MATHEMATICS PRE-COURSE NOTES
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1.One Variable Calculus Foundations

To find the effect of one variable (like money supply or government spending) on the other (e.g. interest rate or total production) is important to understand the relationships between the economic variables. This effect is captured by the "slope" of the linear functions (the "derivative" of the nonlinear functions).

1.1 Some Definitions

- *A real number* is a value of a continuous quantity that can represent a distance along a line. It can be rational (like 3/4 or -232) or irrational (like $\pi \approx$3.14159265... or $\sqrt{2} \approx$1.41421356...)
- A *function (*f(x)) of a real variable x with domain D is a rule that assigns a unique real number to each number x in D.

  y=f(x) $x \xrightarrow{f} y$ x: independent(exogeneous) variable y:dependent (endogenous) variable
- The *domain* is the set of numbers x at which f(x) is defined. When the domain is not specified, it is assumed that it includes all the real numbers for which the function takes for which the function takes meaningful values.

  For example for $\frac{1}{x-5}$ the domain is $\mathbb{R}$ excluding 5.
- The *range* (or *co-domain*) of a function is the set of all the possible values of it.

  e.g. for $|x|$ the domain is $\mathbb{R}$ but the co-domain is $\mathbb{R}^+$

1.2 Function Types

- *Polynomials* : Obtained by the addition of monomials $y = ax^k$ like $h(x) = 3x^5 - 2x^2 + x + 3$ $g(x) = 2x^2 - x$

  The highest exponent defines the order of the polynomial.

  -Constant function is polynomial of order zero: $y = a$

  -Linear functions is polynomial of order one: $y = mx + n$

  -Quadratic function (parabola) is a polynomial of order two: $y = mx^2 + nx + o$

  -Power function is a monomial of order k: $y = ax^k$
- *Rational functions* : Ratios of polynomials $f(x) = \frac{h(x)}{g(x)}$

  -A simple example is Hyperbola (constant over a monomial of order one) : $y = a/x$
- *Exponential functions :* $f(x) = e^x$ or $f(m) = 10^{-m}$
- *Trigonometric functions* : $f(x) = \sin(x)$ or $f(y) = \cos(y)$

[1] Using mainly C. Simon-L. Blume. (1994)

1.3 Basic geometric properties of a function

The basic geometric properties of a function are whether it is increasing or decreasing and the location of its local and global minimum and maximum (if exists)

- A function f is increasing if $x_1 > x_2$ implies $f(x_1) > f(x_2)$ wheras a function f is decreasing if $x_1 > x_2$ implies $f(x_2) > f(x_1)$
- The point where the function turns from decreasing to increasing is a minimum for the function and the point where the function turns from decreasing to increasing is a maximum for the function. If there is no greater (smaller) value of the function in its range from that maximum (minimum) then the maximum (minimum) is called global maximum (minimum).

  $f(x) = 2x + 1$ is always increasing

  $f(x) = -2x + 3$ is always decreasing

  $f(x) = 3x^4$ before 0 decreasing after zero increasing so it has a minimum at x=0

  $f(x) = -2x^2$ before 0 increasing after zero decreasing so it has a maximum at x=0

1.4 Linear Functions

$y = mx + n$ if y is distance, x is hours m denotes the velocity;
if y is utility, x is income m denotes marginal utility of income

- for $x = 0$ $y = n$ is the y-intercept, for $y = 0$ $x = \frac{-n}{m}$ is the x-inctercept
- Knowing two points (x_1, y_1) and (x_2, y_2) the unknowns m and n for the function can easily be found by:

  $m = \frac{y_2 - y_1}{x_2 - x_1}$ substituting one point $n = y_1 - mx_1$ or $n = y_2 - mx_2$

  Exercise 1 *If 0 °C equals to 32 °F and 100 °C equals to 212 °F, find °C as a linear function of °F.*

1.5 The slope of nonlinear functions

The slope of a nonlinear function of f at point $(x_0, f(x_0))$ is the slope of the tangent line to its graph at that point. It is the rate of change (marginal effect) of f with respect to x at that point.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \frac{df}{dx}(x_0)$$

Example 1 *The slope of* $f(x) = x^2$ *at* $x = 3$:

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0} 6 + h = 6$$

1.6 Rules for computing derivatives

- $(x^k)' = kx^{k-1}$

  Example 2 $f(x) = 7x^3 \longrightarrow f'(x) = 21x^2$

- $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

Example 3 $f(x) = 2x^7 \; g(x) = 4x^{\frac{1}{2}}$
$(f-g)(x) = 2x^7 - 4x^{\frac{1}{2}} \longrightarrow (f-g)'(x) = f'(x) - g'(x) = 14x^6 - 2x^{-\frac{1}{2}}$

- $(kf)'(x_0) = kf'(x_0)$

- $(f.g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) \longleftarrow$ *Product rule:* $(f.g)' = f'g + fg'$

Example 4 $f(x) = x - 1 \; g(x) = x^2 + x + 1$
$(x^3 - 1)' = (f.g)'(x) = 1.(x^2 + x + 1) + (x - 1).(2x + 1) = 3x^2$

- $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2} \longleftarrow$ *Quotient rule:* $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$

Example 5 $f(x) = x - 1 \; g(x) = x + 1$

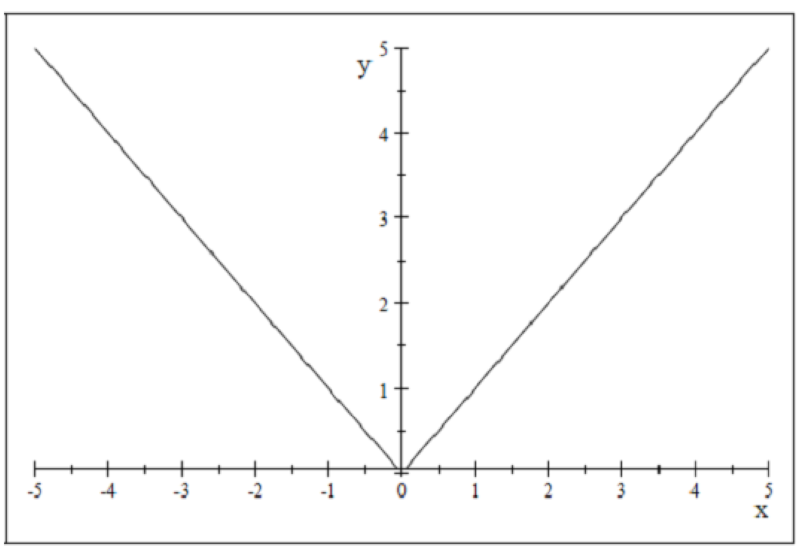
$\left(\frac{x-1}{x+1}\right)' = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$

- $([f(x)]^n)' = n\,[f(x)]^{n-1}\, f'(x) \longleftarrow$ *From Chain rule:* $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$

1.7 Differentiability

- If $f(x)$ has a tangent line at point x_0 or $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists then $f(x)$ is diffentiable at point x_0.
- If a function is differentiable at every point in its domain we say that it is a differentiable (smooth) function.

Example 6 $f(x) = |x|$ *is not differentiable*
because $\lim\limits_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = \frac{h-0}{h} = 1$
wheras $\lim\limits_{h\to 0^-} \frac{f(x_0)-f(x_0+h)}{h} = \frac{0-h}{h} = -1$



$|x|$

1.8 Continuity

- A function is continuous if its graph has no breaks (no jumps for the same point) in its domain.
- For a function to be differentiable, it must be continuous but not vice versa. For instance $f(x) = |x|$ is not differentiable but it is continous.

  For continuity, for every points in the domain (let x_0) the equality $\lim_{x \to x_0} f(x) = f(x_0)$ must hold.
- A *continuously differentiable function* is a function whose derivative is continuous (C^1). Every polynomial is C^1.

