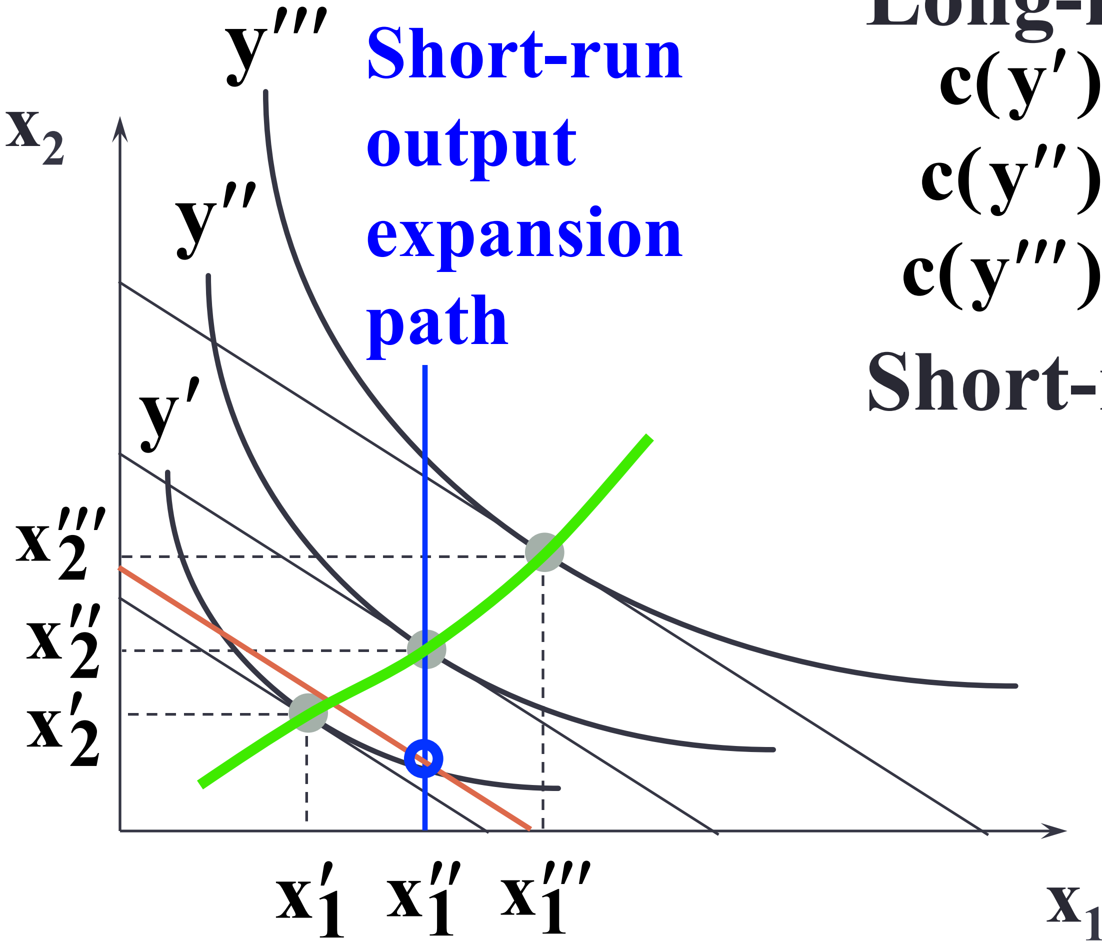
Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$
$$c(y'') = w_1x_1'' + w_2x_2''$$
$$c(y''') = w_1x_1''' + w_2x_2'''$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$

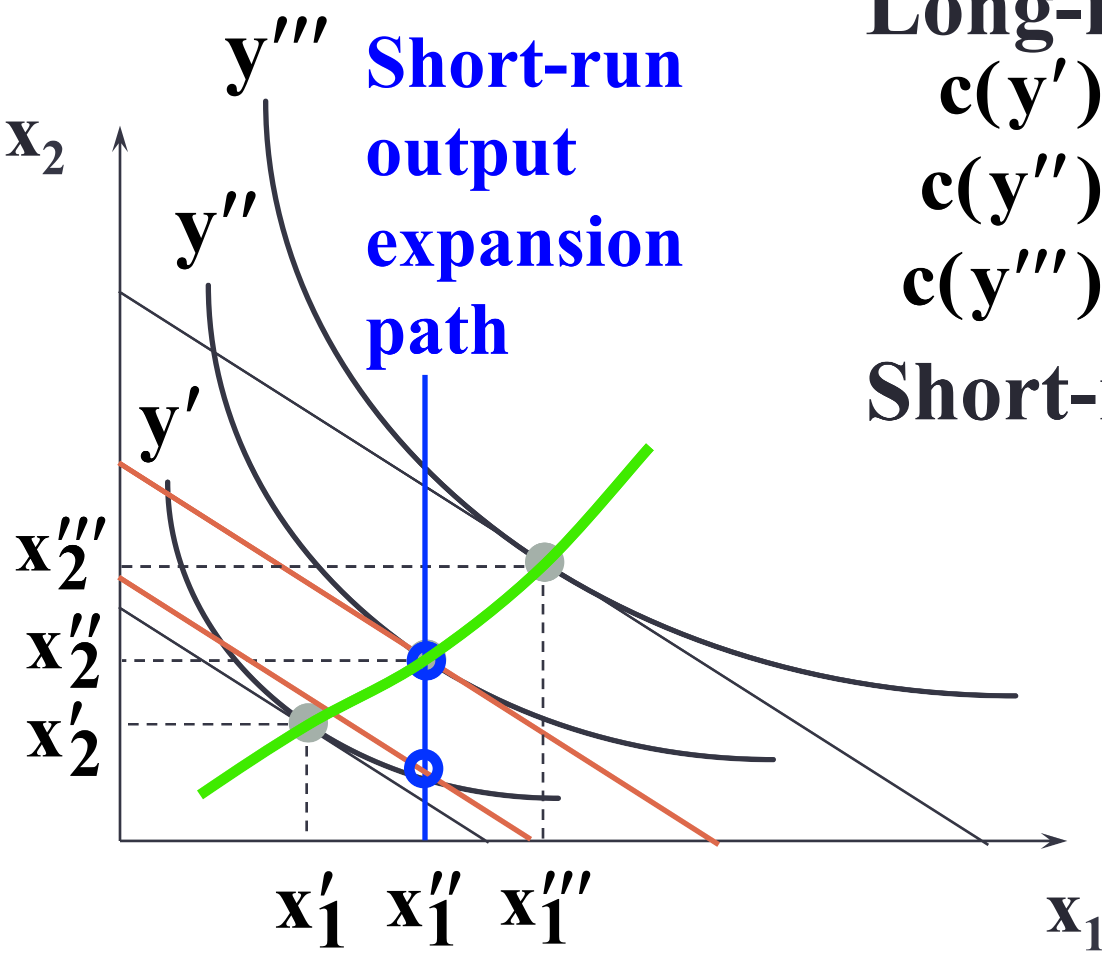
$$c(y'') = w_1x_1'' + w_2x_2'''$$

$$c(y''') = w_1x_1''' + w_2x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$

Short-Run & Long-Run Total Costs



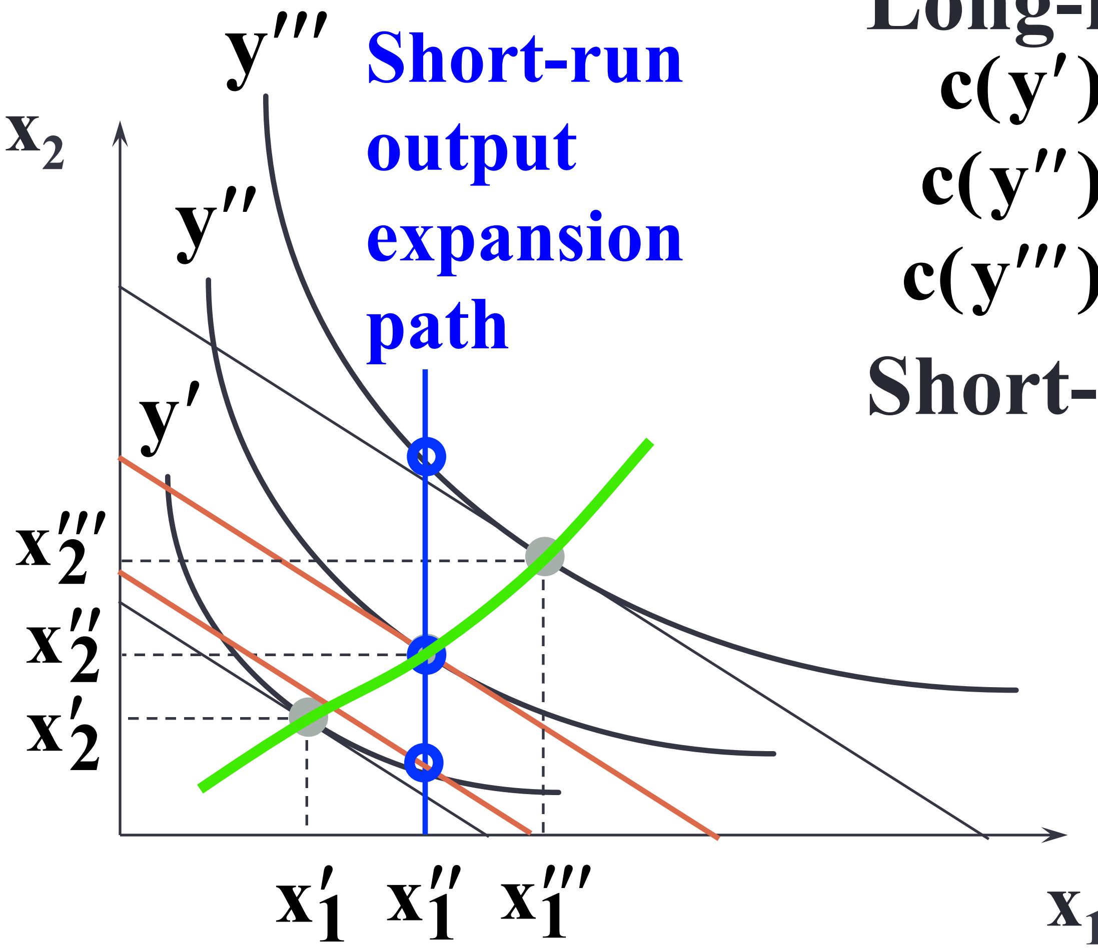
Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$
$$c(y'') = w_1x_1'' + w_2x_2'''$$
$$c(y''') = w_1x_1''' + w_2x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$
$$c_s(y'') = c(y'')$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$
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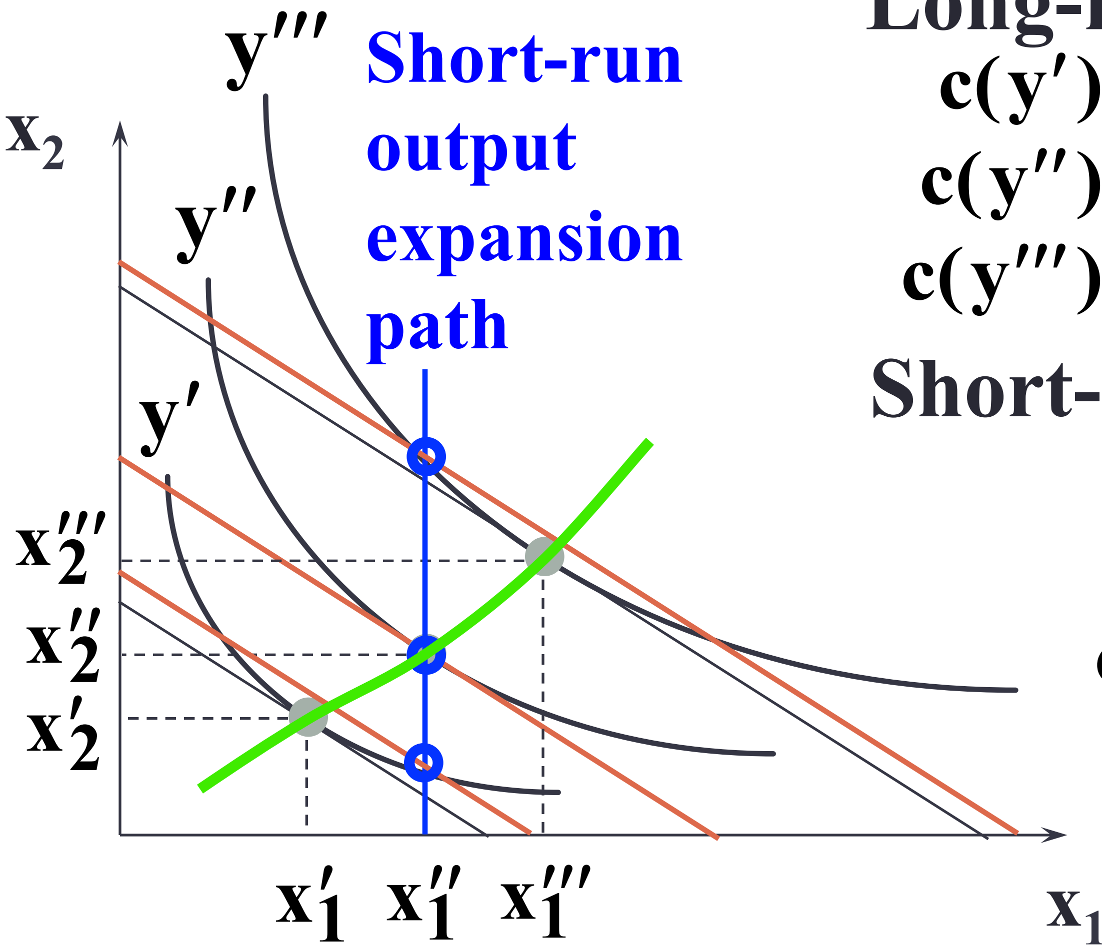
$$c(y''') = w_1x_1''' + w_2x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$
$$c_s(y'') = c(y'')$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$
$$c(y'') = w_1x_1'' + w_2x_2''$$
$$c(y''') = w_1x_1''' + w_2x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$
$$c_s(y'') = c(y'')$$
$$c_s(y''') > c(y''')$$

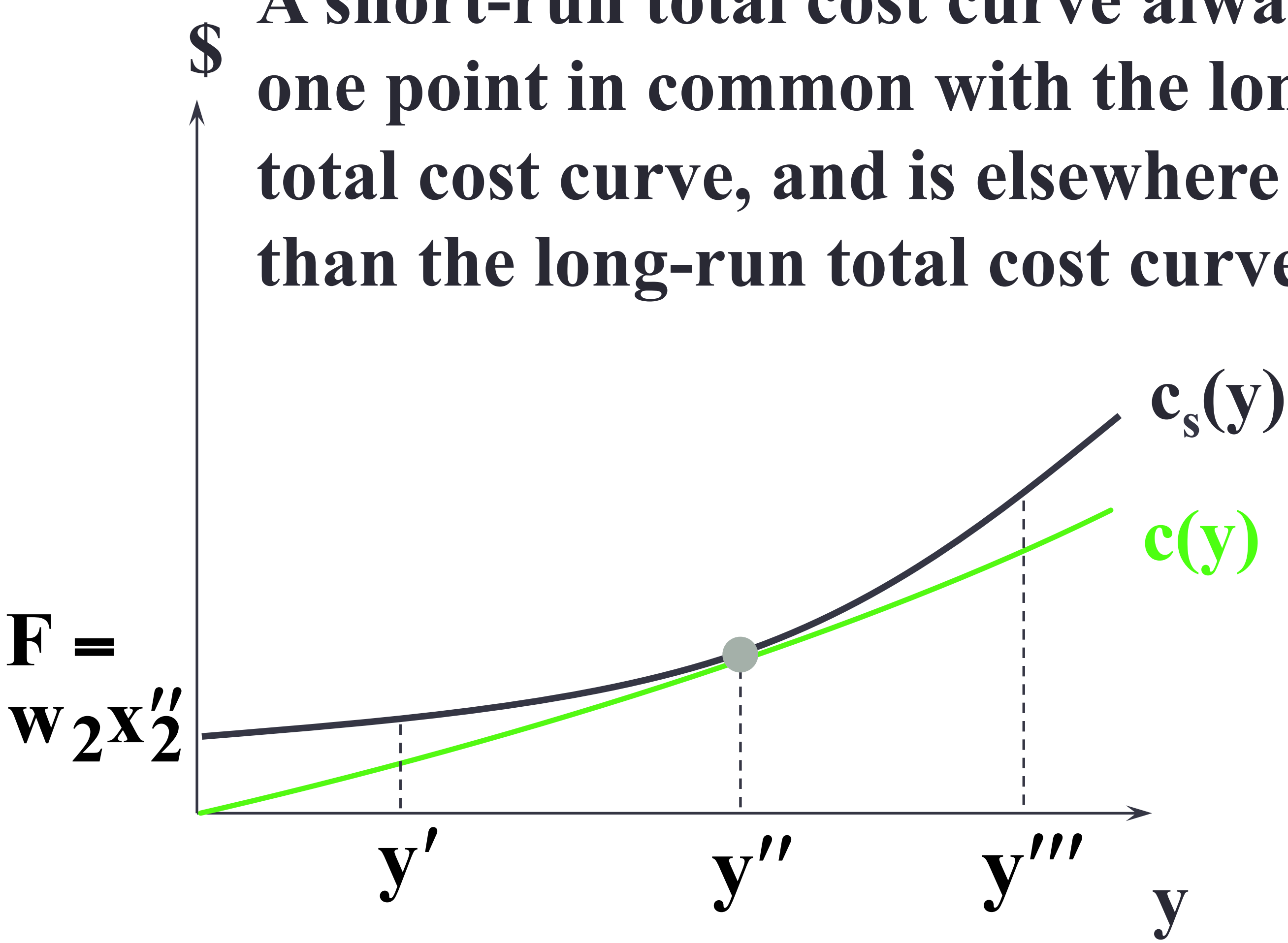
Short-Run & Long-Run Total Costs

- Short-run total cost exceeds long-run total cost except for the output level where the short-run input level restriction is the long-run input level choice.

- This says that the long-run total cost curve always has one point in common with any particular short-run total cost curve.

Short-Run & Long-Run Total Costs

A short-run total cost curve always has one point in common with the long-run total cost curve, and is elsewhere higher than the long-run total cost curve.



Summary

- For any chosen level of output, a firm will want to produce that output at the **minimum** total cost.
  - With **constant returns to scale** technology, total costs grow linearly with output and average cost is constant.
  - With **increasing returns to scale** technology, total costs grow sub-linearly with output and average cost is decreasing.
  - With **decreasing returns to scale** technology, total costs grow super-linearly and average cost is increasing.
- Short run total costs are always at least as large as long run costs.

22

Cost Curves

Varian, H. 2010. *Intermediate Microeconomics*, W.W. Norton.

Types of Cost Curves

- A total cost curve is the graph of a firm's total cost function.
- A variable cost curve is the graph of a firm's variable cost function.
- An average total cost curve is the graph of a firm's average total cost function.

Types of Cost Curves

- An average variable cost curve is the graph of a firm's average variable cost function.
- An average fixed cost curve is the graph of a firm's average fixed cost function.
- A marginal cost curve is the graph of a firm's marginal cost function.

Types of Cost Curves

- How are these cost curves related to each other?
- How are a firm's long-run and short-run cost curves related?

Fixed, Variable & Total Cost Functions

- F is the total cost to a firm of its short-run fixed inputs. F, the firm's fixed cost, does not vary with the firm's output level.

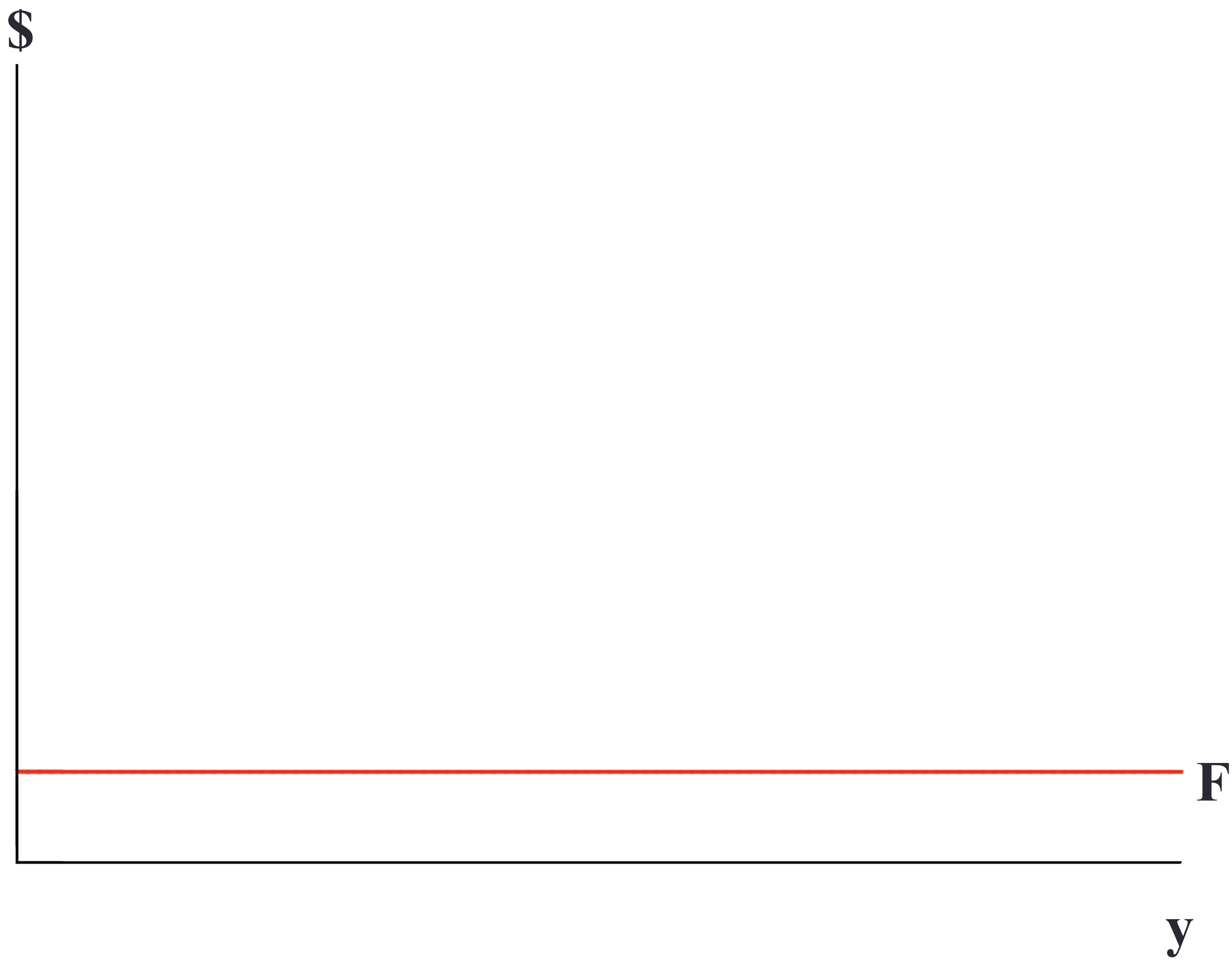
- $c_v(y)$ is the total cost to a firm of its variable inputs when producing y output units. $c_v(y)$ is the firm's variable cost function.