Example 7

$$f(x) = \left\{ \begin{array}{c} x^2,\ x \geq 0 \\ -x^2,\ x < 0 \end{array} \right\} \textit{ is continuous} \longrightarrow f'(x) = \left\{ \begin{array}{c} 2x,\ x \geq 0 \\ -2x,\ x < 0 \end{array} \right\} \textit{is continuous so } f(x) \textit{ is } C^1$$

Example 8 *$f(x) = x^{\frac{2}{3}}$ i s continous but not differentiable*

$$f'(x) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{\frac{2}{3}}}{h} = h^{-\frac{1}{3}} \textit{ do not exists} \begin{pmatrix} \to \infty \textit{ if } h > 0 \\ \to -\infty \textit{ if } h < 0 \end{pmatrix} \textit{ so f(x) is not } C^1$$

1.9 Higher order derivatives

- If $f(x)$ is C^1 then we can ask whether $f'(f'(x))$ exists that is $f''(x) = \frac{d}{dx}(\frac{df}{dx})$
- Polynomials are C^1

Example 9

$$f(x) = 3x^7 + 2x^3 \longrightarrow f'(x) = 21x^6 + 6x^2 \longrightarrow f''(x) = 126x^5 + 12x$$

Example 10

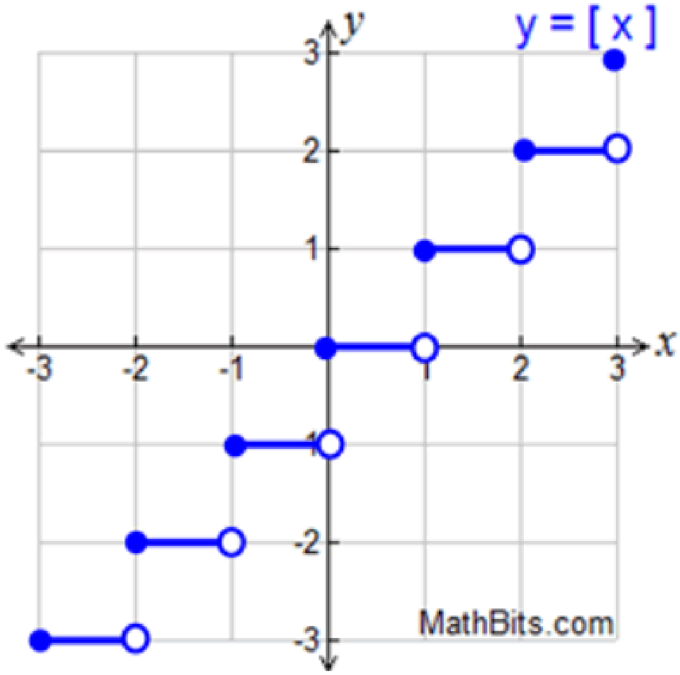
$$f(x) = \left\{ \begin{array}{c} x^2,\ x \geq 0 \\ -x^2,\ x < 0 \end{array} \right\} \text{ is continuous and } C^1 \text{ but} \longrightarrow f''(x) = \left\{ \begin{array}{c} 2,\ x \geq 0 \\ -2,\ x < 0 \end{array} \right\} \text{ does not exist}$$

- If f'' is continuous then f is C^2 (twice continosly differentiable)
- If $f^{(n)}$ is continuous then f is C^n. All polynomials are C^∞

Example 11 *$f(x) = 3x^{\frac{-3}{2}}$ is continuous in its domain $\mathbb{R}^+$*

$$f'(x) = \frac{-9}{2} x^{\frac{-5}{2}} \longrightarrow f''(x) = \frac{45}{4} x^{\frac{-7}{2}} \textit{ are also continuous in } \mathbb{R}^+$$

Example 12 *$f(x) = [x]$ that is the largest integer $\leq x$, not continous and not C^1 in its domain but excluding integers it is C^∞*



1.10 Approximation by Differentials

- Without limit $\frac{f(x_0+h)-f(x_0)}{h} \approx f'(x_0)$ or $f(x_0+h) \approx f(x_0) + f'(x_0).h$

  Example 13 *Assume a firm's production function is $f(x) = \frac{1}{2}\sqrt{x}$ Suppose the firm is currently using 100 unit of labor input x. Find the firms current marginal product of labor that is the additional output that can be achieved by hiring one more unit of labor.*

  actual result is $f'(100) = \frac{1}{4}100^{-\frac{1}{2}} = 0.025$

  approximate result is $\frac{f(101)-f(100)}{1} = 0.02494$

  Example 14 *Estimate the cube root of 1001.5 (true value=10.004998)*

  $f(x) = x^{1/3}$ *and hence* $f'(x) = \frac{1}{3}x^{-2/3}$*.We know* $f(1000) = 10$ *and* $f'(1000) = \frac{1}{300}$