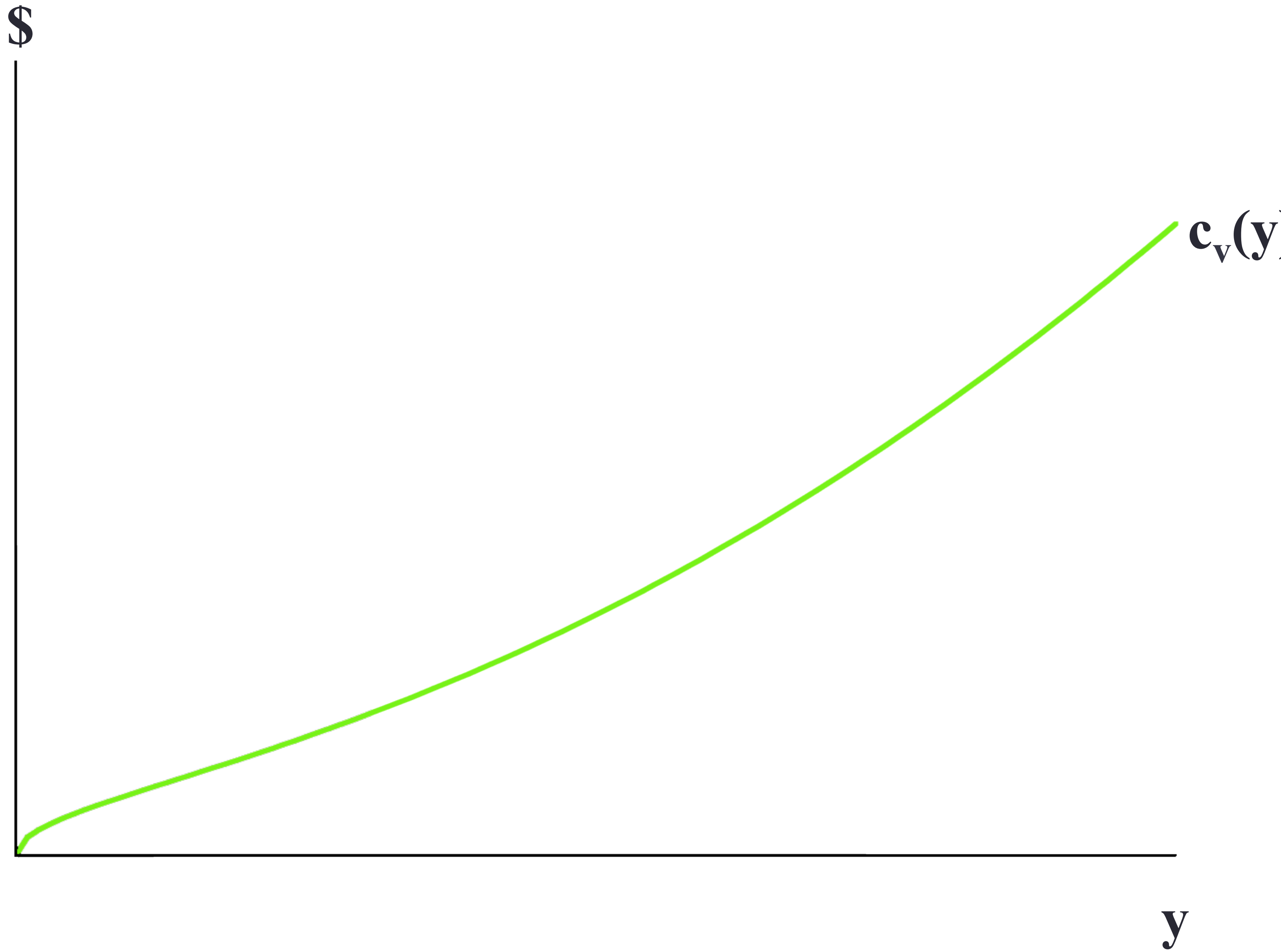
- $c_v(y)$ depends upon the levels of the fixed inputs.

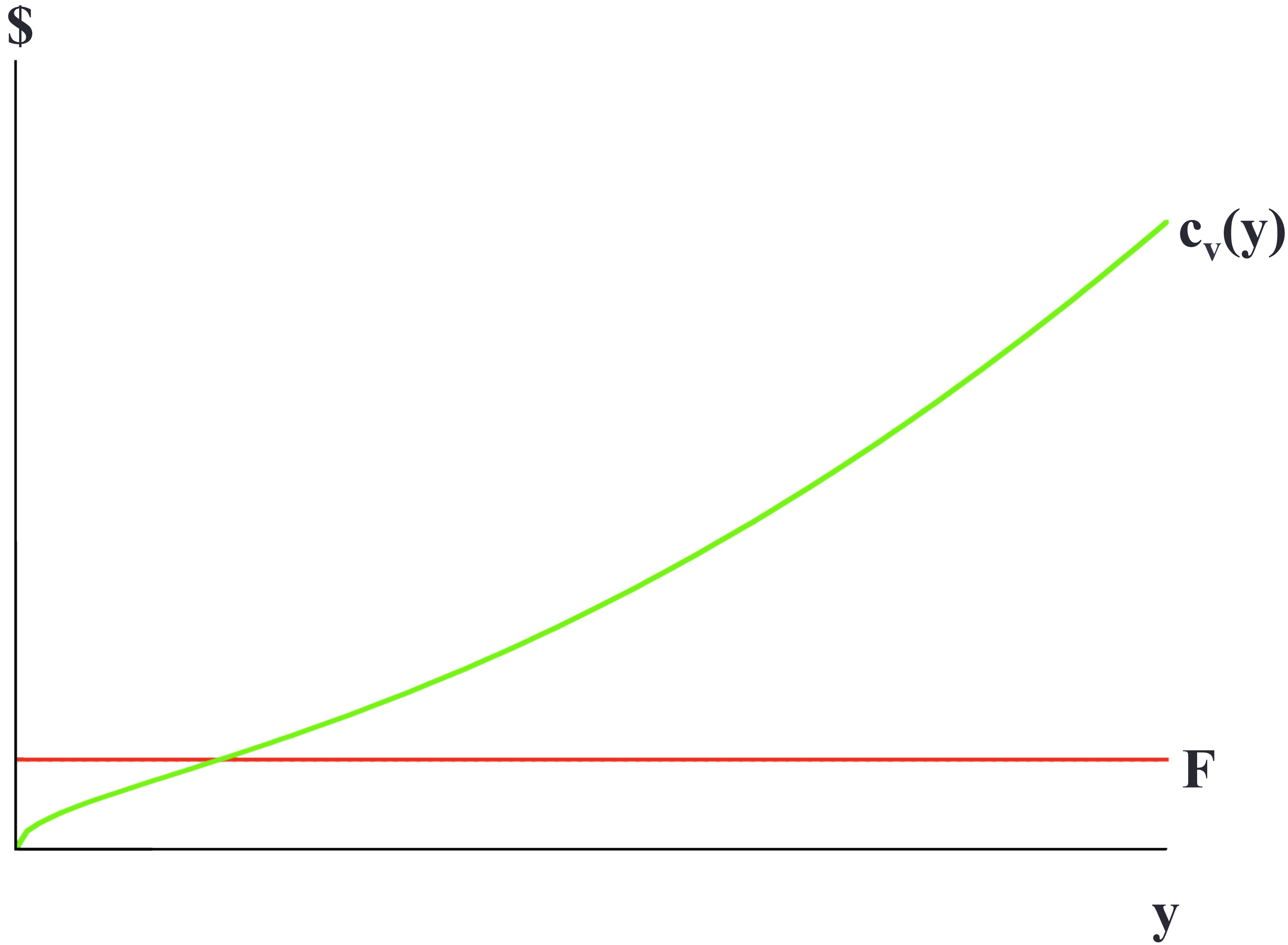
Fixed, Variable & Total Cost Functions

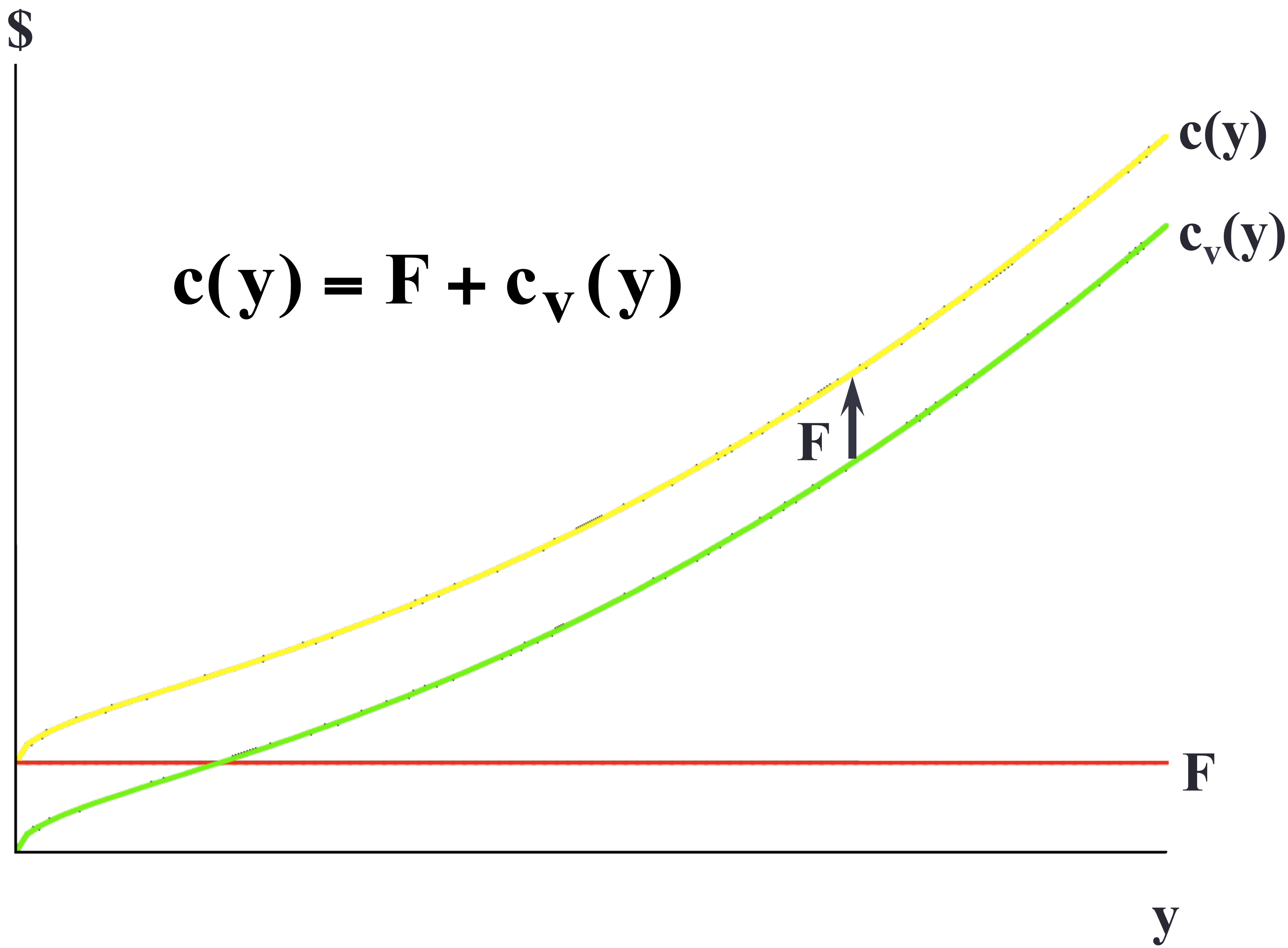
- c(y) is the total cost of all inputs, fixed and variable, when producing y output units. c(y) is the firm's total cost function;

$$c(y) = F + c_v(y).$$


$
F
y


$
$c_v(y)$
y


$
$c_v(y)$
F
y


$
c(y)
$c_v(y)$
$c(y) = F + c_v(y)$
F
F
y

Av. Fixed, Av. Variable & Av. Total Cost Curves

- The firm's total cost function is

$$c(y) = F + c_v(y).$$

- For y > 0, the firm's average total cost function is

$$AC(y) = \frac{F}{y} + \frac{c_v(y)}{y}$$

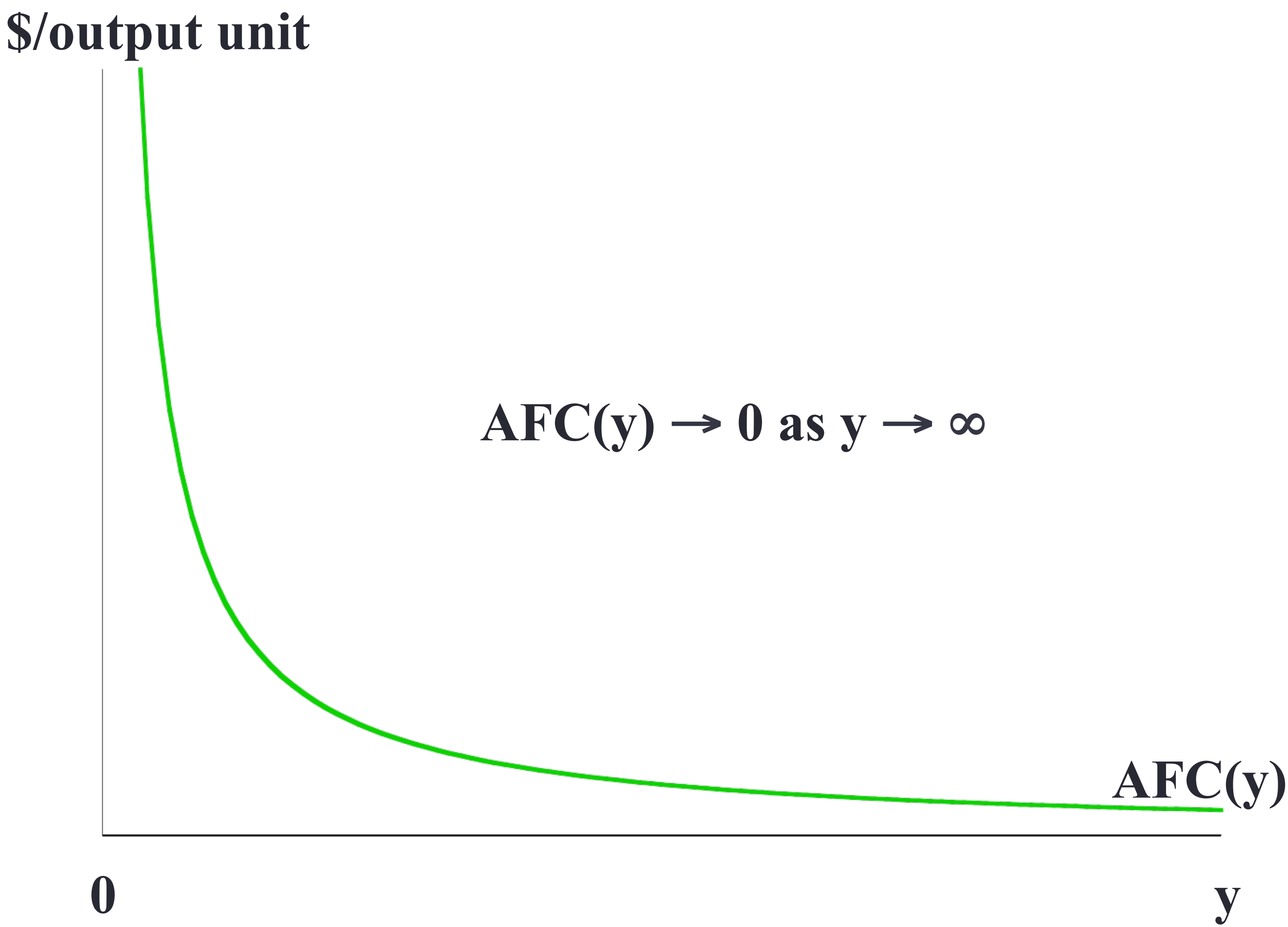
$$= AFC(y) + AVC(y).$$

Av. Fixed, Av. Variable & Av. Total Cost Curves

- What does an average fixed cost curve look like?

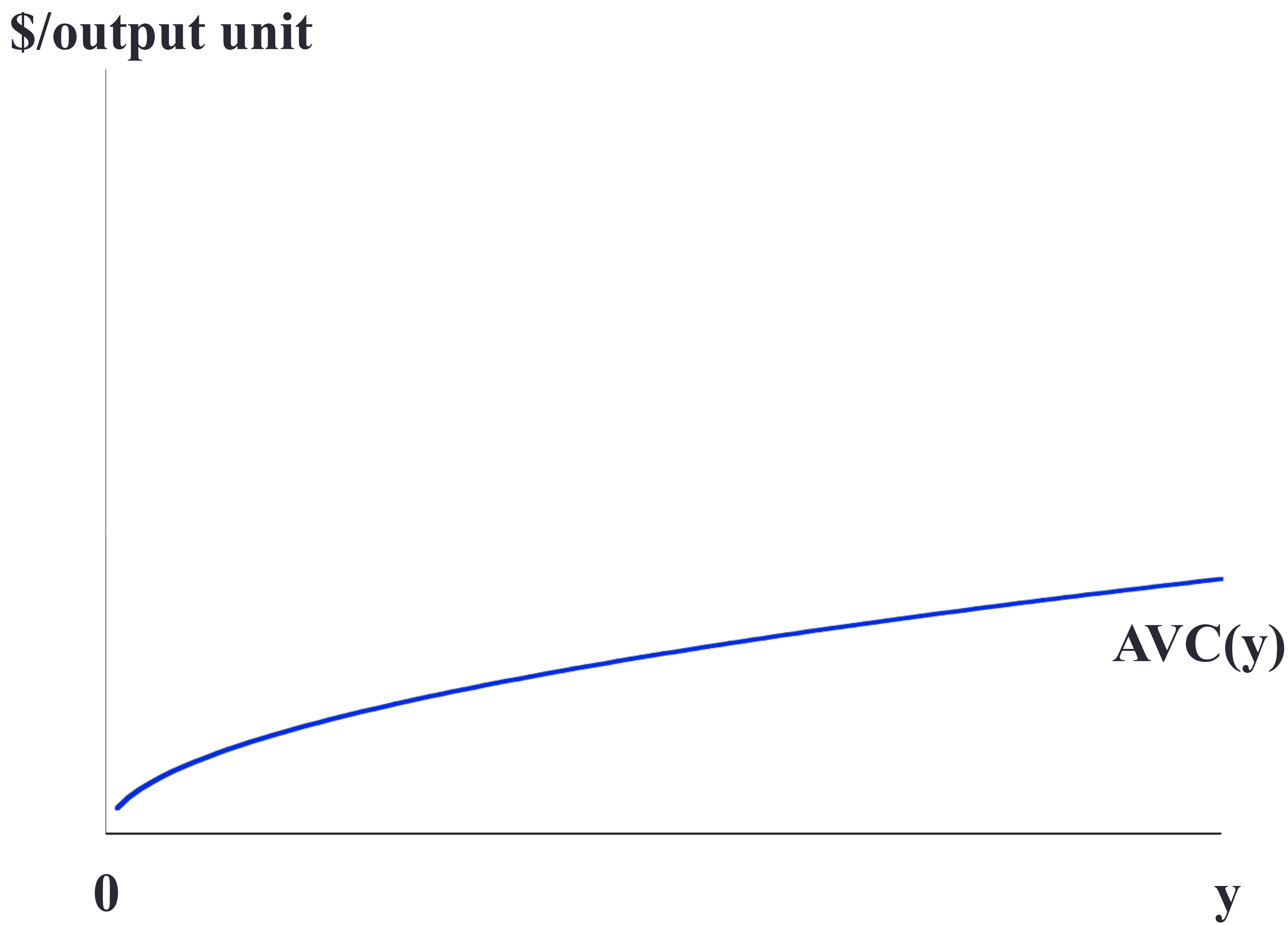
$$\mathbf{AFC(y) = \frac{F}{y}}$$

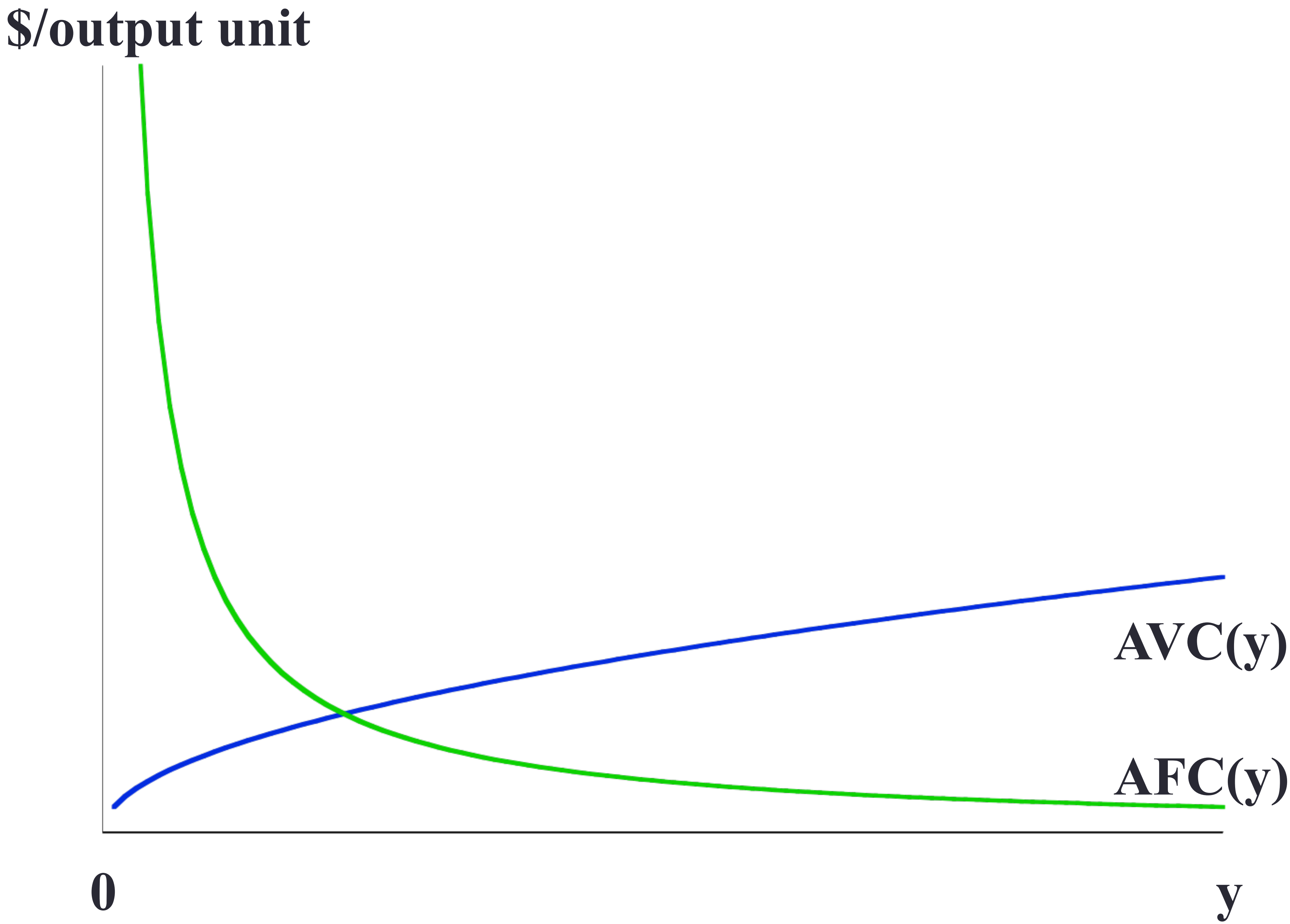
- AFC(y) is a rectangular hyperbola so its graph looks like ...


$/output unit
AFC(y) → 0 as y → ∞
AFC(y)
0
y

Av. Fixed, Av. Variable & Av. Total Cost Curves

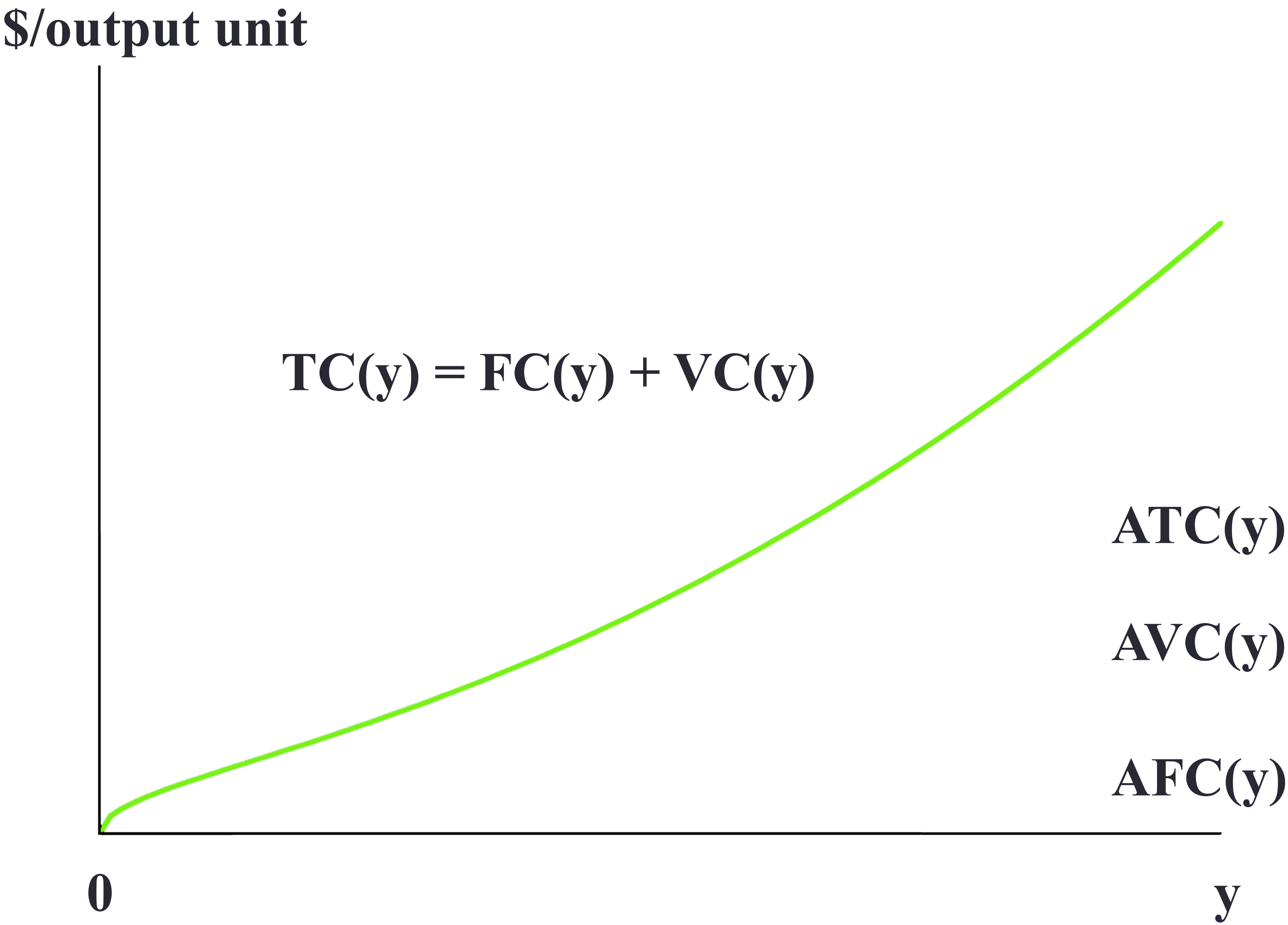
- In a short-run with a fixed amount of at least one input, the Law of Diminishing (Marginal) Returns must apply, causing the firm's average variable cost of production to increase eventually.


$/output unit
AVC(y)
0
y

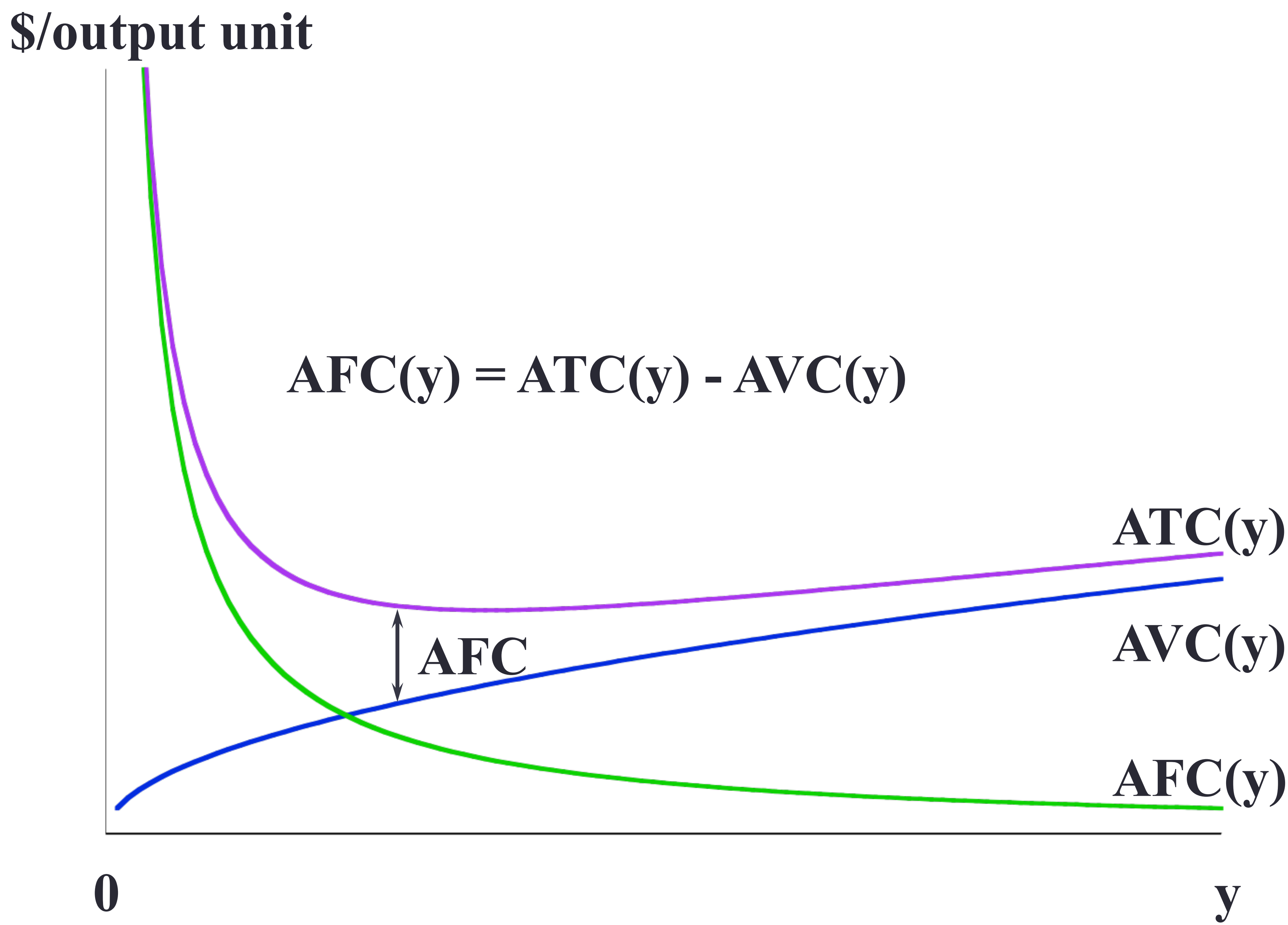

$/output unit
AVC(y)
AFC(y)
0
y

Av. Fixed, Av. Variable & Av. Total Cost Curves

- And $ATC(y) = AFC(y) + AVC(y)$

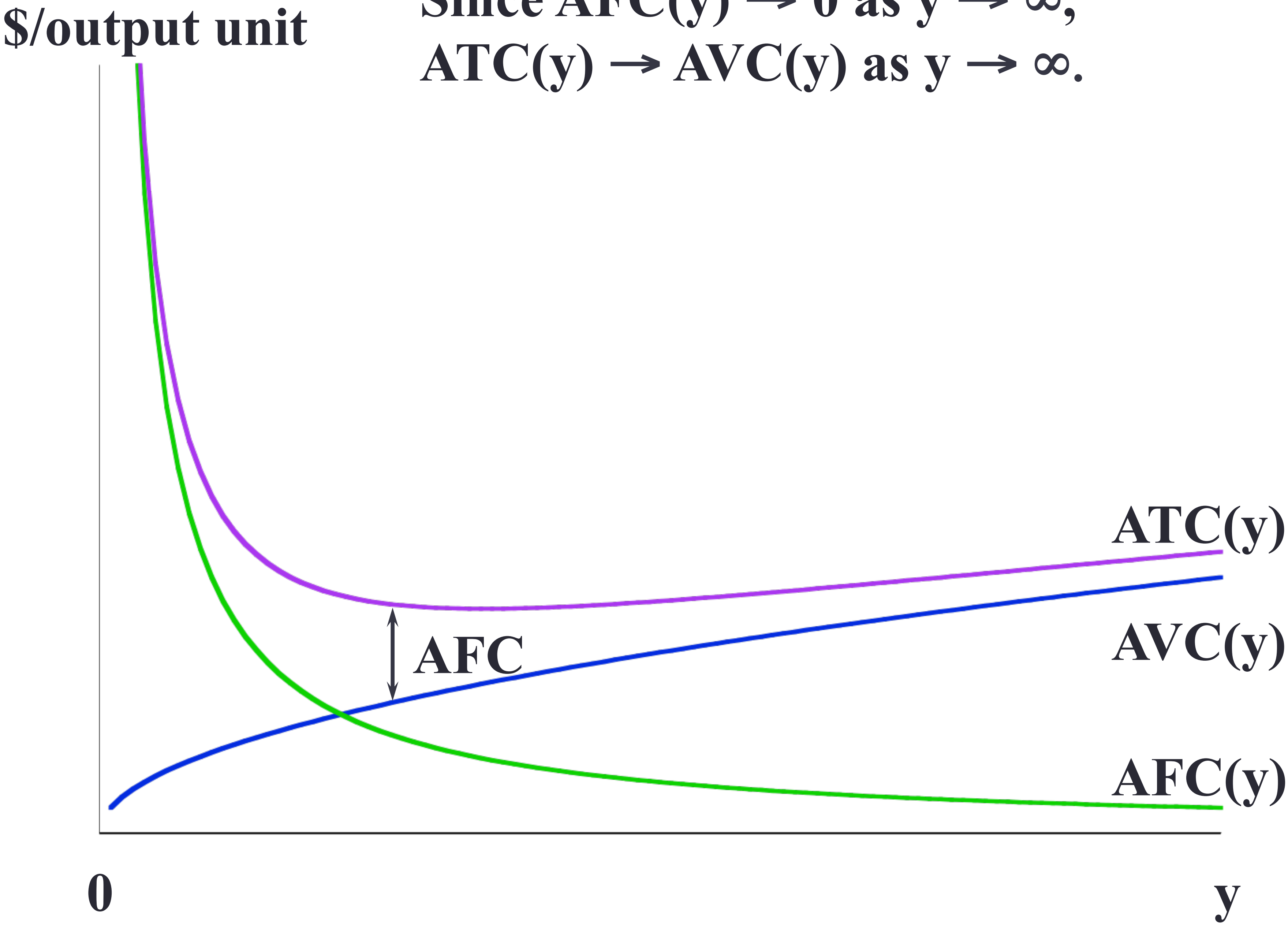


$/output unit
TC(y) = FC(y) + VC(y)
ATC(y)
AVC(y)
AFC(y)
0
y



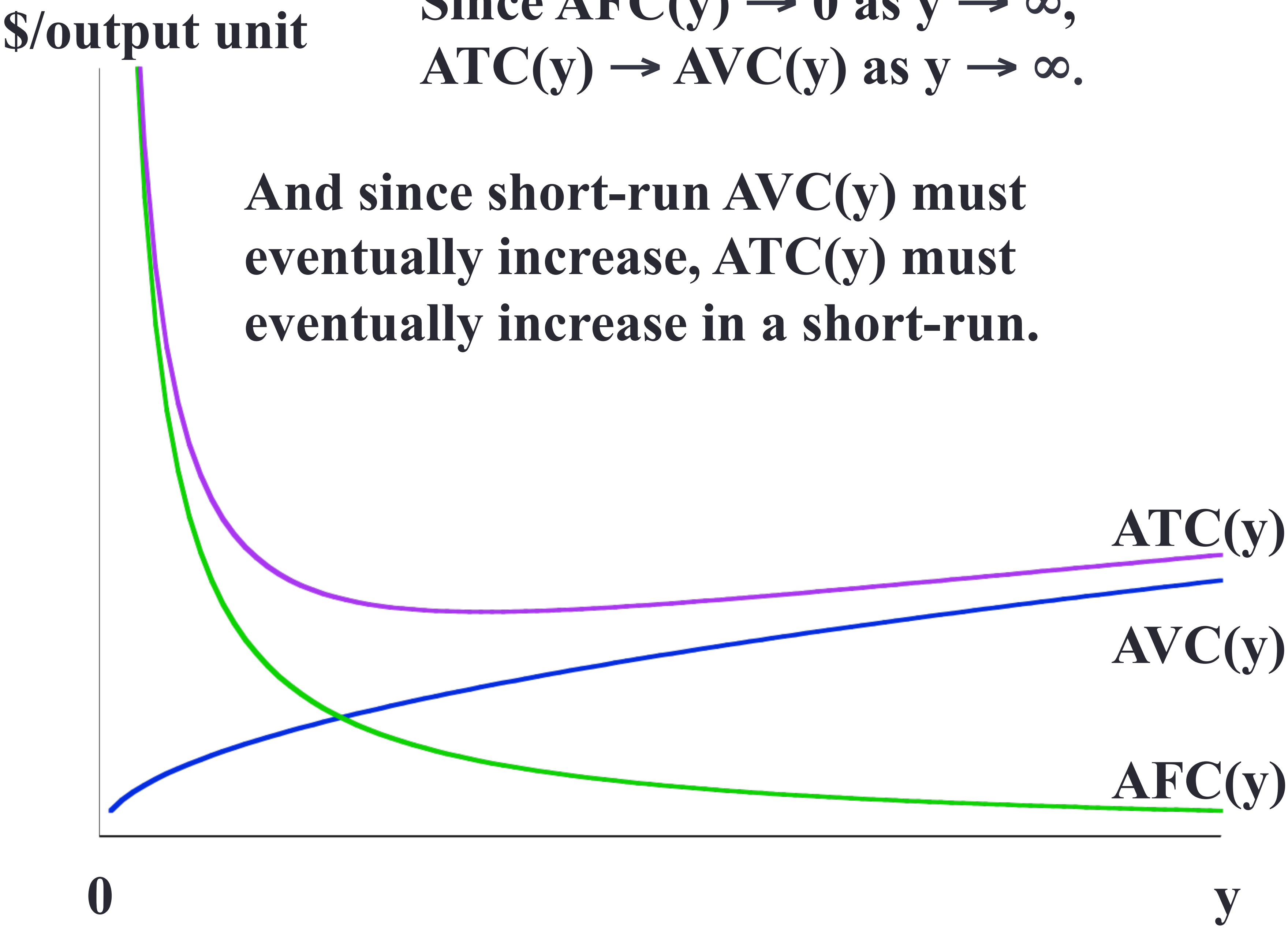
$/output unit
AFC(y) = ATC(y) - AVC(y)
ATC(y)
AFC
AVC(y)
AFC(y)
0
y

Since $AFC(y) \to 0$ as $y \to \infty$,
$ATC(y) \to AVC(y)$ as $y \to \infty$.



Since AFC(y) → 0 as y → ∞,
ATC(y) → AVC(y) as y → ∞.

And since short-run AVC(y) must eventually increase, ATC(y) must eventually increase in a short-run.



Marginal Cost Function

- Marginal cost is the rate-of-change of variable production cost as the output level changes. That is,

$$\mathbf{MC(y) = \frac{\partial\, c_v(y)}{\partial\, y}.}$$

Marginal Cost Function

- The firm's total cost function is

$$c(y) = F + c_v(y)$$

- and the fixed cost F does not change with the output level y, so

$$MC(y) = \frac{\partial\, c_v(y)}{\partial\, y} = \frac{\partial\, c(y)}{\partial\, y}.$$

- MC is the slope of both the variable cost and the total cost functions.

Marginal and Variable Cost Functions

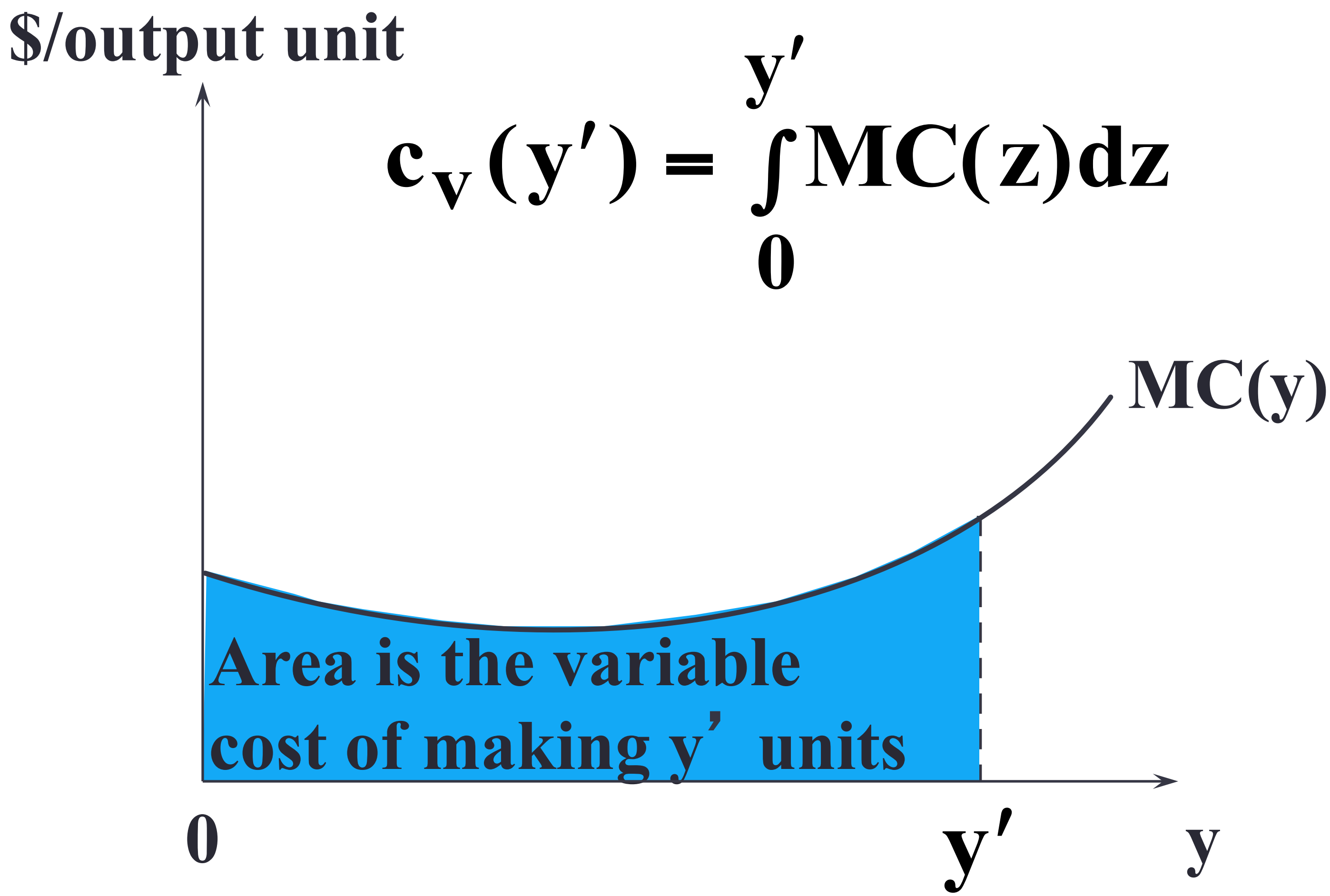
- Since MC(y) is the derivative of $c_v(y)$, $c_v(y)$ must be the integral of MC(y). That is,

$$MC(y) = \frac{\partial c_v(y)}{\partial y}$$

$$\Rightarrow \quad c_v(y) = \int_0^y MC(z)dz.$$

Marginal and Variable Cost Functions

$$c_v(y') = \int_0^{y'} MC(z)\,dz$$



Marginal & Average Cost Functions

- How is marginal cost related to average variable cost?

Marginal & Average Cost Functions

Since $$\mathbf{AVC(y) = \frac{c_v(y)}{y}},$$

$$\frac{\partial\, AVC(y)}{\partial\, y} = \frac{y \times MC(y) - 1 \times c_v(y)}{y^2}.$$

Marginal & Average Cost Functions

Since $$\text{AVC}(y) = \frac{c_v(y)}{y},$$

$$\frac{\partial\, \text{AVC}(y)}{\partial\, y} = \frac{y \times \text{MC}(y) - 1 \times c_v(y)}{y^2}.$$

Therefore,

$$\frac{\partial\, \text{AVC}(y)}{\partial\, y} \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \text{as} \quad y \times \text{MC}(y) \begin{matrix} > \\ = \\ < \end{matrix} c_v(y).$$

Marginal & Average Cost Functions

Since $$\text{AVC}(y) = \frac{c_v(y)}{y},$$

$$\frac{\partial\, \text{AVC}(y)}{\partial\, y} = \frac{y \times \text{MC}(y) - 1 \times c_v(y)}{y^2}.$$

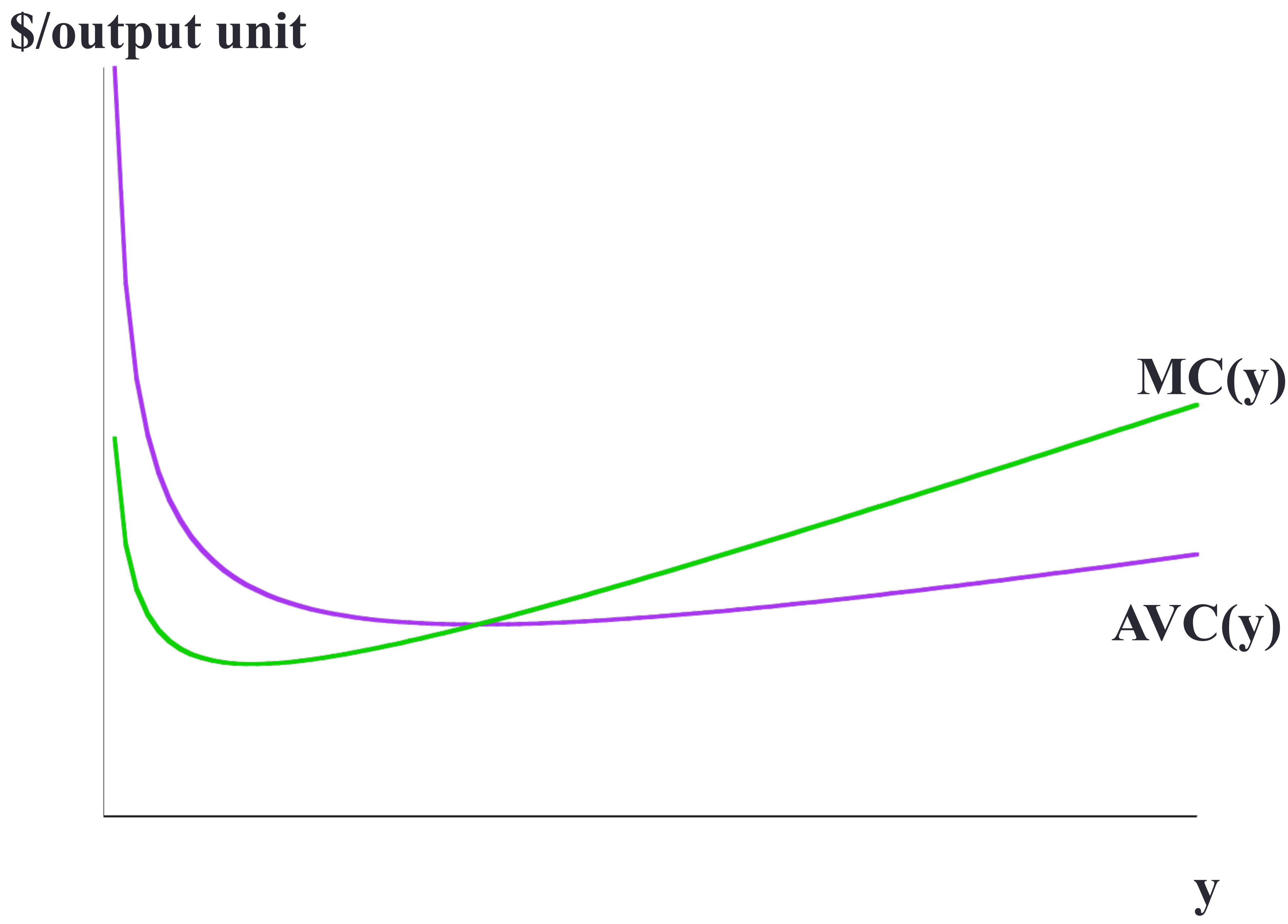
Therefore,

$$\frac{\partial\, \text{AVC}(y)}{\partial\, y} \underset{<}{\overset{>}{=}} 0 \quad \text{as} \quad y \times \text{MC}(y) \underset{<}{\overset{>}{=}} c_v(y).$$

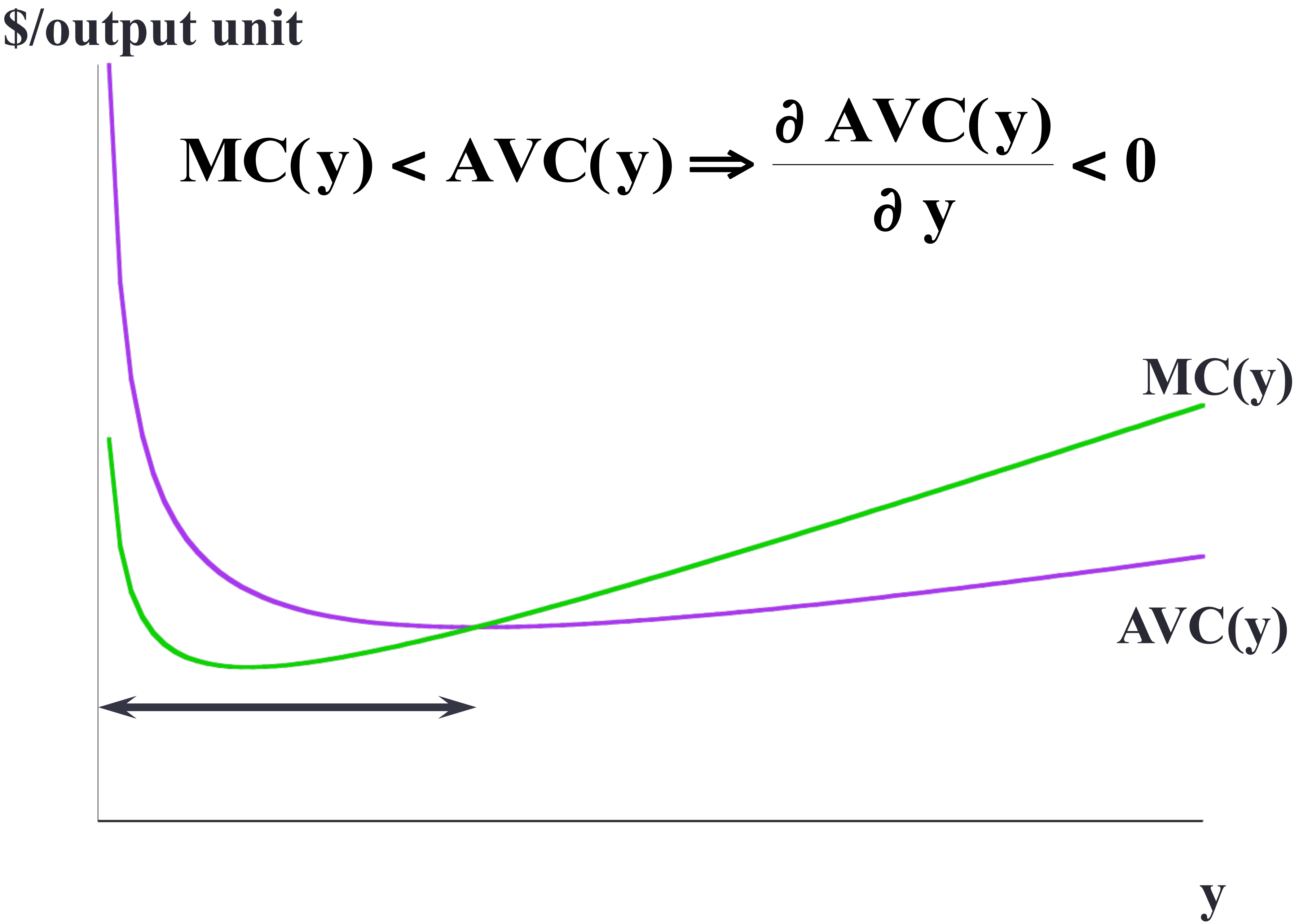
$$\frac{\partial\, \text{AVC}(y)}{\partial\, y} \underset{<}{\overset{>}{=}} 0 \quad \text{as} \quad \text{MC}(y) \underset{<}{\overset{>}{=}} \frac{c_v(y)}{y} = \text{AVC}(y).$$

Marginal & Average Cost Functions

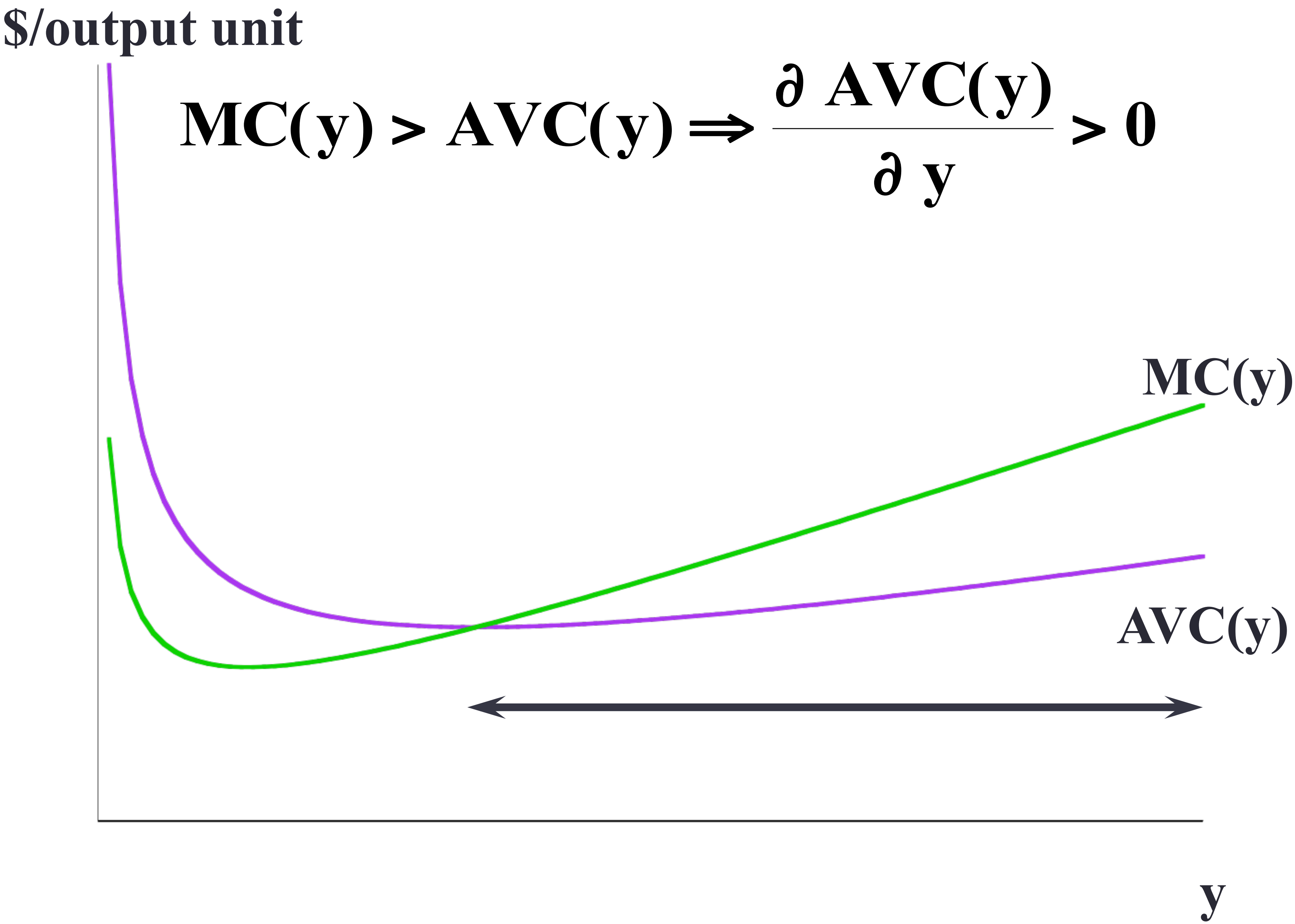
$$\frac{\partial\, AVC(y)}{\partial\, y} \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \text{as} \quad MC(y) \begin{matrix} > \\ = \\ < \end{matrix} AVC(y).$$


$/output unit
MC(y)
AVC(y)
y

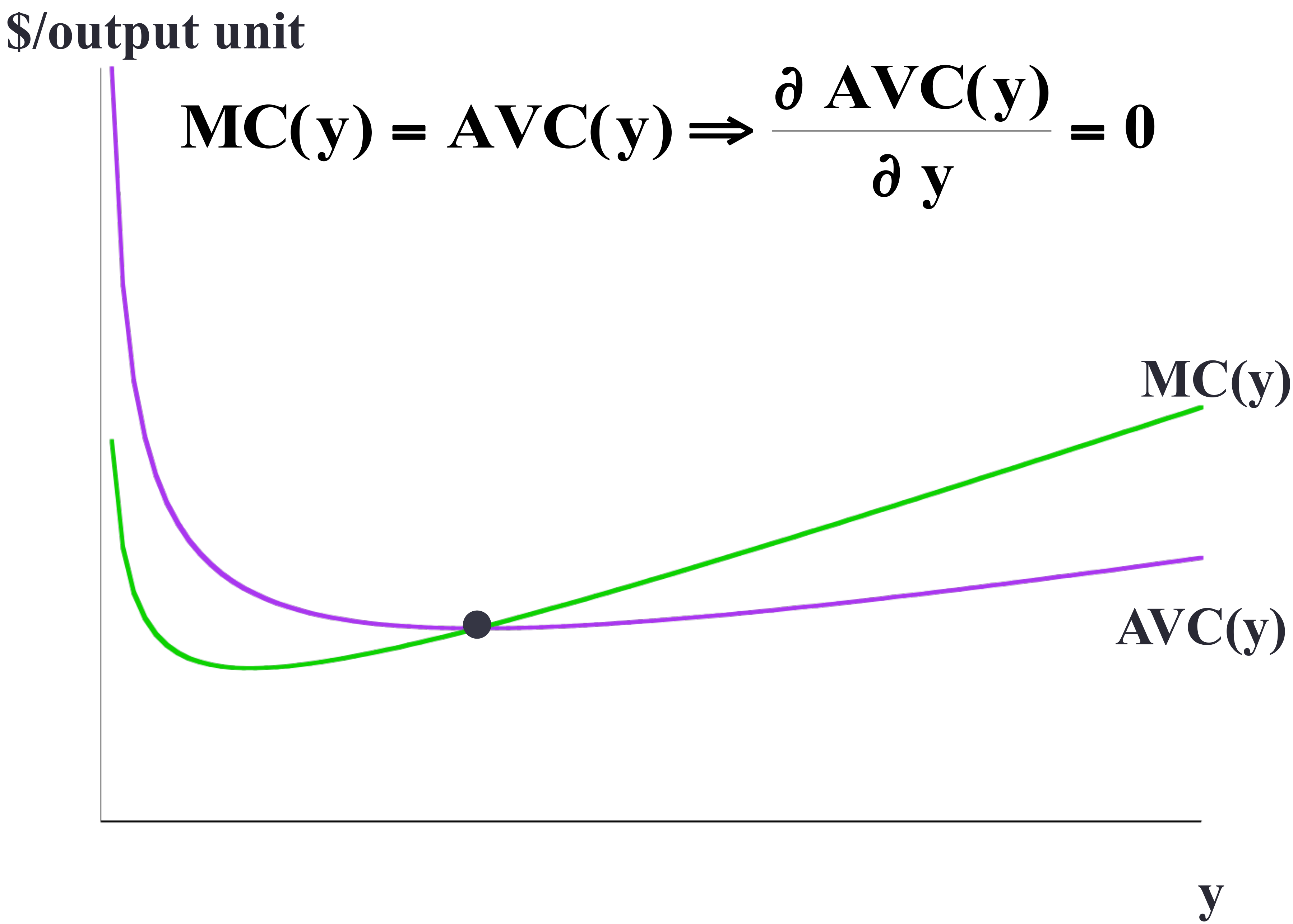
$$MC(y) < AVC(y) \Rightarrow \frac{\partial\, AVC(y)}{\partial\, y} < 0$$



$$MC(y) > AVC(y) \Rightarrow \frac{\partial\, AVC(y)}{\partial\, y} > 0$$

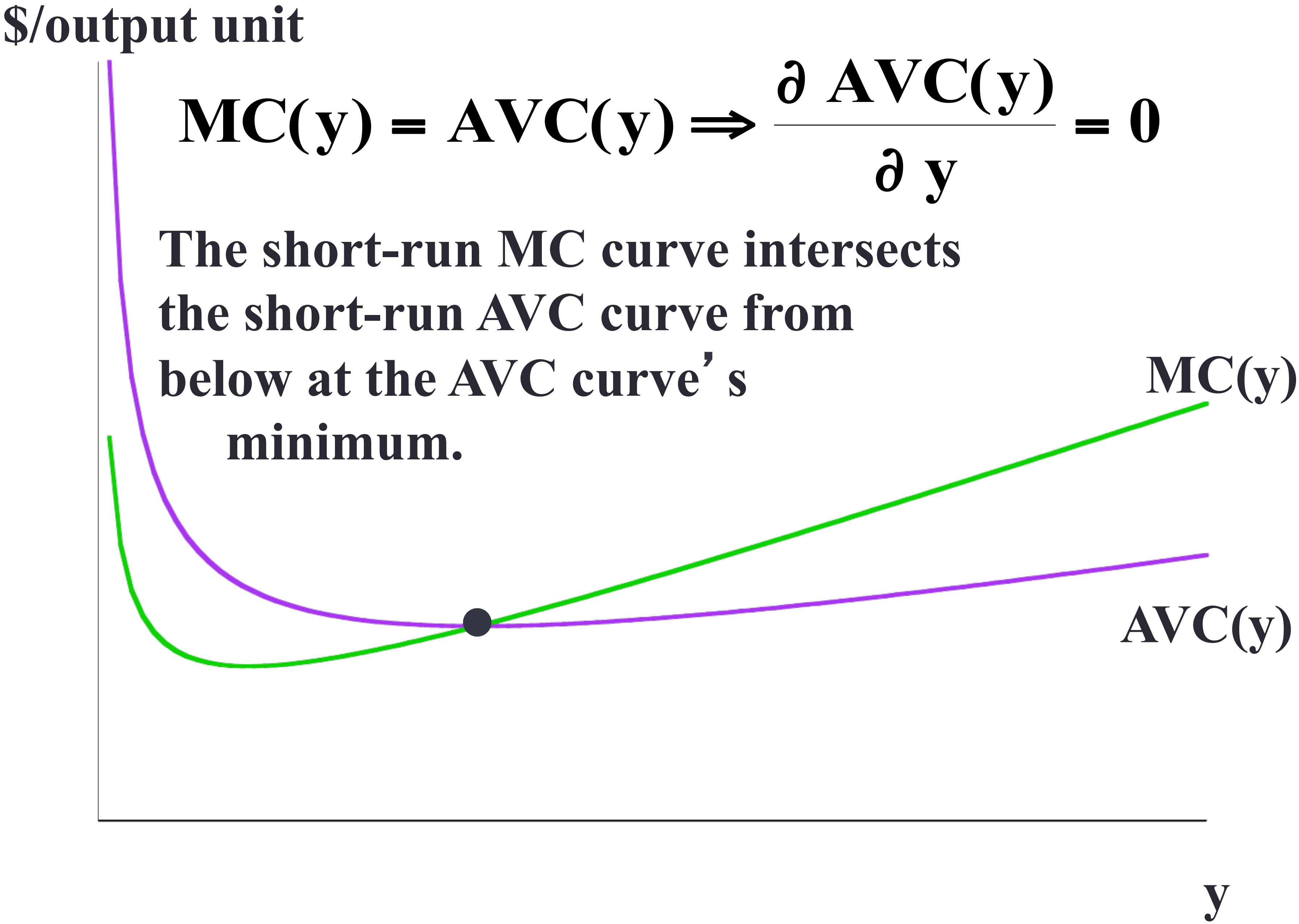


$$MC(y) = AVC(y) \Rightarrow \frac{\partial\, AVC(y)}{\partial\, y} = 0$$



$$MC(y) = AVC(y) \Rightarrow \frac{\partial\, AVC(y)}{\partial\, y} = 0$$

The short-run MC curve intersects the short-run AVC curve from below at the AVC curve's minimum.



Marginal & Average Cost Functions

Similarly, since $$\mathbf{ATC(y) = \frac{c(y)}{y}},$$

$$\frac{\partial\ \mathbf{ATC(y)}}{\partial\ \mathbf{y}} = \frac{\mathbf{y \times MC(y) - 1 \times c(y)}}{\mathbf{y^2}}.$$

Marginal & Average Cost Functions

Similarly, since $$\mathrm{ATC}(y) = \frac{c(y)}{y},$$

$$\frac{\partial\ \mathrm{ATC}(y)}{\partial\ y} = \frac{y \times \mathrm{MC}(y) - 1 \times c(y)}{y^2}.$$

Therefore,

$$\frac{\partial\ \mathrm{ATC}(y)}{\partial\ y} \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \text{as} \quad y \times \mathrm{MC}(y) \begin{matrix} > \\ = \\ < \end{matrix} c(y).$$

Marginal & Average Cost Functions

Similarly, since $\mathrm{ATC}(y) = \dfrac{c(y)}{y}$,

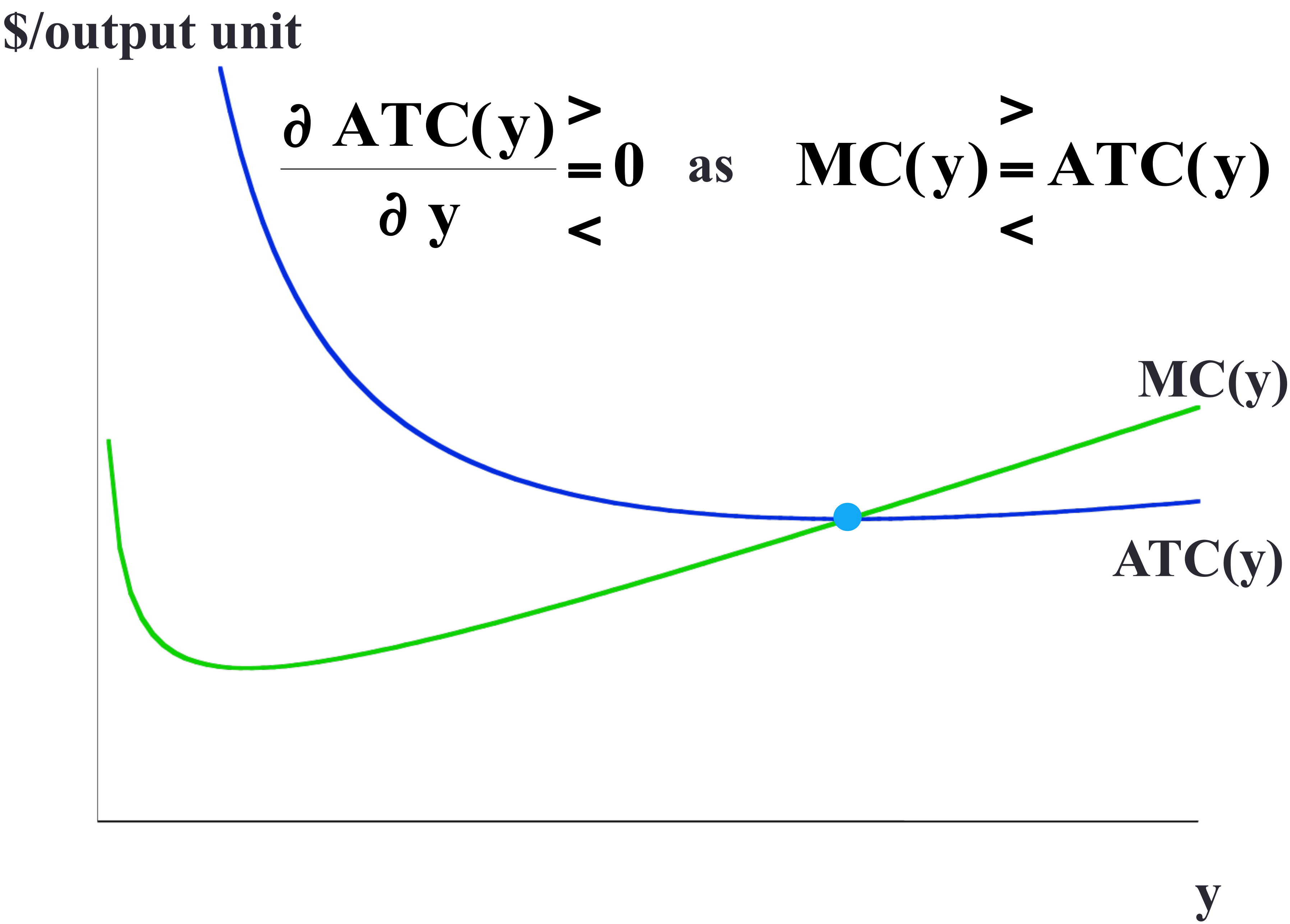
$$\frac{\partial\, \mathrm{ATC}(y)}{\partial\, y} = \frac{y \times \mathrm{MC}(y) - 1 \times c(y)}{y^2}.$$

Therefore,

$$\frac{\partial\, \mathrm{ATC}(y)}{\partial\, y} \begin{array}{c} > \\ = \\ < \end{array} 0 \quad \text{as} \quad y \times \mathrm{MC}(y) \begin{array}{c} > \\ = \\ < \end{array} c(y).$$

$$\frac{\partial\, \mathrm{ATC}(y)}{\partial\, y} \begin{array}{c} > \\ = \\ < \end{array} 0 \quad \text{as} \quad \mathrm{MC}(y) \begin{array}{c} > \\ = \\ < \end{array} \frac{c(y)}{y} = \mathrm{ATC}(y).$$

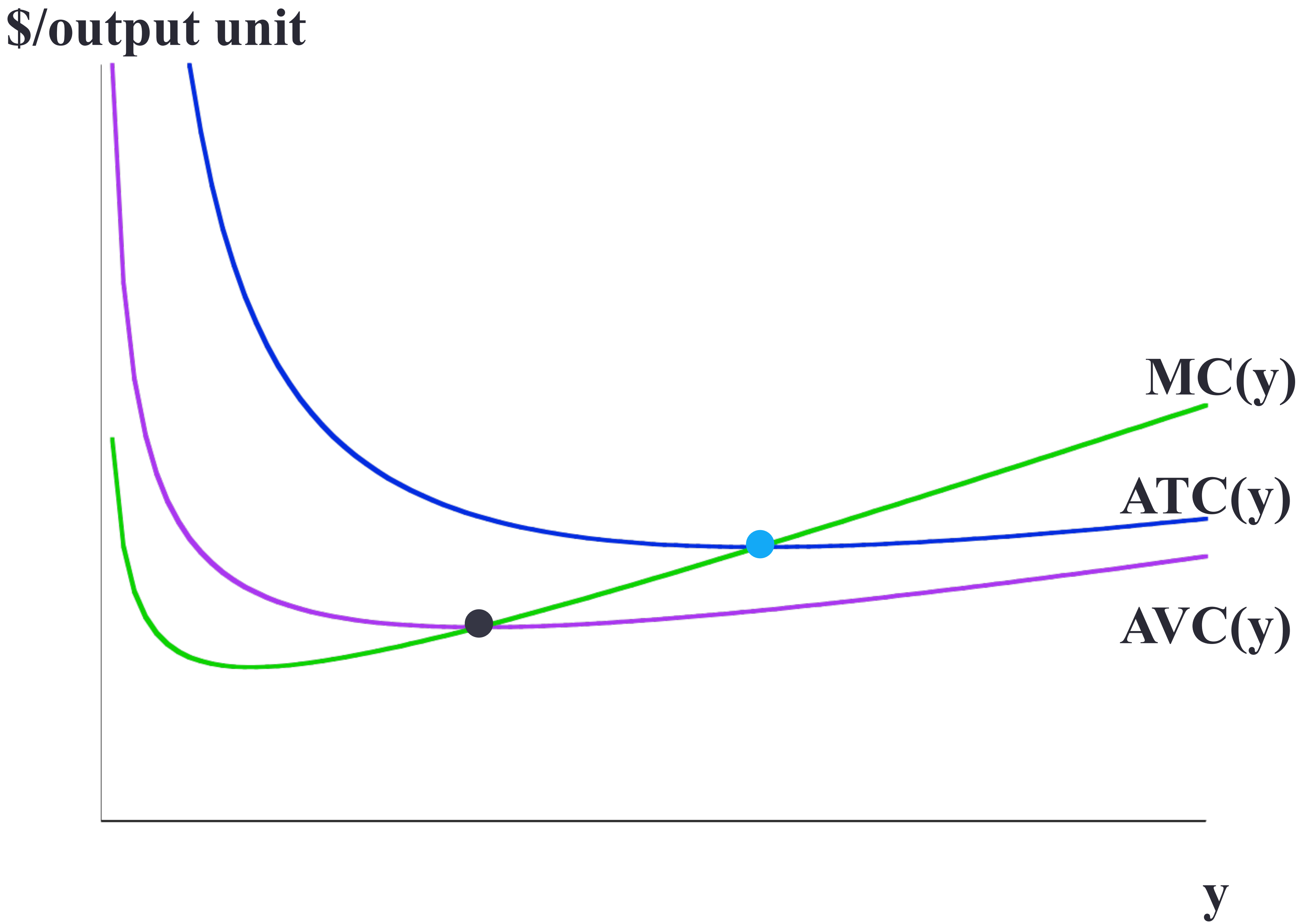
$$\frac{\partial\, ATC(y)}{\partial\, y} \;\substack{>\\=\\<}\; 0 \quad \text{as} \quad MC(y) \;\substack{>\\=\\<}\; ATC(y)$$



Marginal & Average Cost Functions

- The short-run MC curve intersects the short-run AVC curve from below at the AVC curve's minimum.

- And, similarly, the short-run MC curve intersects the short-run ATC curve from below at the ATC curve's minimum.

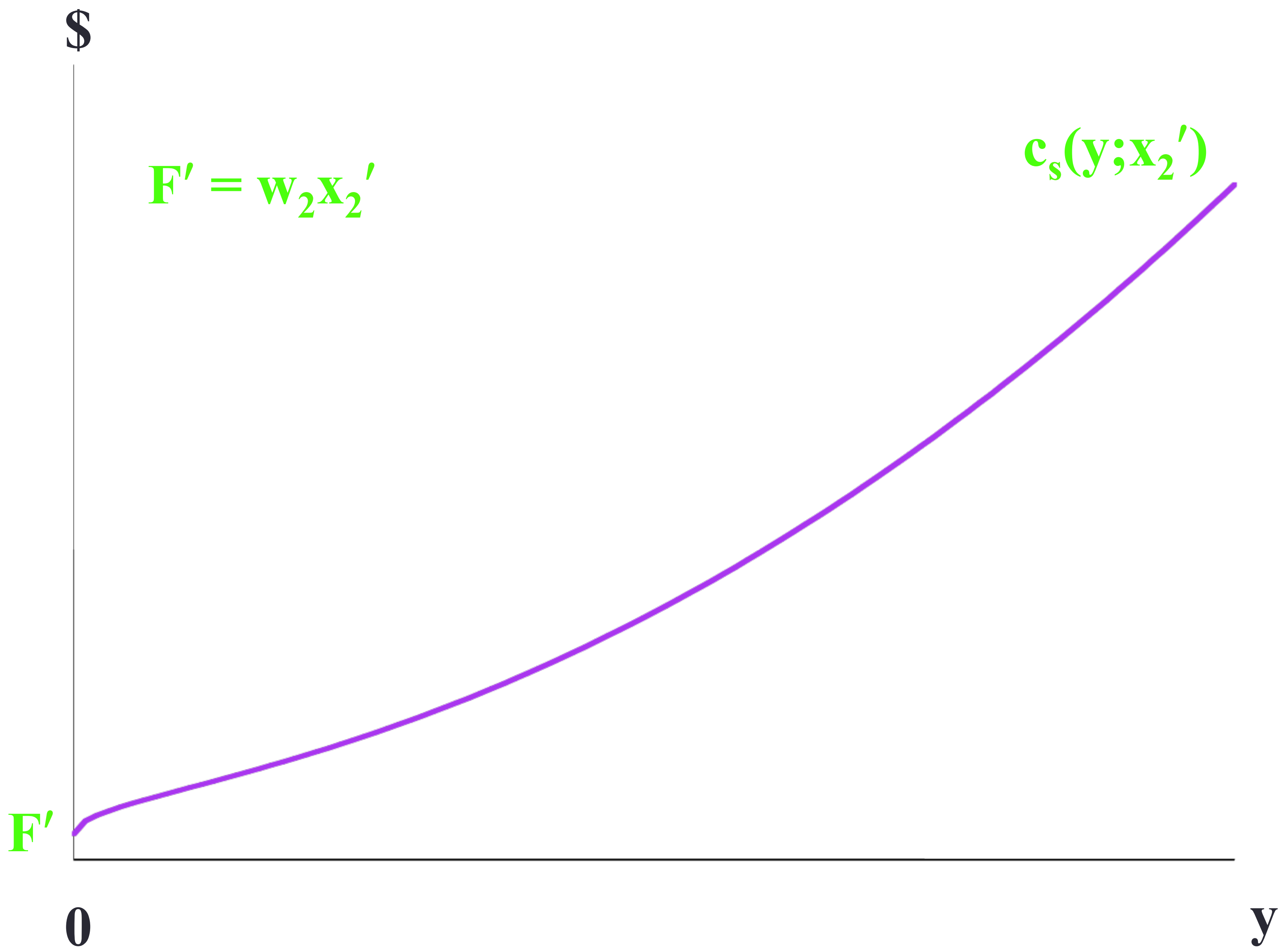

$/output unit
MC(y)
ATC(y)
AVC(y)
y

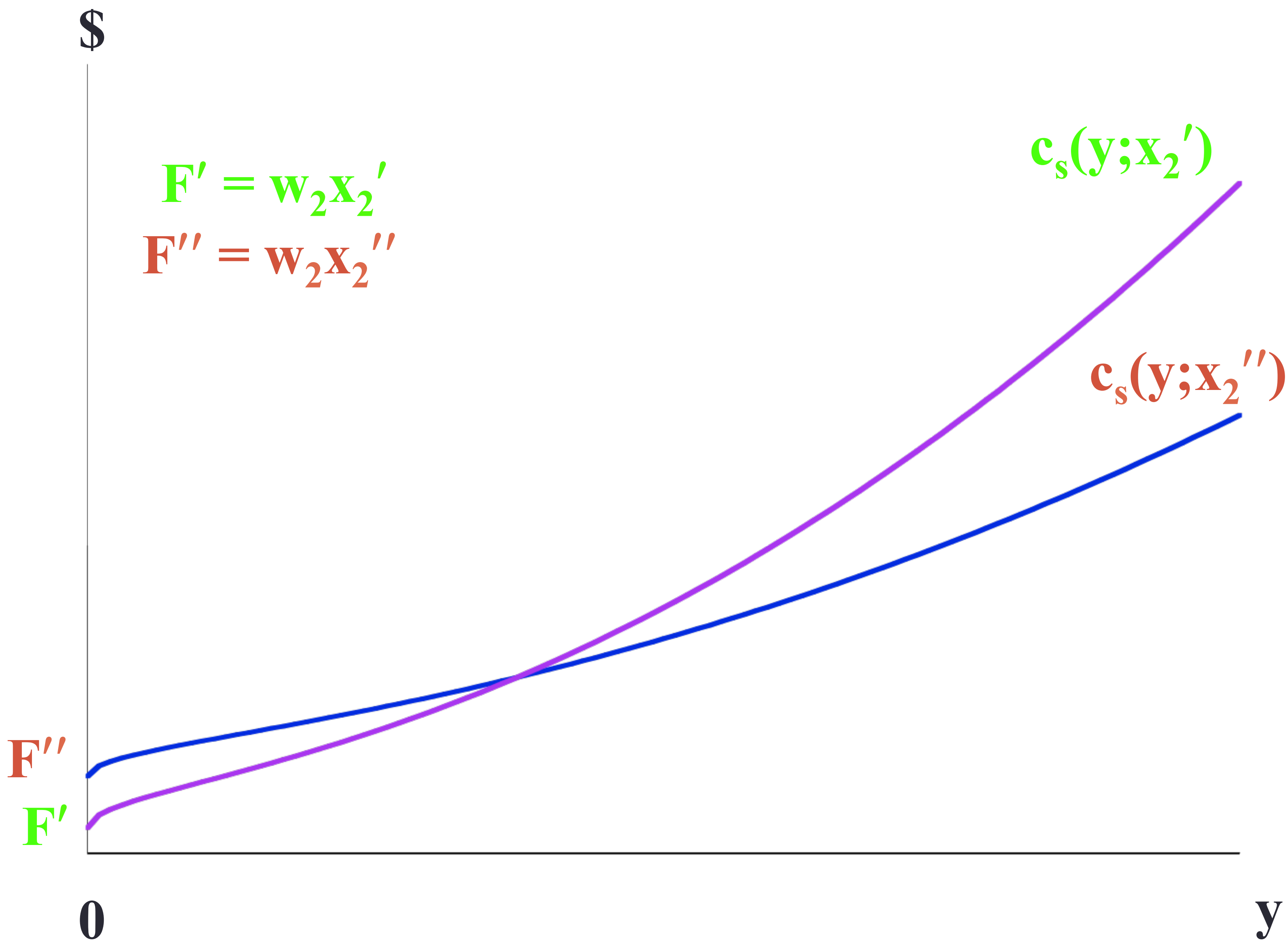
Short-Run & Long-Run Total Cost Curves

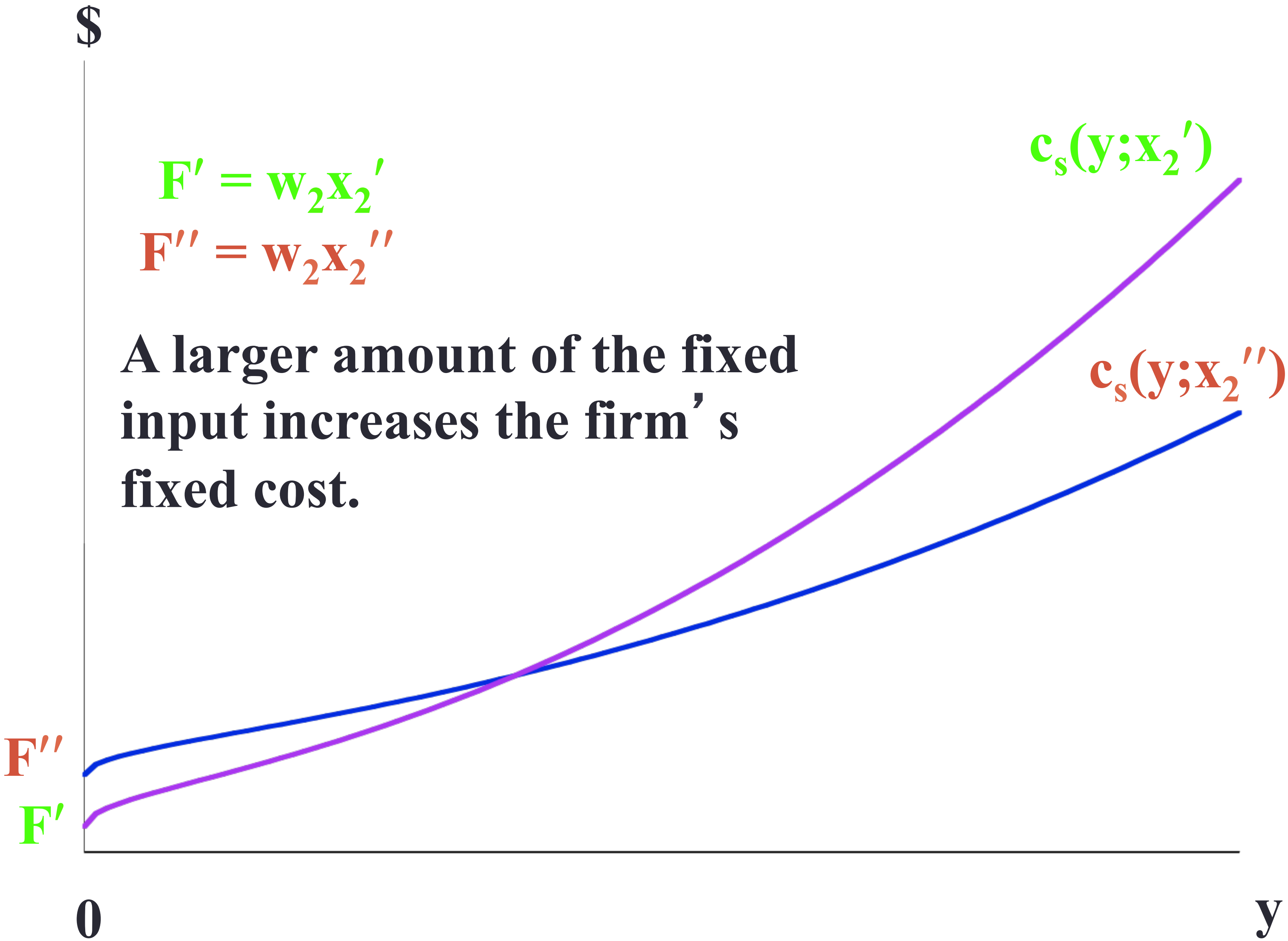
- A firm has a different short-run total cost curve for each possible short-run circumstance.

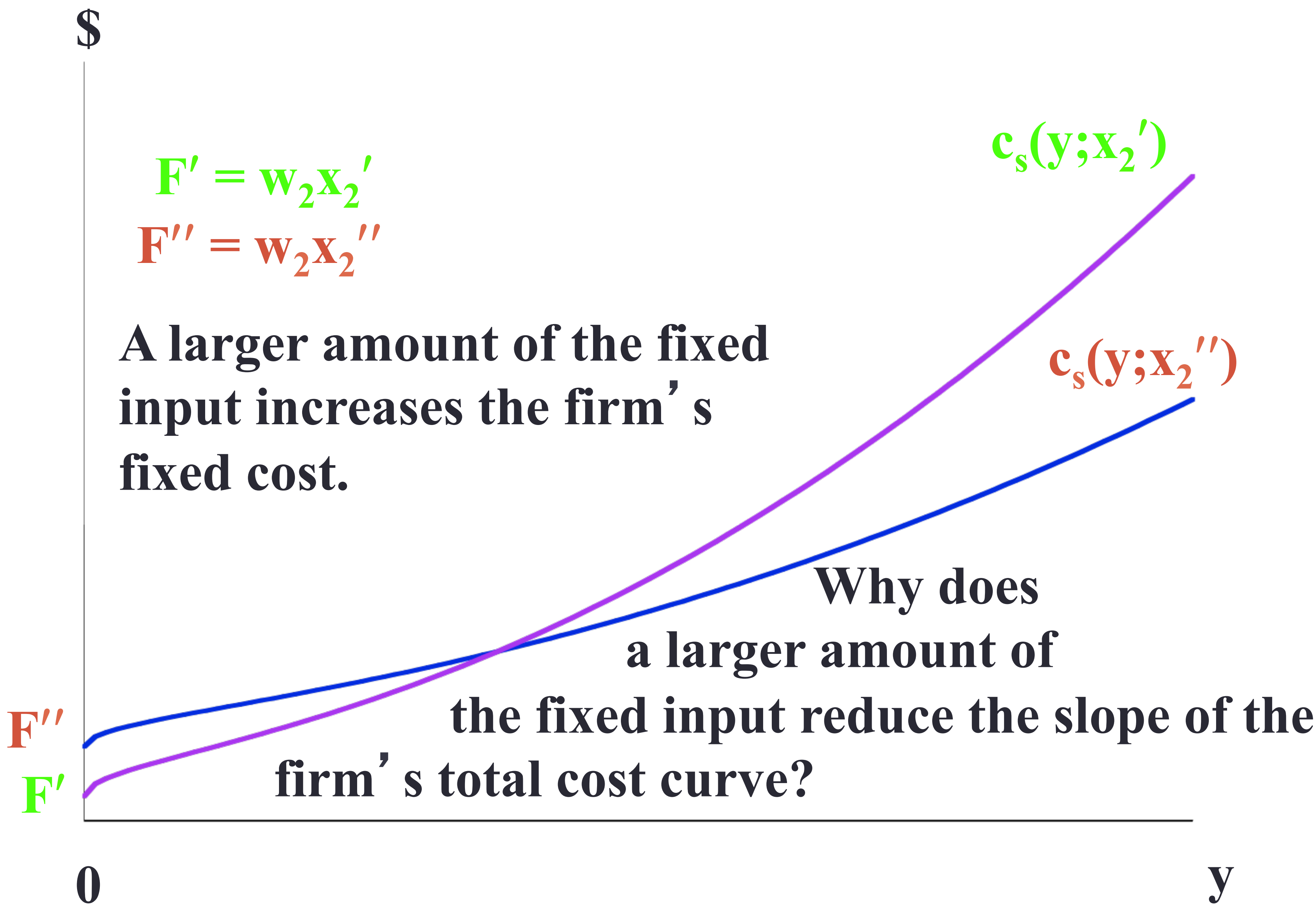
- Suppose the firm can be in one of just three short-runs;
- $x_2 = x_2'$
- or $x_2 = x_2''$
- or $x_2 = x_2'''$.

- Where $x_2' < x_2'' < x_2'''$.


$
$F' = w_2x_2'$
$c_s(y;x_2')$
F'
0
y


$
$F' = w_2x_2'$
$F'' = w_2x_2''$
$c_s(y;x_2')$
$c_s(y;x_2'')$
F''
F'
0
y


$
F′ = w2x2′
F″ = w2x2″
A larger amount of the fixed input increases the firm’s fixed cost.
cs(y;x2′)
cs(y;x2″)
F″
F′
0
y


$
F′ = w2x2′
F″ = w2x2″
A larger amount of the fixed input increases the firm’s fixed cost.
cs(y;x2′)
cs(y;x2″)
Why does a larger amount of the fixed input reduce the slope of the firm’s total cost curve?
F″
F′
0
y

Short-Run & Long-Run Total Cost Curves

MP_1 is the marginal physical productivity of the variable input 1, so one extra unit of input 1 gives MP_1 extra output units. Therefore, the extra amount of input 1 needed for 1 extra output unit is

Short-Run & Long-Run Total Cost Curves

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Short-Run & Long-Run Total Cost Curves

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$$1/MP_1 \text{ units of input 1.}$$

Each unit of input 1 costs w_1, so the firm's extra cost from producing one extra unit of output is

Short-Run & Long-Run Total Cost Curves

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$1/MP_1$ units of input 1.

Each unit of input 1 costs w_1, so the firm's extra cost from producing one extra unit of output is $$MC = \frac{w_1}{MP_1}.$$

Short-Run & Long-Run Total Cost Curves

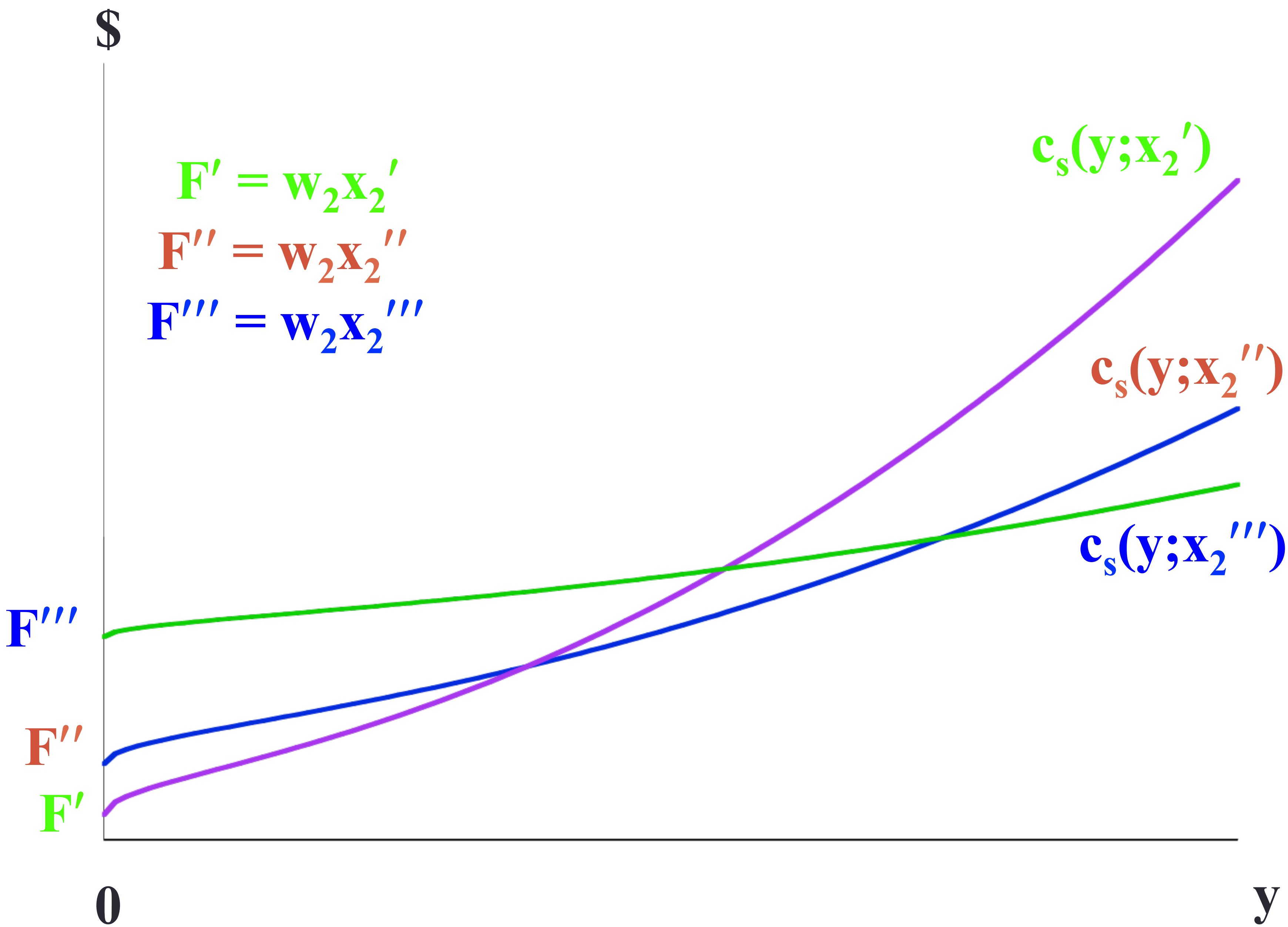
$MC = \dfrac{w_1}{MP_1}$ **is the slope of the firm's total cost curve.**

Short-Run & Long-Run Total Cost Curves

$$MC = \frac{w_1}{MP_1}$$ is the slope of the firm's total cost curve.

If input 2 is a complement to input 1 then MP_1 is higher for higher x_2.
Hence, MC is lower for higher x_2.

That is, a short-run total cost curve starts higher and has a lower slope if x_2 is larger.

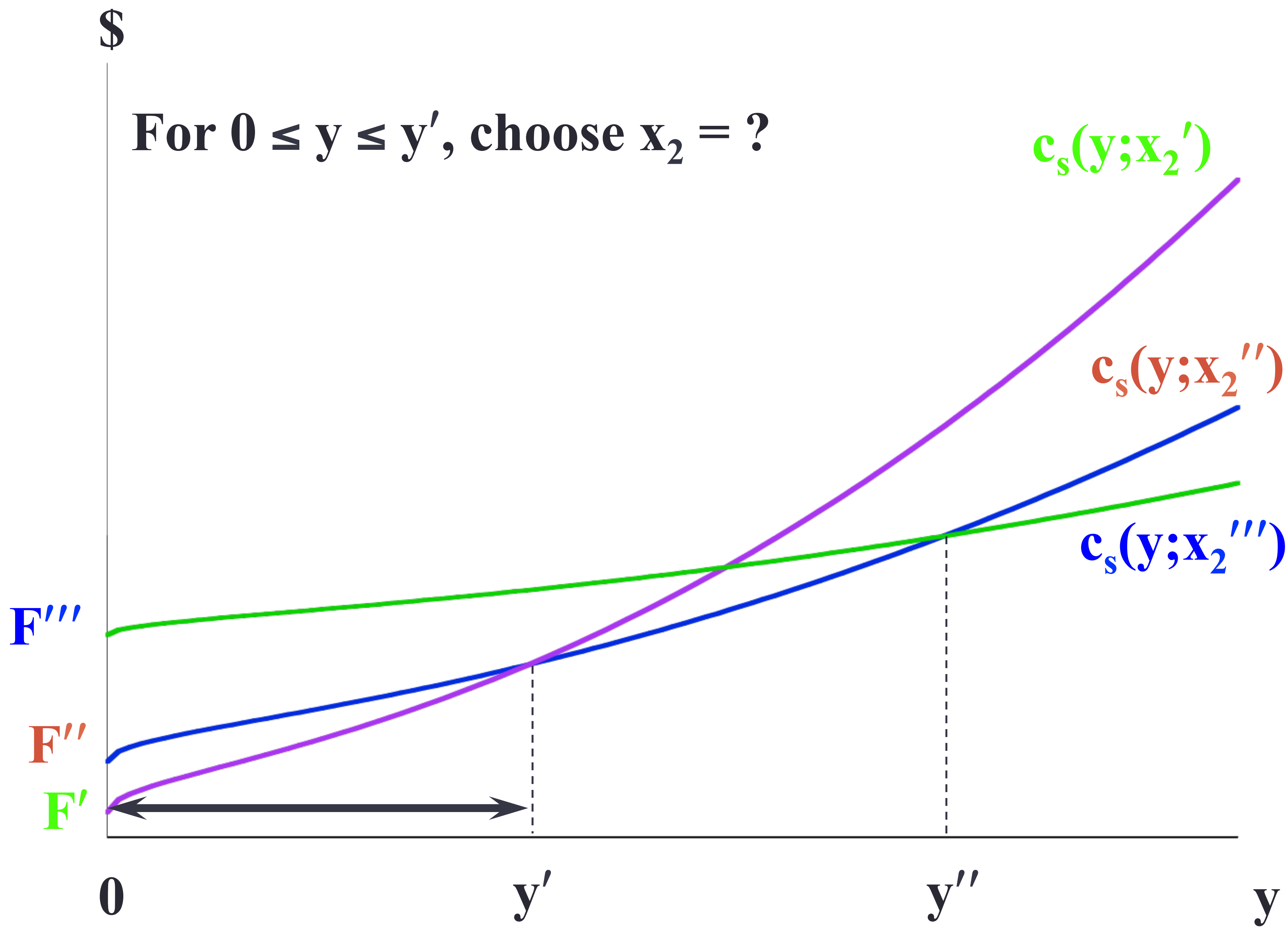


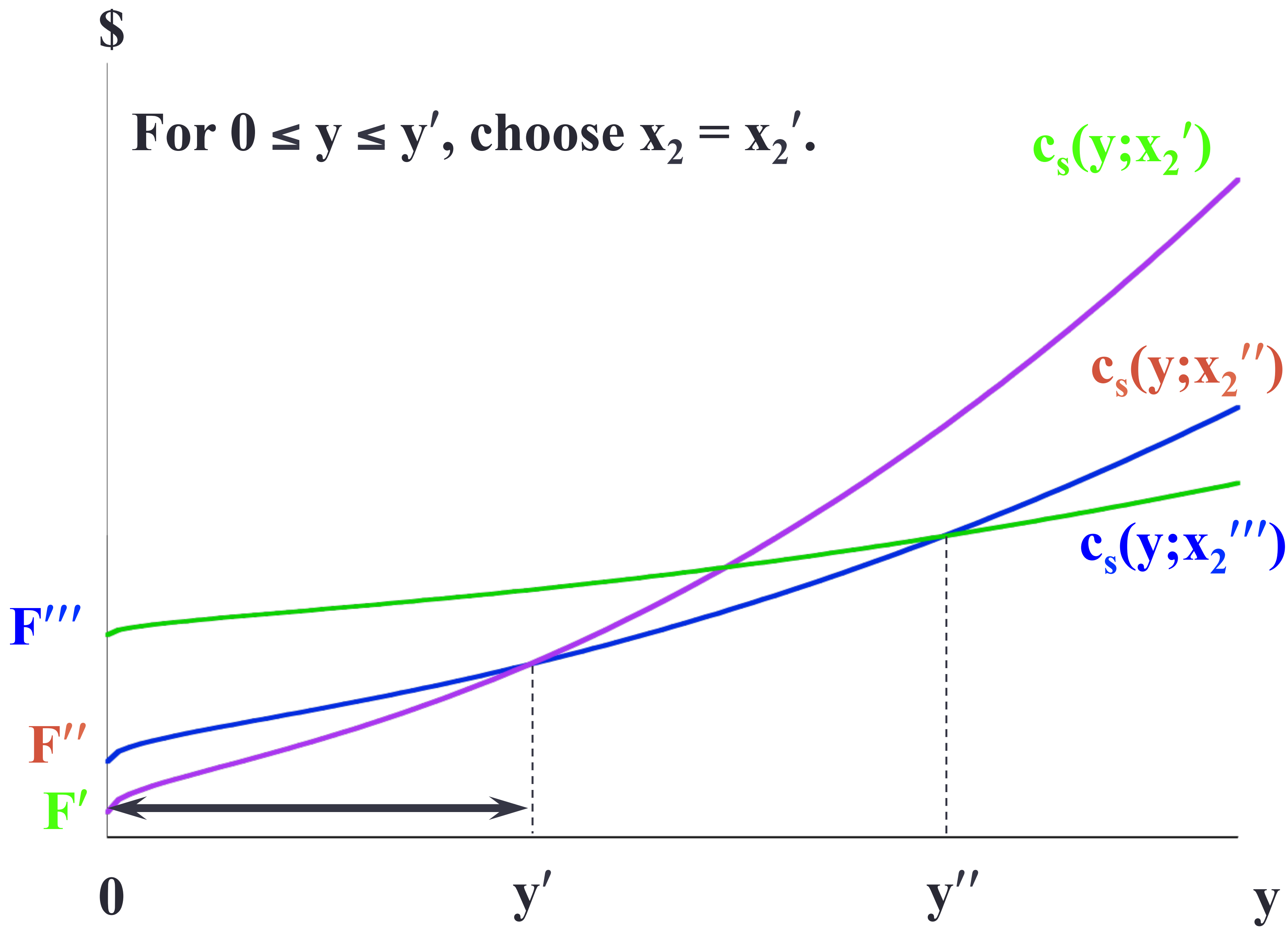
$
$F' = w_2x_2'$
$F'' = w_2x_2''$
$F''' = w_2x_2'''$
$c_s(y;x_2')$
$c_s(y;x_2'')$
$c_s(y;x_2''')$
F'''
F''
F'
0
y

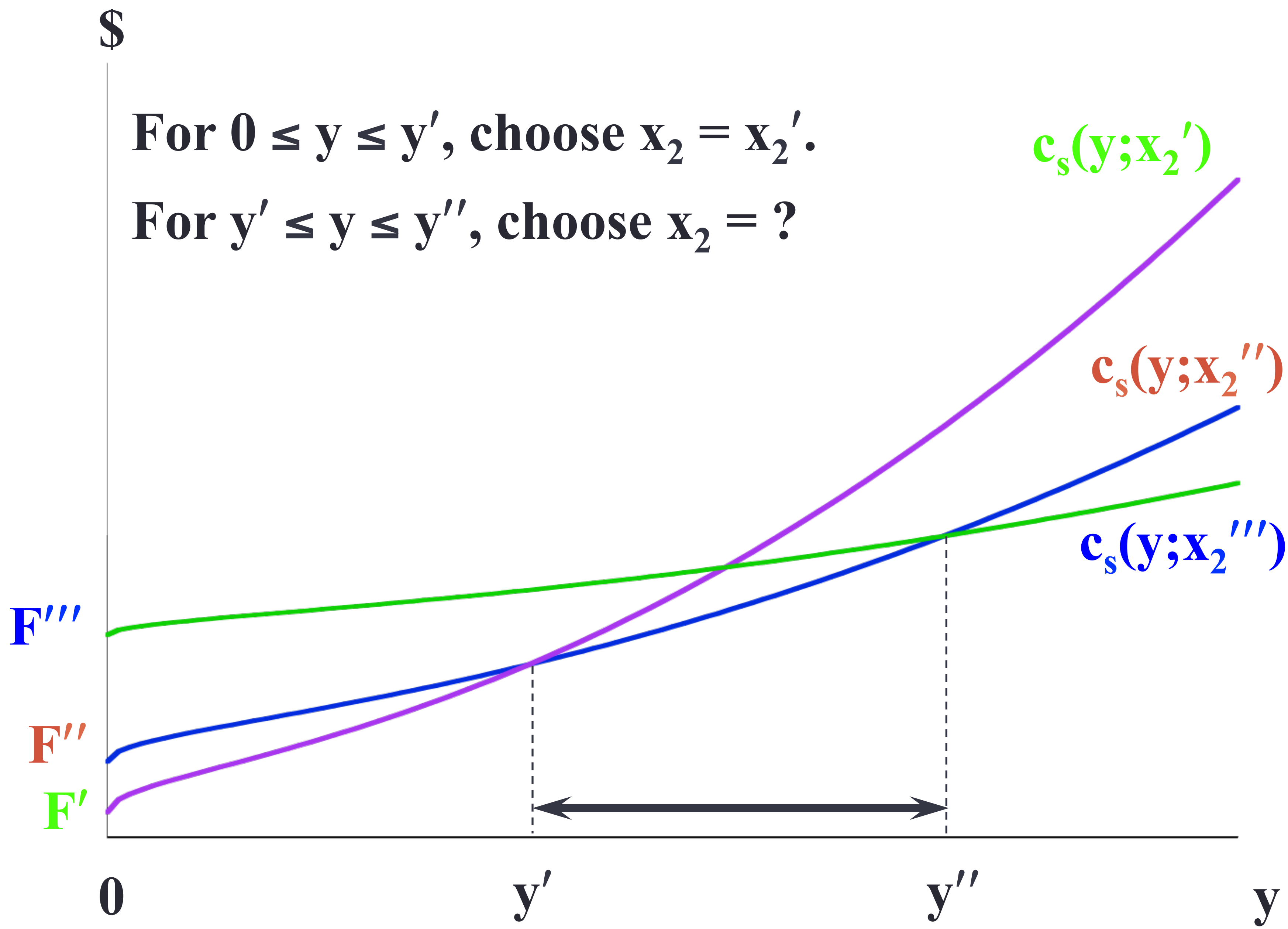
Short-Run & Long-Run Total Cost Curves

- The firm has three short-run total cost curves.
- In the long-run the firm is free to choose amongst these three since it is free to select x_2 equal to any of x_2', x_2'', or x_2'''.
- How does the firm make this choice?

For $0 \le y \le y'$, choose $x_2 = ?$

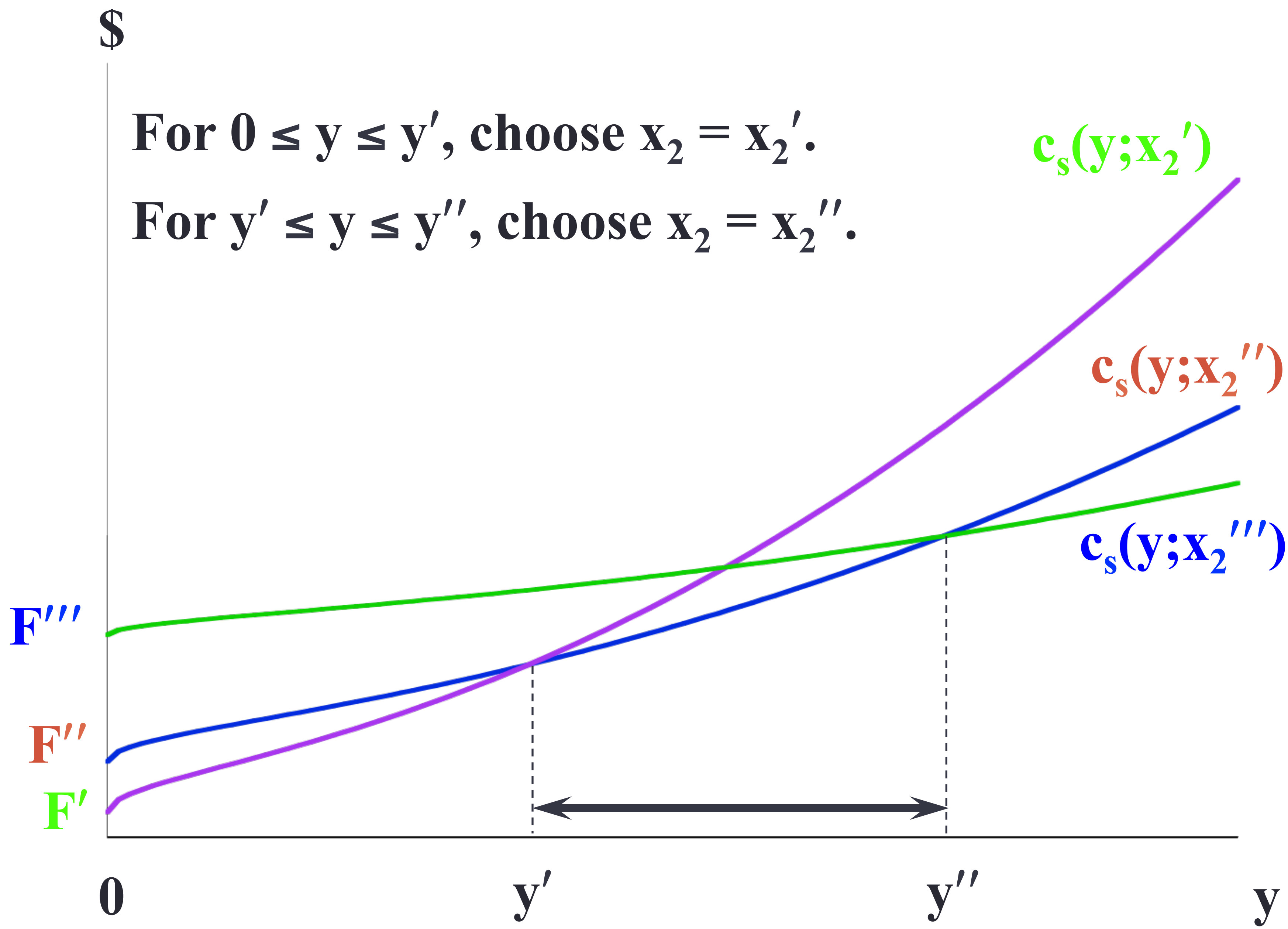


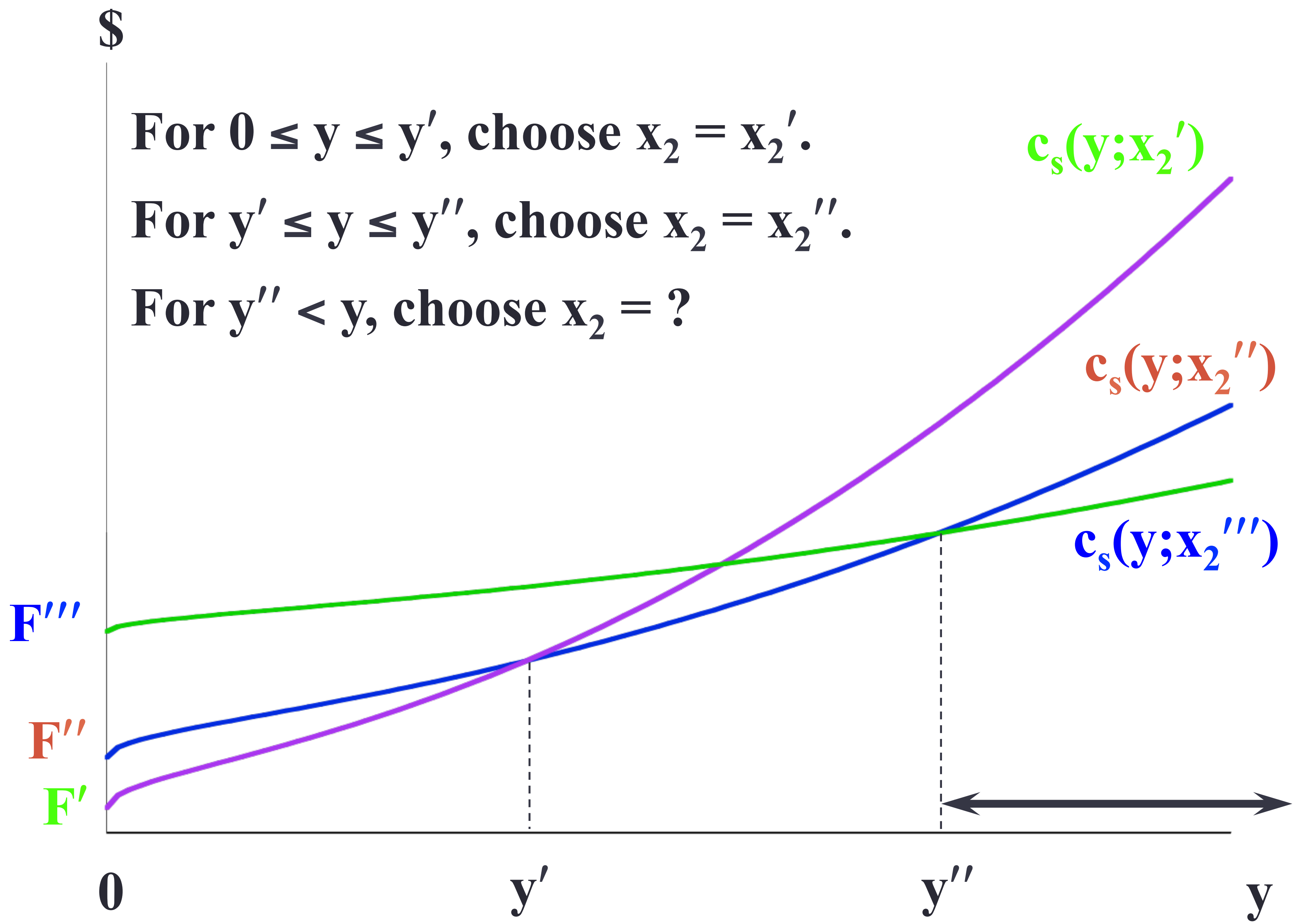

$
For 0 ≤ y ≤ y′, choose x₂ = x₂′.
c_s(y;x_2′)
c_s(y;x_2″)
c_s(y;x_2‴)
F‴
F″
F′
0
y′
y″
y

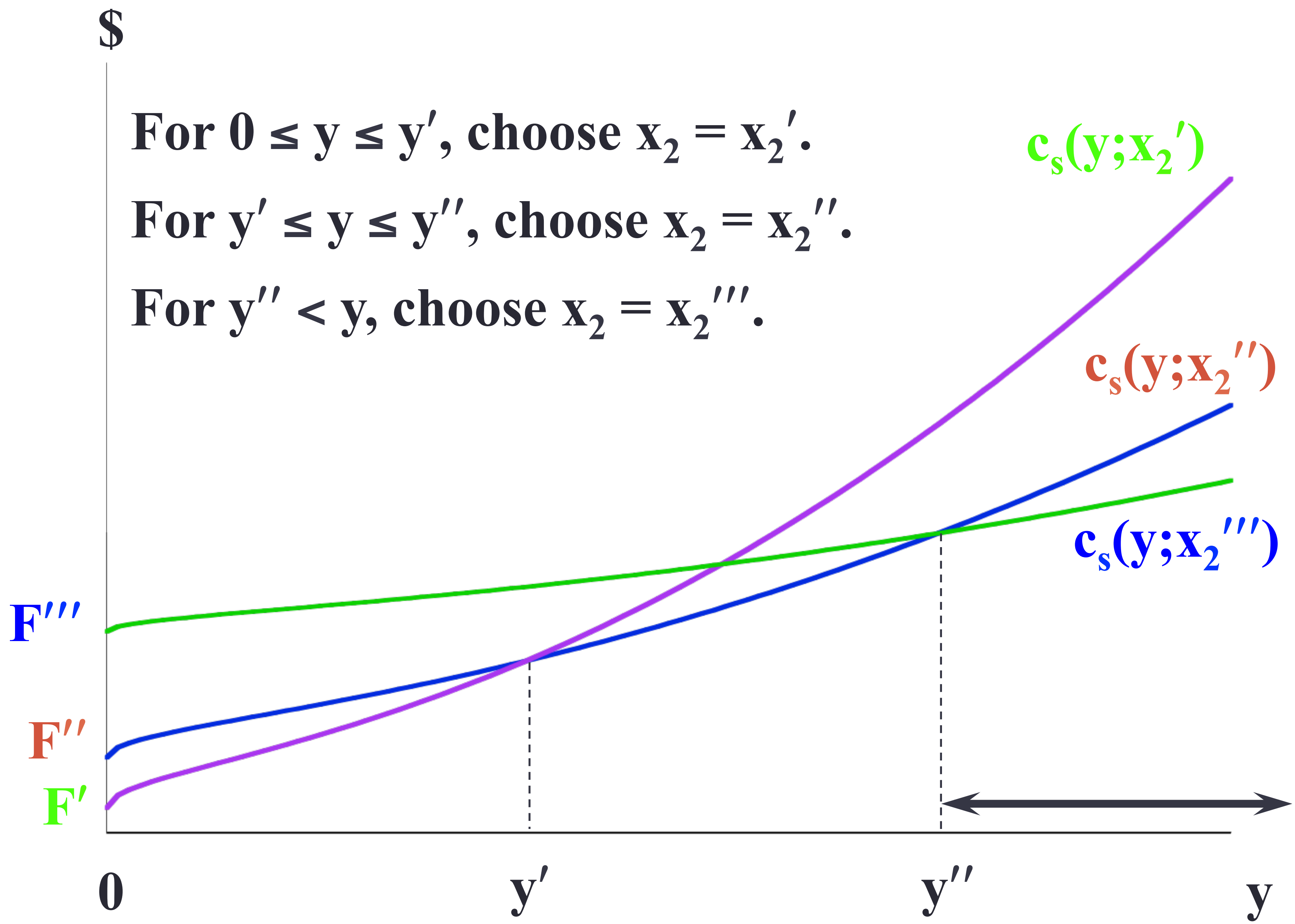

$
For $0 \le y \le y'$, choose $x_2 = x_2'$.
For $y' \le y \le y''$, choose $x_2 = ?$
$c_s(y;x_2')$
$c_s(y;x_2'')$
$c_s(y;x_2''')$
F'''
F''
F'
0
y'
y''
y

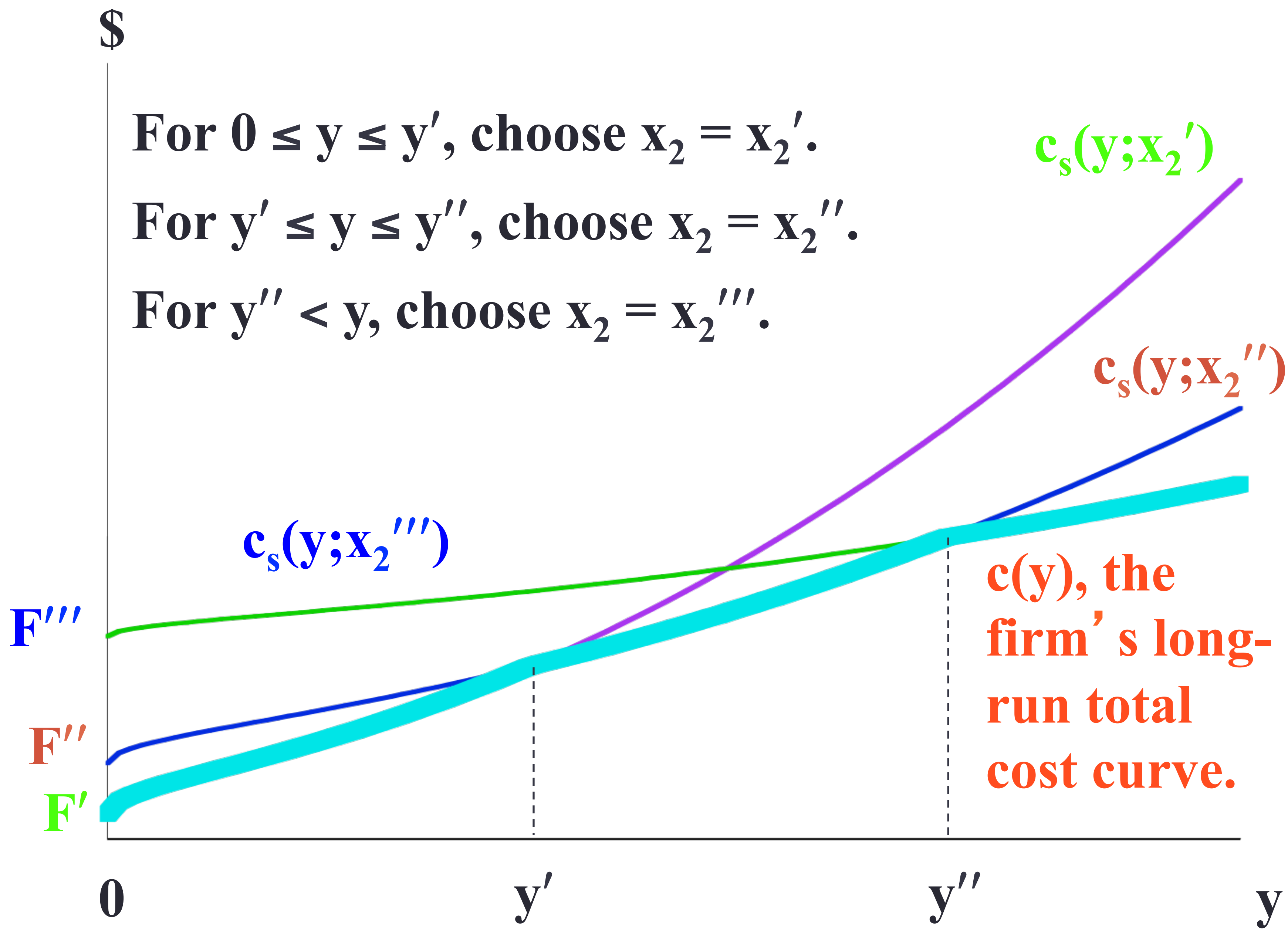
For $0 \leq y \leq y'$, choose $x_2 = x_2'$.

For $y' \leq y \leq y''$, choose $x_2 = x_2''$.




$
For 0 ≤ y ≤ y′, choose x₂ = x₂′.
For y′ ≤ y ≤ y″, choose x₂ = x₂″.
For y″ < y, choose x₂ = ?
c_s(y;x_2′)
c_s(y;x_2″)
c_s(y;x_2‴)
F‴
F″
F′
0
y′
y″
y


$
For 0 ≤ y ≤ y′, choose x₂ = x₂′.
For y′ ≤ y ≤ y″, choose x₂ = x₂″.
For y″ < y, choose x₂ = x₂‴.
c_s(y;x₂′)
c_s(y;x₂″)
c_s(y;x₂‴)
F‴
F″
F′
0
y′
y″
y

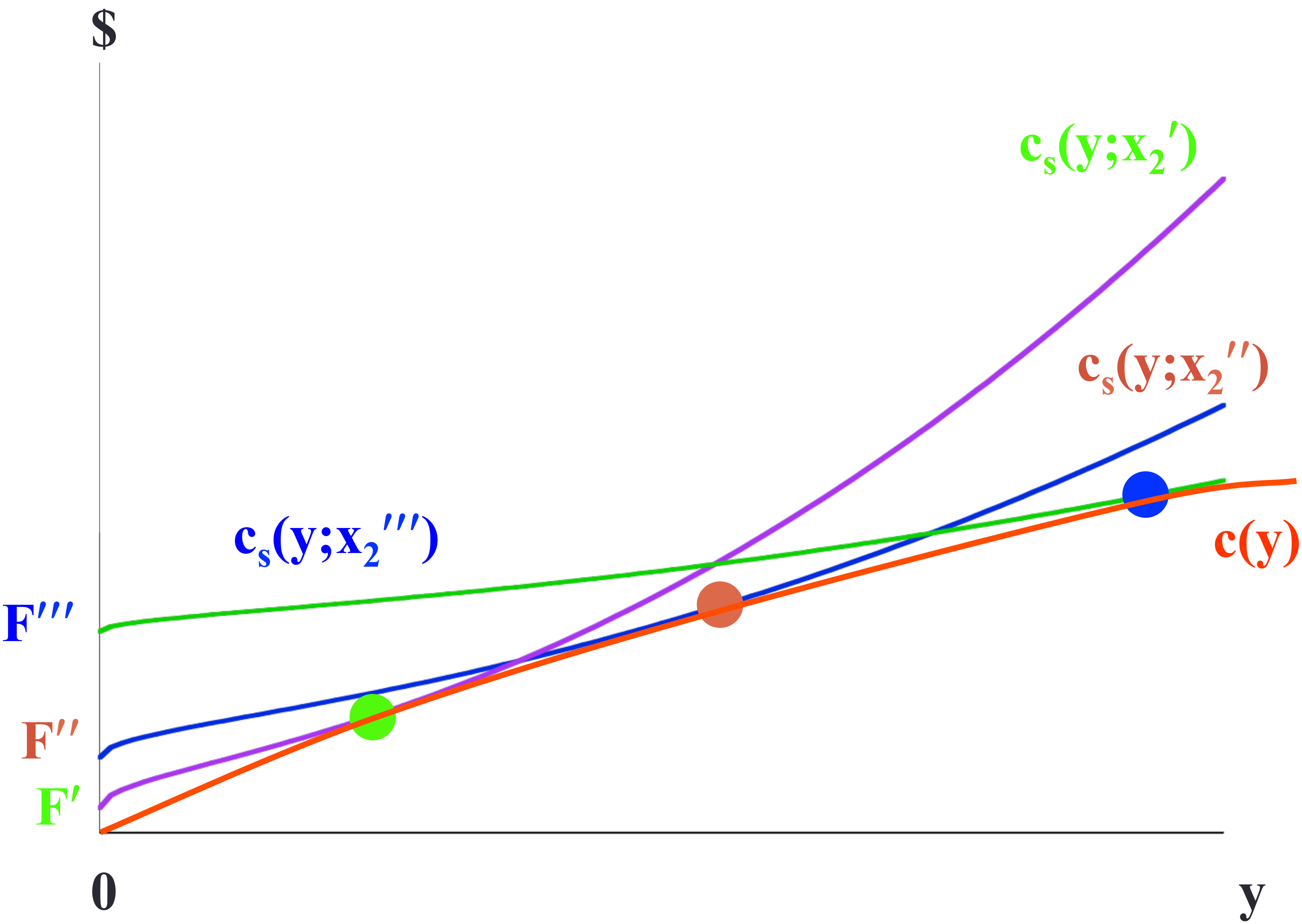

$
For 0 ≤ y ≤ y′, choose $x_2 = x_2'$.
For y′ ≤ y ≤ y″, choose $x_2 = x_2''$.
For y″ < y, choose $x_2 = x_2'''$.
$c_s(y;x_2')$
$c_s(y;x_2'')$
$c_s(y;x_2''')$
c(y), the firm's long-run total cost curve.
F‴
F″
F′
0
y′
y″
y

Short-Run & Long-Run Total Cost Curves

- The firm's long-run total cost curve consists of the lowest parts of the short-run total cost curves. The long-run total cost curve is the **lower envelope** of the short-run total cost curves.

Short-Run & Long-Run Total Cost Curves

- If input 2 is available in continuous amounts then there is an infinity of short-run total cost curves but the long-run total cost curve is *still* the lower envelope of all of the short-run total cost curves.


$
$c_s(y;x_2')$
$c_s(y;x_2'')$
$c_s(y;x_2''')$
c(y)
F'''
F''
F'
0
y

Short-Run & Long-Run Average Total Cost Curves

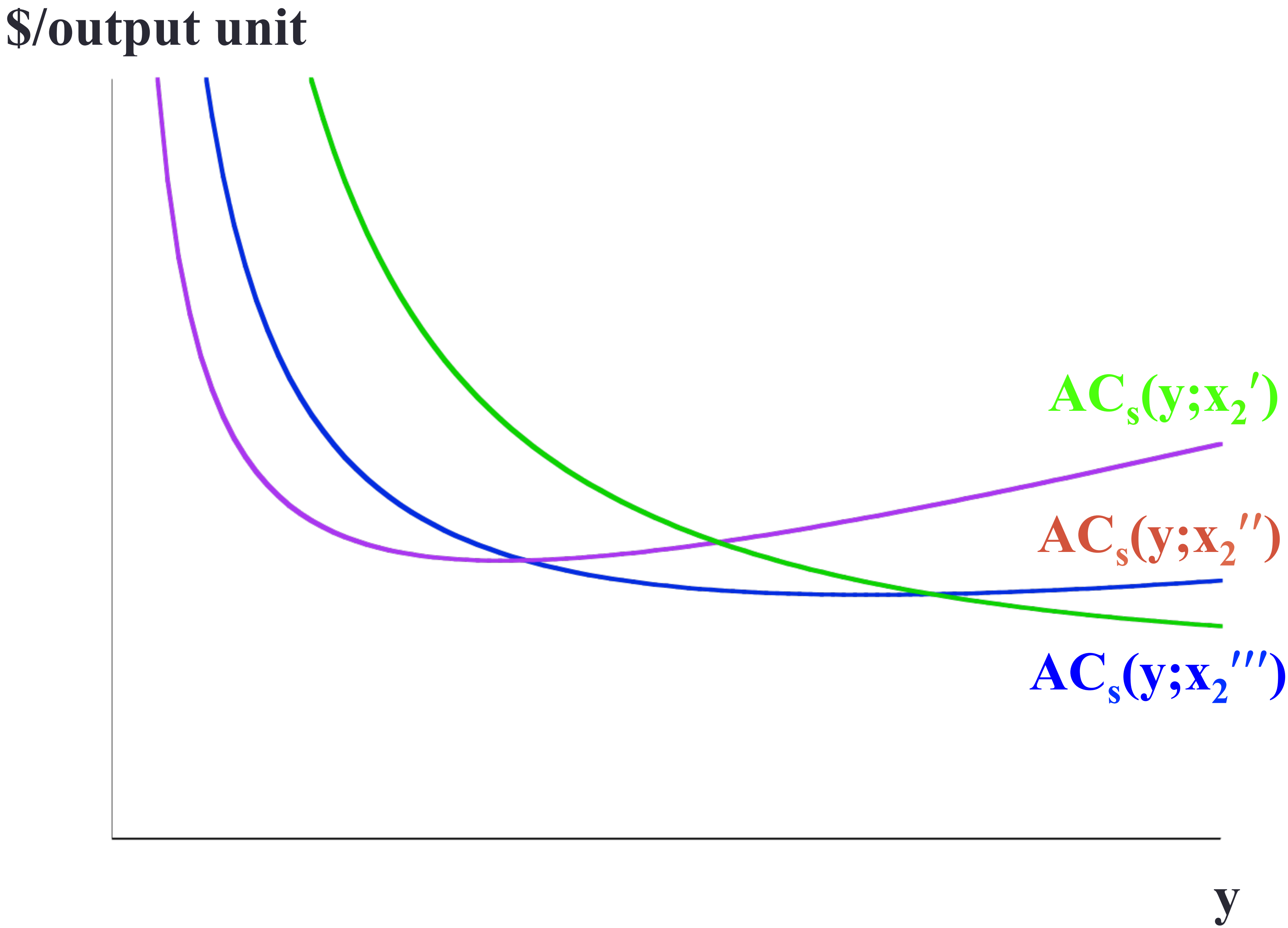
- For any output level y, the long-run total cost curve always gives the lowest possible total production cost.
- Therefore, the long-run av. total cost curve must always give the lowest possible av. total production cost.
- The long-run av. total cost curve must be the lower envelope of all of the firm's short-run av. total cost curves.

Short-Run & Long-Run Average Total Cost Curves

- E.g. suppose again that the firm can be in one of just three short-runs;

- $x_2 = x_2'$
  or $\quad x_2 = x_2''$ $\qquad (x_2' < x_2'' < x_2''')$
  or $\quad x_2 = x_2'''$

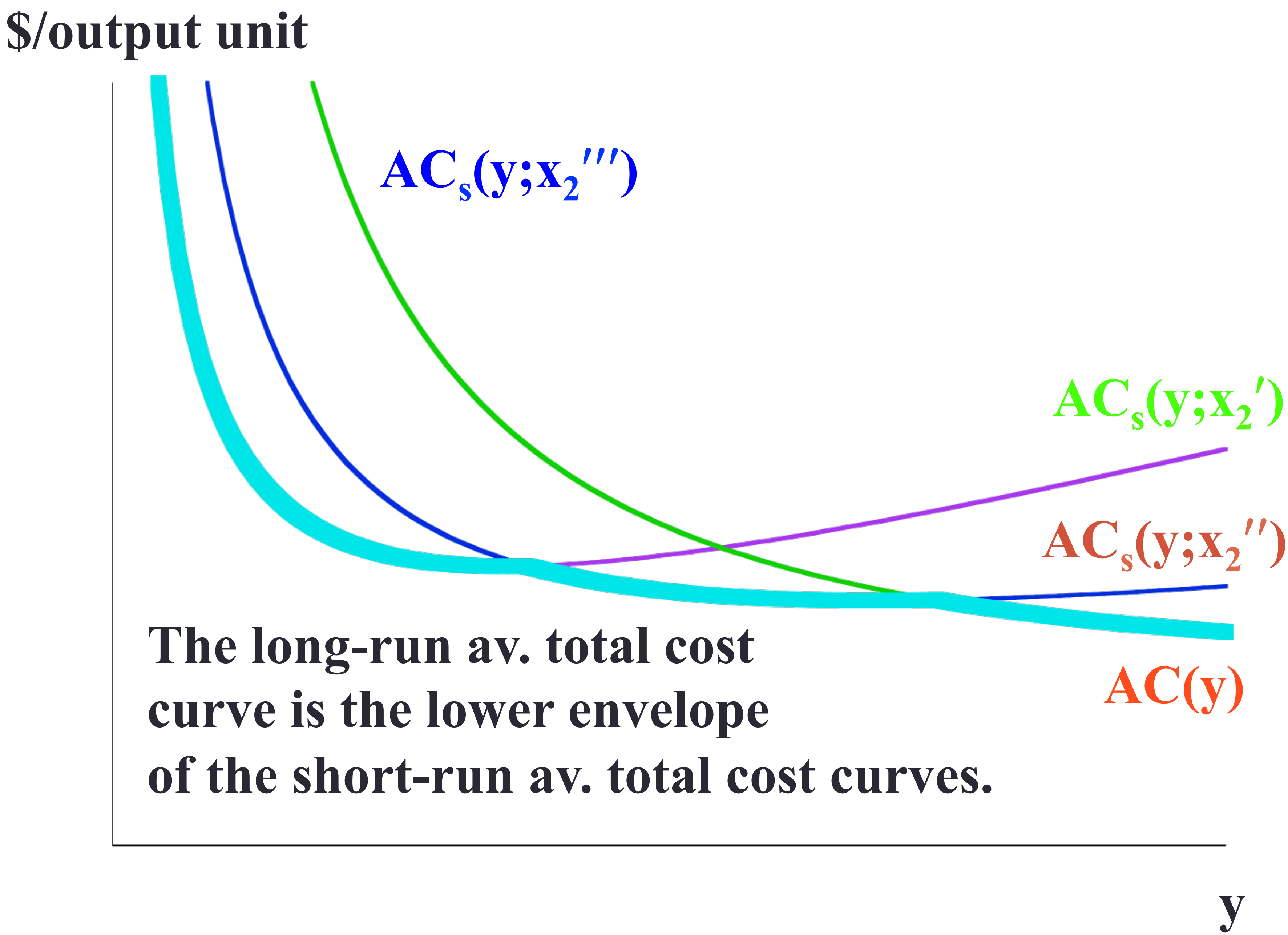
- then the firm's three short-run average total cost curves are ...



$/output unit
ACs(y;x2′)
ACs(y;x2″)
ACs(y;x2‴)
y

Short-Run & Long-Run Average Total Cost Curves

- The firm's long-run average total cost curve is the lower envelope of the short-run average total cost curves ...


$/output unit
$AC_s(y;x_2''')$
$AC_s(y;x_2')$
$AC_s(y;x_2'')$
$AC(y)$
The long-run av. total cost
curve is the lower envelope
of the short-run av. total cost curves.
y

Short-Run & Long-Run Marginal Cost Curves

- Q: Is the long-run marginal cost curve the lower envelope of the firm's short-run marginal cost curves?

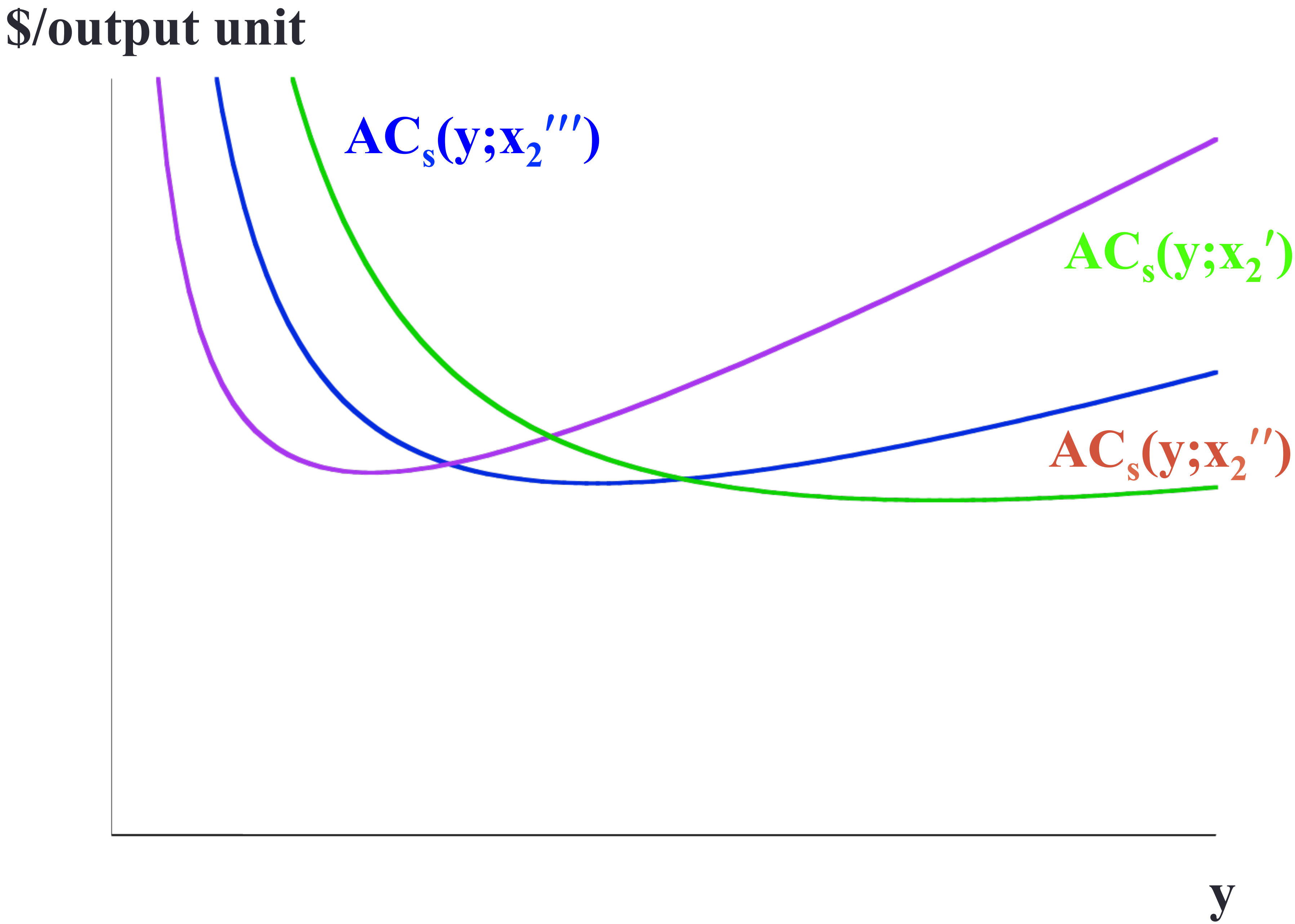
Short-Run & Long-Run Marginal Cost Curves

- Q: Is the long-run marginal cost curve the lower envelope of the firm's short-run marginal cost curves?

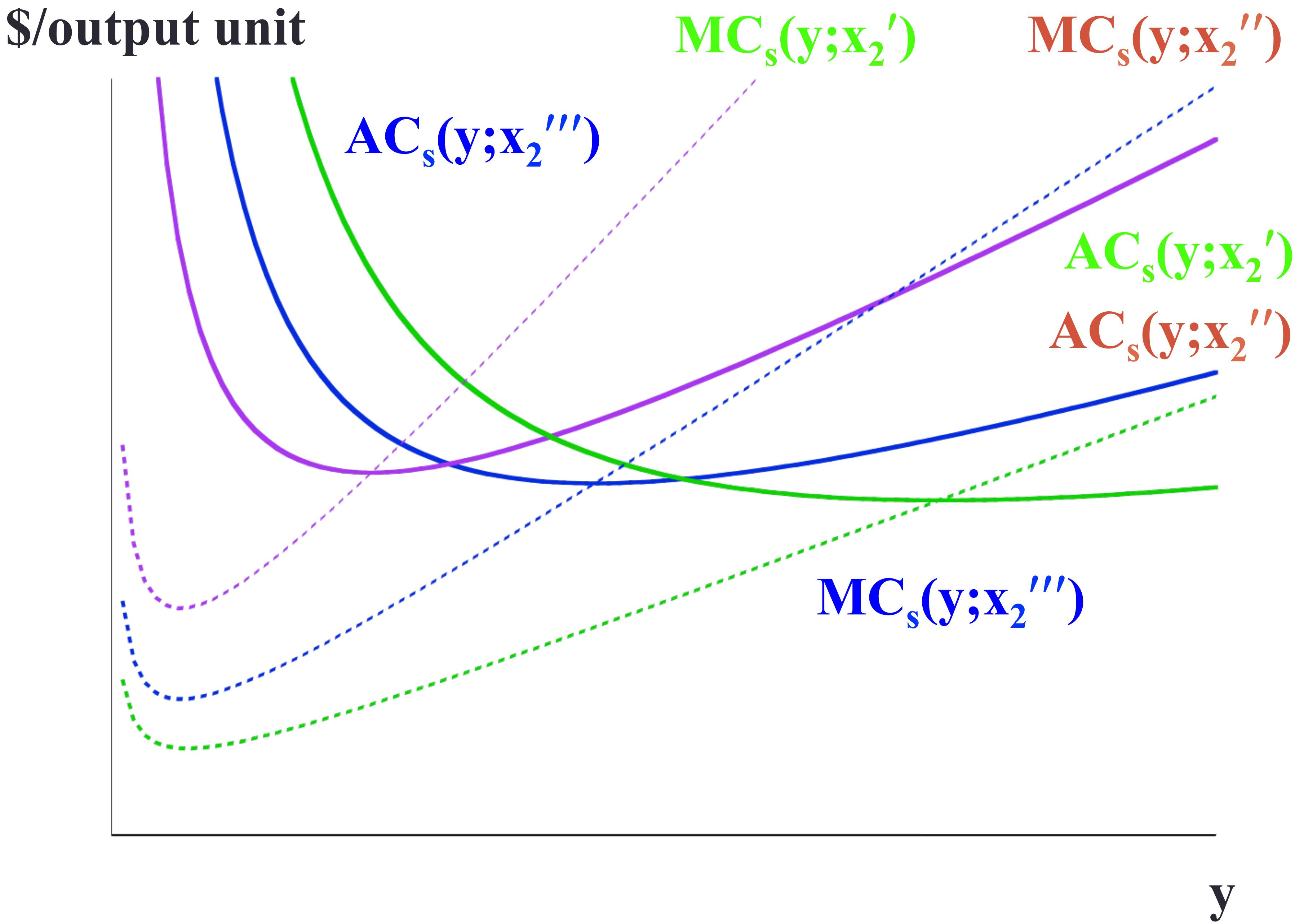
- A: No.

Short-Run & Long-Run Marginal Cost Curves

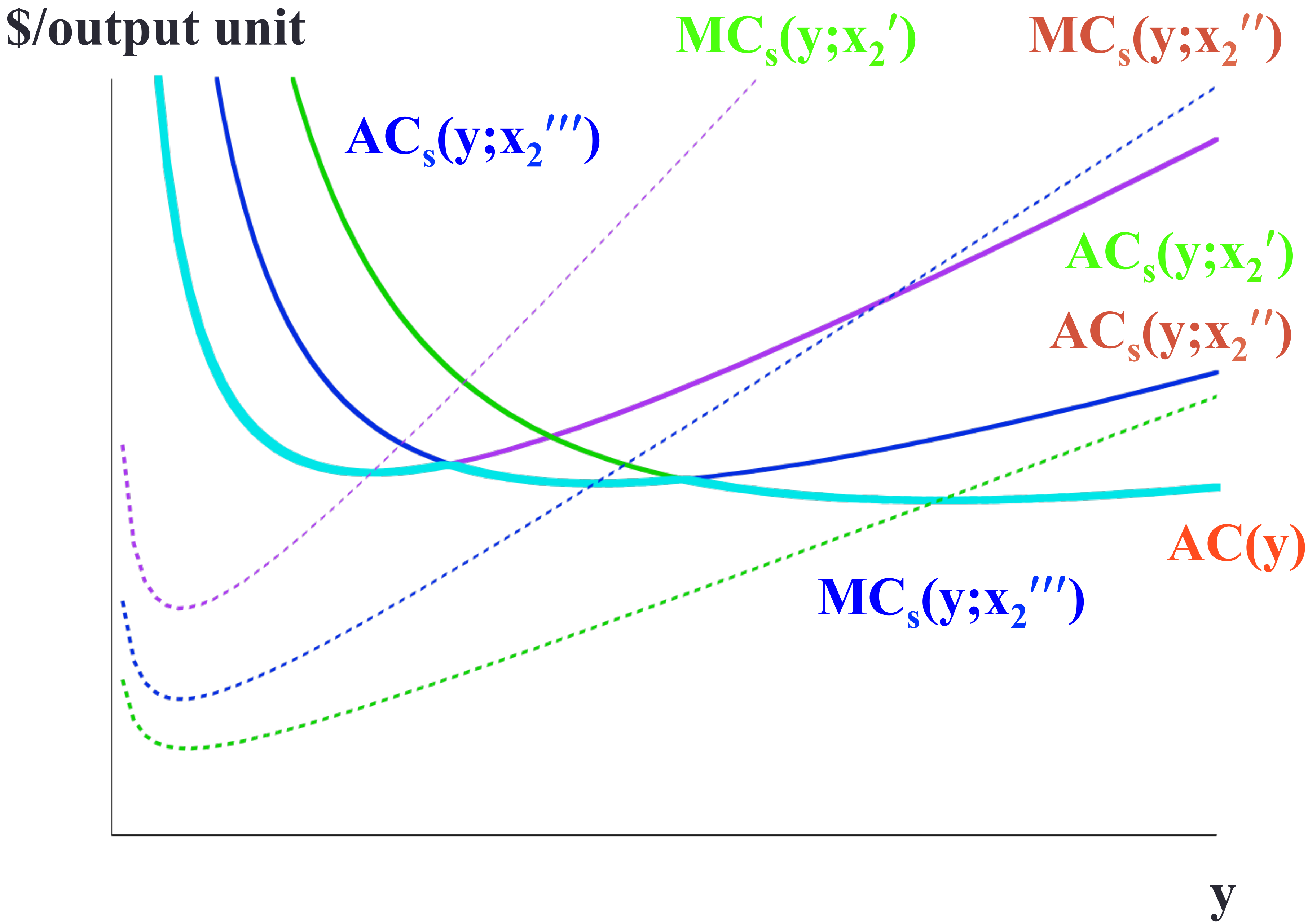
- The firm's three short-run average total cost curves are ...



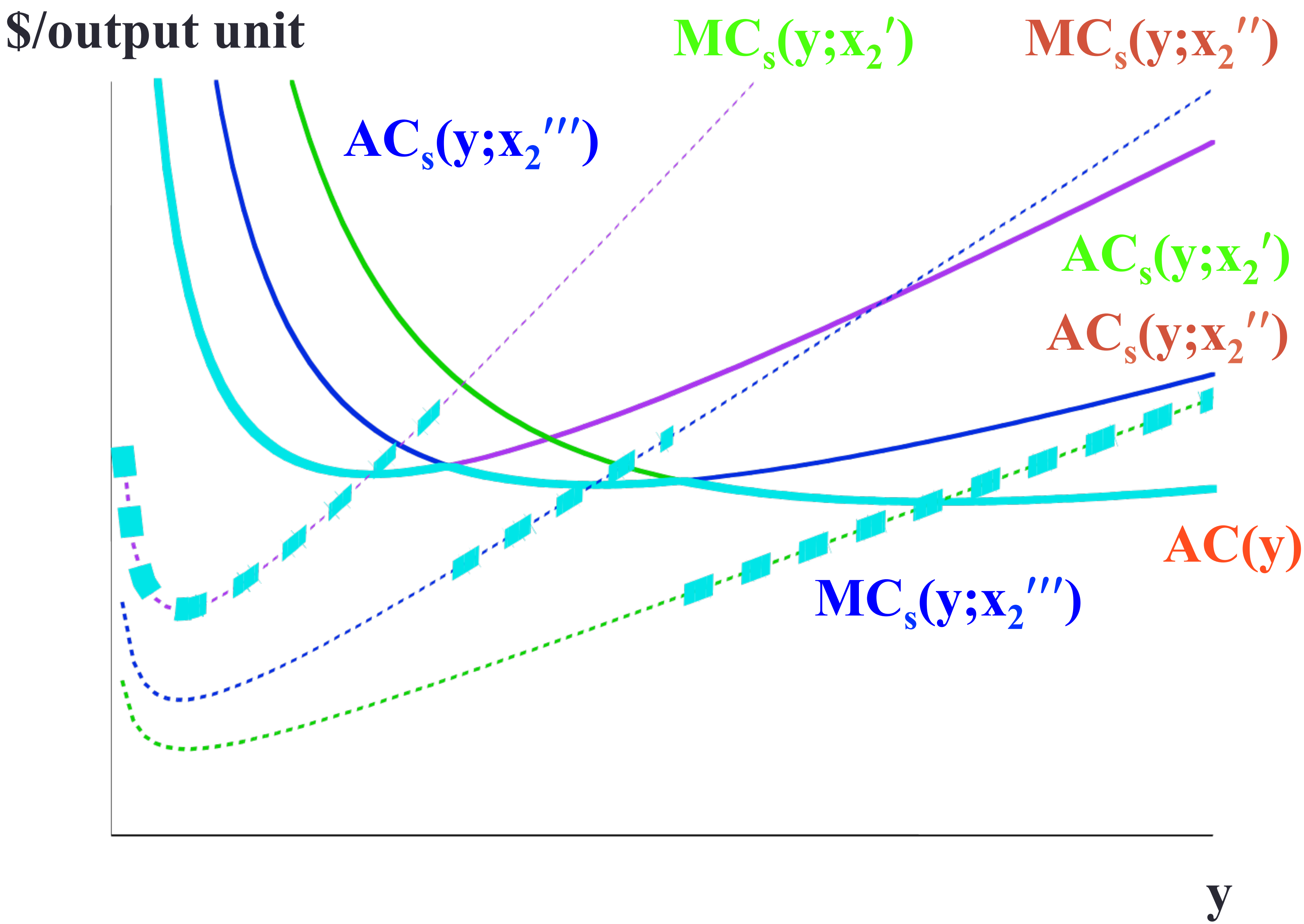
$/output unit
$AC_s(y;x_2''')$
$AC_s(y;x_2')$
$AC_s(y;x_2'')$
y



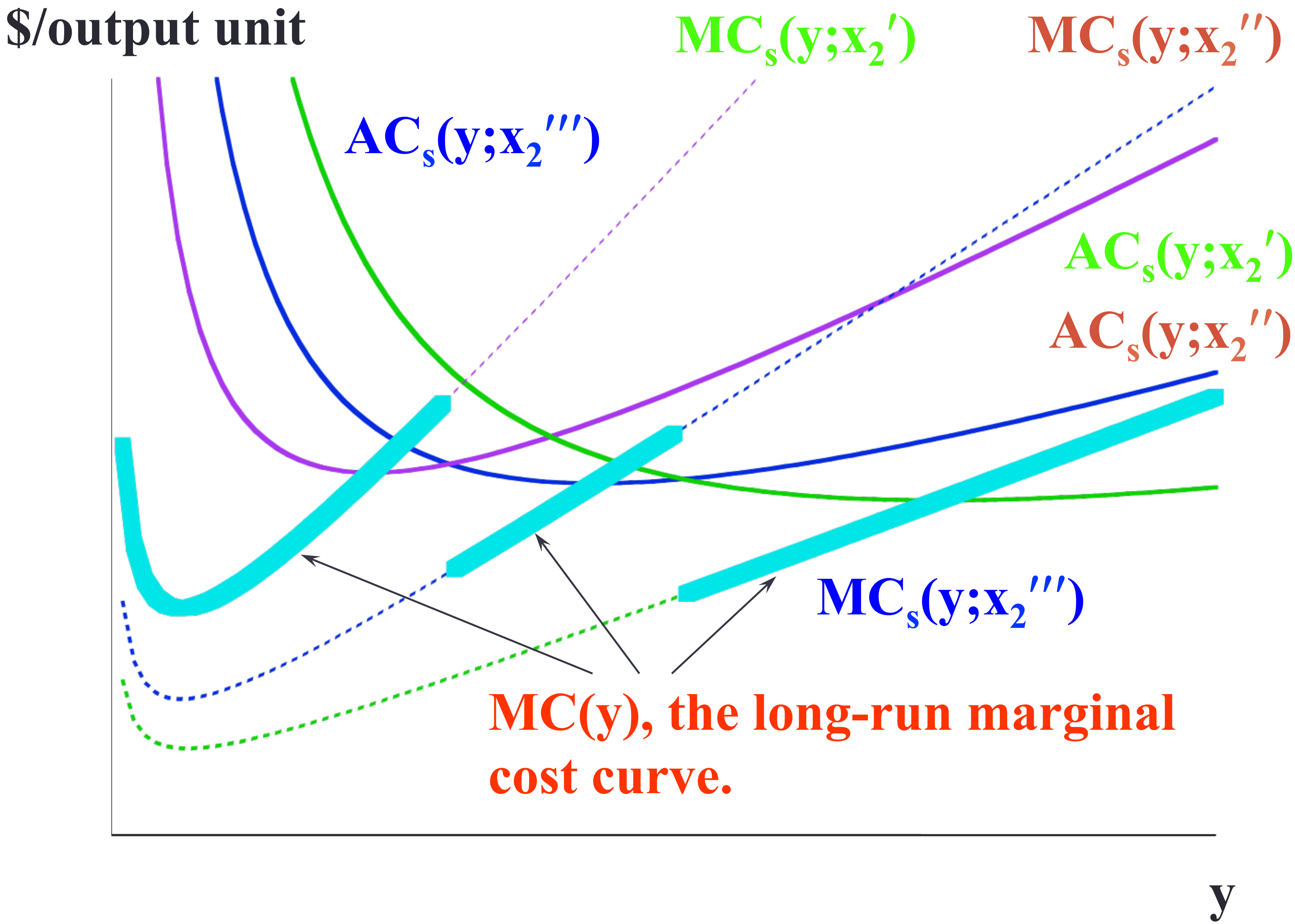
$/output unit
MCs(y;x2′)
MCs(y;x2″)
ACs(y;x2‴)
ACs(y;x2′)
ACs(y;x2″)
MCs(y;x2‴)
y



$/output unit
MCs(y;x2′)
MCs(y;x2″)
ACs(y;x2‴)
ACs(y;x2′)
ACs(y;x2″)
AC(y)
MCs(y;x2‴)
y



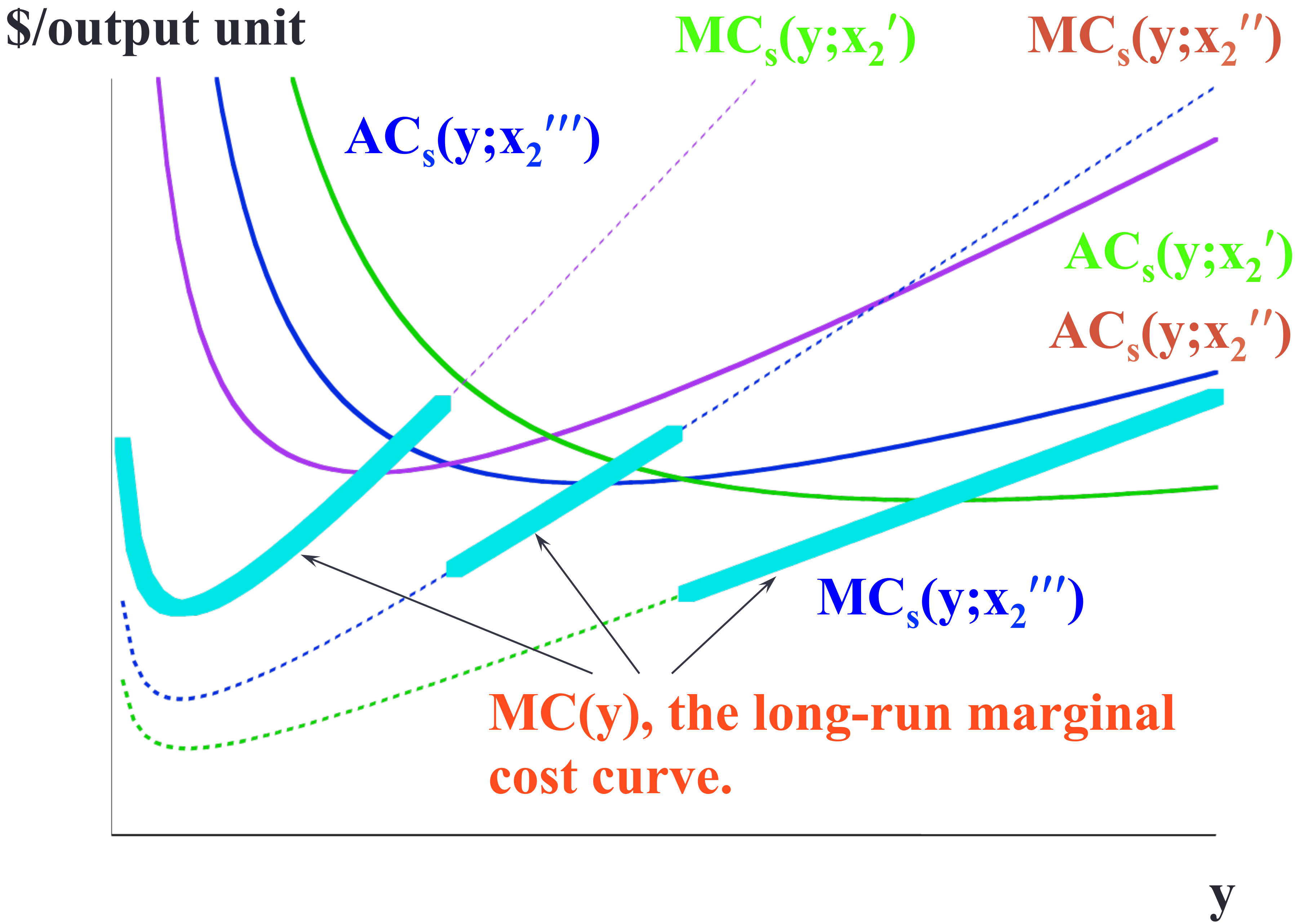
$/output unit
MC_s(y;x_2′)
MC_s(y;x_2″)
AC_s(y;x_2‴)
AC_s(y;x_2′)
AC_s(y;x_2″)
AC(y)
MC_s(y;x_2‴)
y



$/output unit
$MC_s(y;x_2')$
$MC_s(y;x_2'')$
$AC_s(y;x_2''')$
$AC_s(y;x_2')$
$AC_s(y;x_2'')$
$MC_s(y;x_2''')$
MC(y), the long-run marginal cost curve.
y

Short-Run & Long-Run Marginal Cost Curves

- For any output level $y > 0$, the long-run marginal cost of production is the marginal cost of production for the short-run chosen by the firm.


$/output unit
$MC_s(y;x_2')$
$MC_s(y;x_2'')$
$AC_s(y;x_2''')$
$AC_s(y;x_2')$
$AC_s(y;x_2'')$
$MC_s(y;x_2''')$
MC(y), the long-run marginal cost curve.
y

Short-Run & Long-Run Marginal Cost Curves

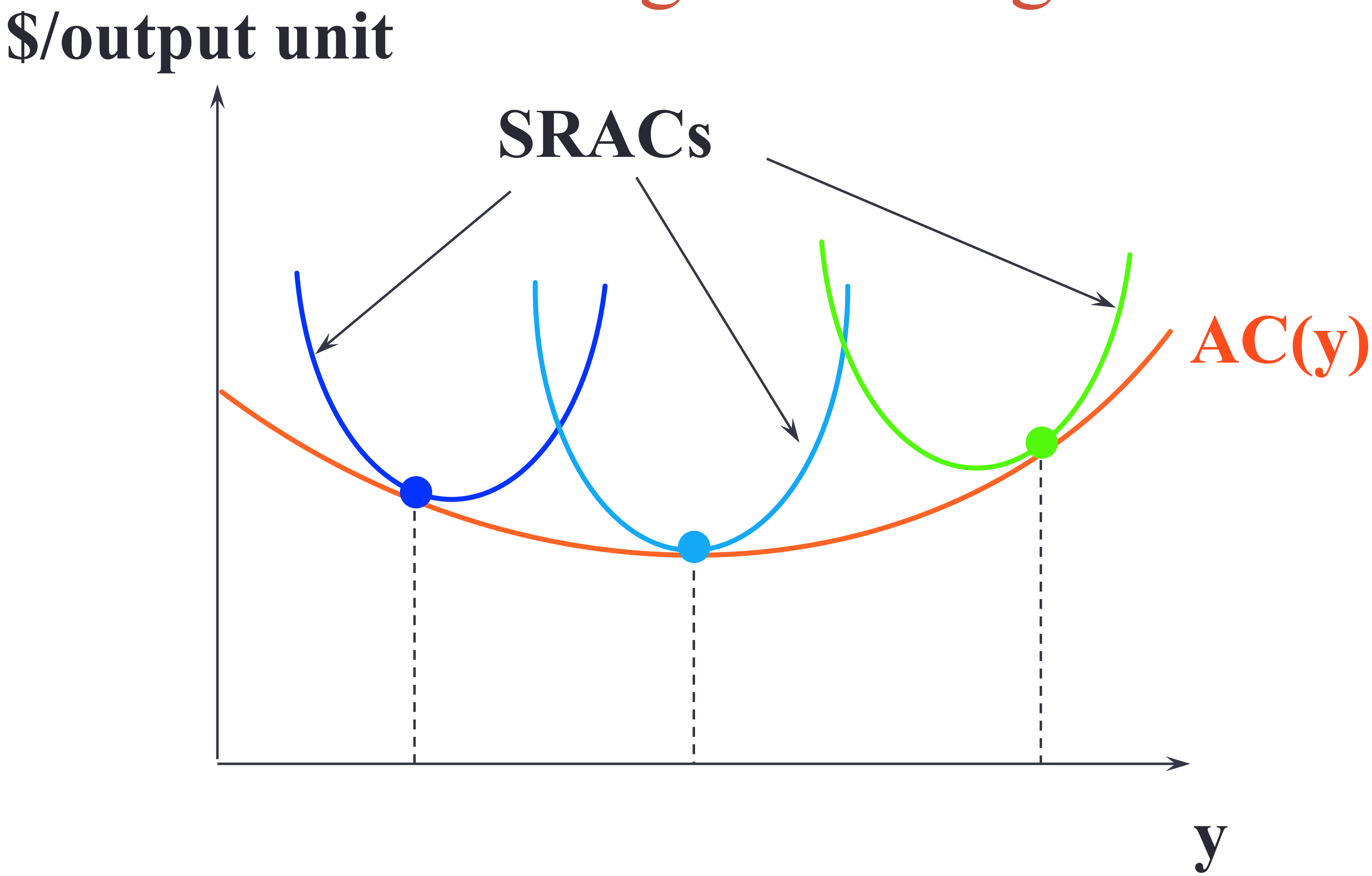
- For any output level $y > 0$, the long-run marginal cost is the marginal cost for the short-run chosen by the firm.
- This is always true, no matter how many and which short-run circumstances exist for the firm.

Short-Run & Long-Run Marginal Cost Curves

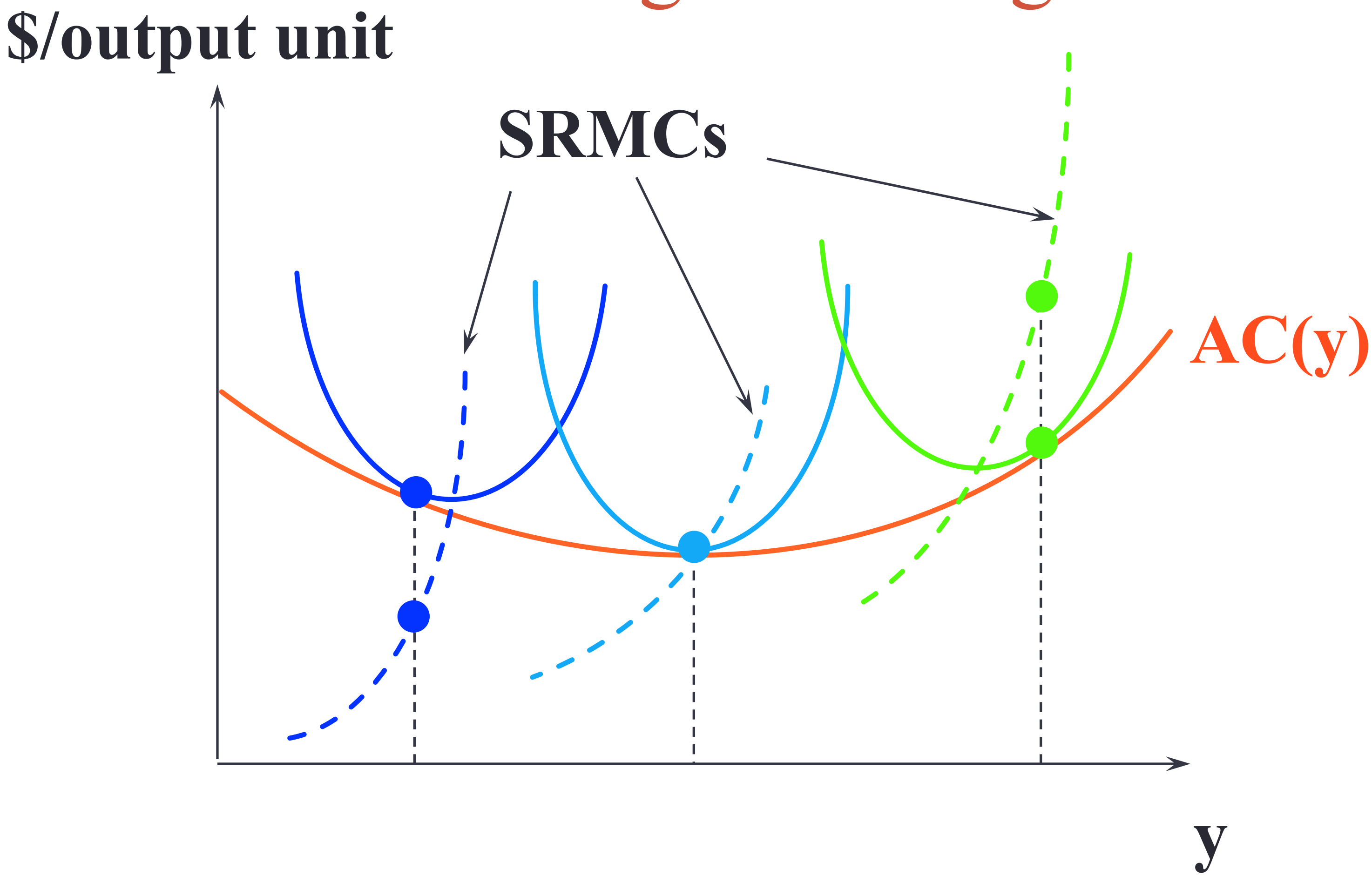
- For any output level $y > 0$, the long-run marginal cost is the marginal cost for the short-run chosen by the firm.

- So for the continuous case, where x_2 can be fixed at any value of zero or more, the relationship between the long-run marginal cost and all of the short-run marginal costs is ...

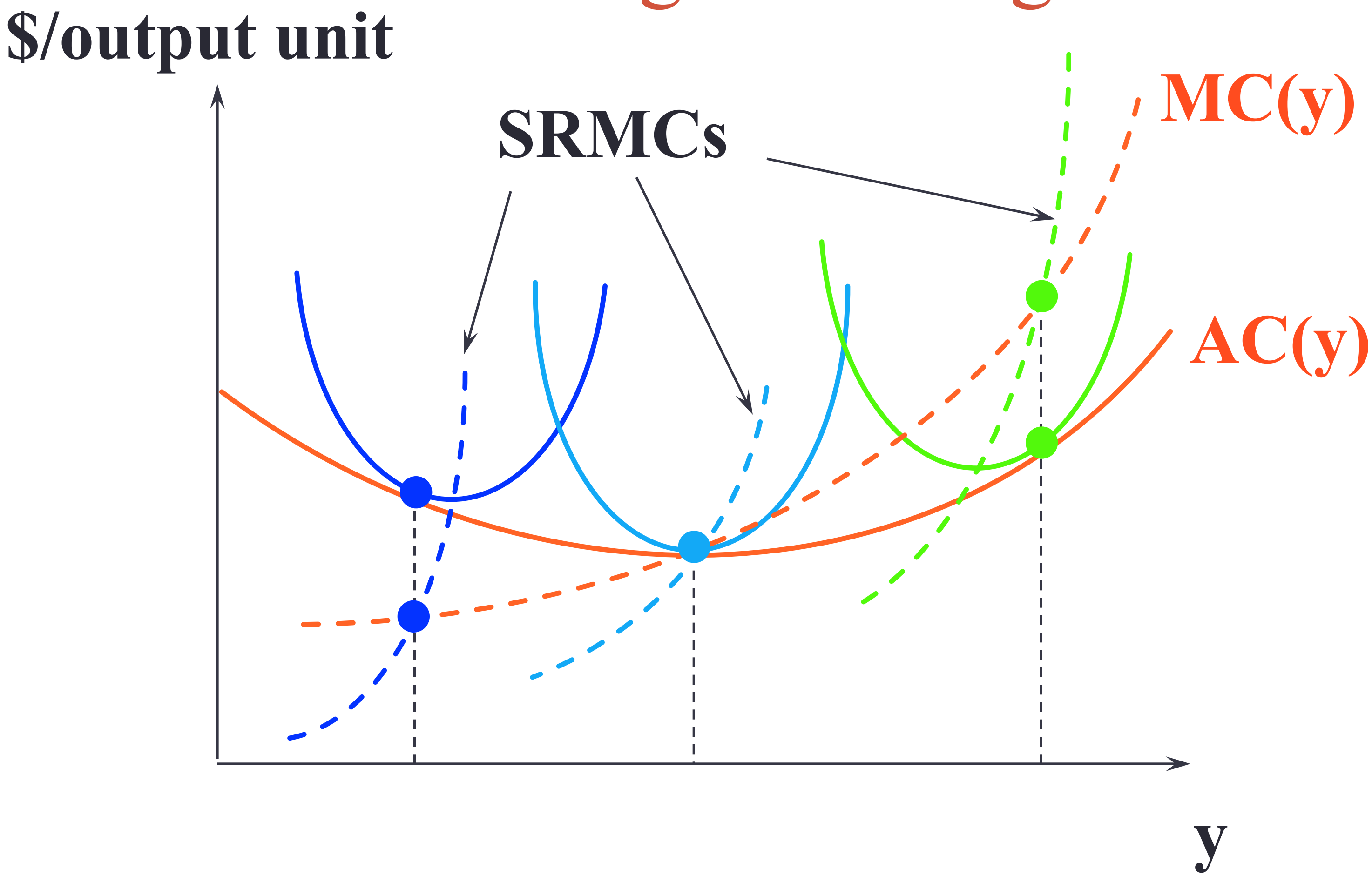
Short-Run & Long-Run Marginal Cost Curves



Short-Run & Long-Run Marginal Cost Curves



Short-Run & Long-Run Marginal Cost Curves



◆For each y > 0, the long-run MC equals the MC for the short-run chosen by the firm.

Summary

- Costs can be decomposed into **fixed** and **variable** components.
  - In the long run, all costs are **variable.**
- Marginal cost is the rate of change of variable cost as output changes.
- The law of diminishing marginal returns says that average variable costs must eventually increase in any short run, and MC intersects AVC at its minimum.
- The long run total cost curve is the **lower envelope** of the short run cost curves.
- The long run marginal cost is the marginal cost for whichever short run the firm chooses to be in.