  We need to find $f(1001.5) \approx f(1000) + f'(1000).1.5 = 10 + \frac{1.5}{300} = 10.005$

  Example 15 *The population is estimated to be t years from now $f(t) = 40 - \frac{8}{t+2}$ .Estimate the population rise after 6 months?*

  $f'(t) = \frac{8}{(t+2)^2} \longrightarrow f'(0) = 2 \longrightarrow$*the population rise* $\approx f'(0).0.5 = 1$

2.One Variable Calculus: Applications

2.1 Using first and second derivatives to sketch graphs

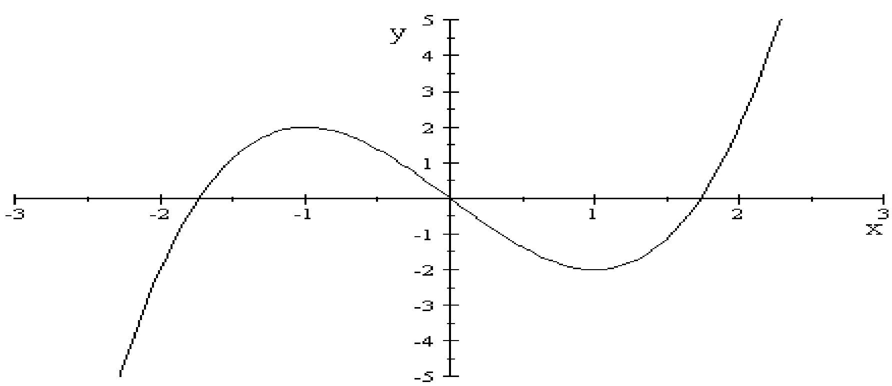
1. Find the critical (stationary) points at which $f'(x) = 0$ or $f'(x)$ is not defined, order those points showing on the x axis as $(-\infty, x_1)$, $(x_1, x_2)...(x_k, \infty)$
2. Find the sign of the $f'(x)$ as x goes to ∞ for (x_k, ∞) and $-\infty$. Alternate the sign for the subsequent interval if the critical point is not even times repeated (double, quadruple etc.). Otherwise do not change the sign.
3. The function $f(x)$ is increasing in the intervals with positive sign whereas it is decreasing in the intervals with the negative sign. If $f'(x)$ is positive (negative) always then $f(x)$ is always increasing (or decreasing).
4. Do the first two steps for $f''(x)$ instead of $f'(x)$.
5. The function $f(x)$ is convex (upward curved, the slope of $f'(x)$ is increasing) in the intervals with positive sign, whereas it is concave (downward curved,the slope of $f'(x)$ is decreasing) in the intervals with the negative sign. If $f''(x)$ is positive (negative) always then $f(x)$ is always convex (or concave).

- **Definition 1** *f is convex in the interval* $[a, b]$ *iff* $f((1-t)a + tb) \leq (1-t)f(a) + tf(b)$ *with* $0 \leq t \leq 1$ *(the secant line joining two points on the funtion is above the graph of the function)*

  Definition 2 *f is concave in the interval* $[a, b]$ *iff* $f((1-t)a + tb) \geq (1-t)f(a) + tf(b)$ *with* $0 \leq t \leq 1$ *(the secant line joining two points on the funtion is below the graph of the f function)*

Example 16 *Sketch* $f(x) = x^3 - 3x$

1. $f'(x) = 3x^2 - 3 \rightarrow$ critical points are -1 and 1
2. $f'(x)$ is positive as x goes to ∞ or $-\infty$, it changes sign and becomes negative between -1 and 1
3. Then $f(x)$ increasing in $(-\infty, -1)$ and $(1, \infty)$ and decreasing between -1 (local ma$x$) and 1 (local min)
4. $f''(x) = 6x \rightarrow$ critical point is 0, $f''(x)$ is positive as x goes to ∞ and negative as x goes to $-\infty$
5. $f(x)$ is concave in $(-\infty, 0)$ and convex in $(0, \infty)$. $x = 0$ is the **inflection point** [2] as the concavity of $f(x)$ changes at that point.



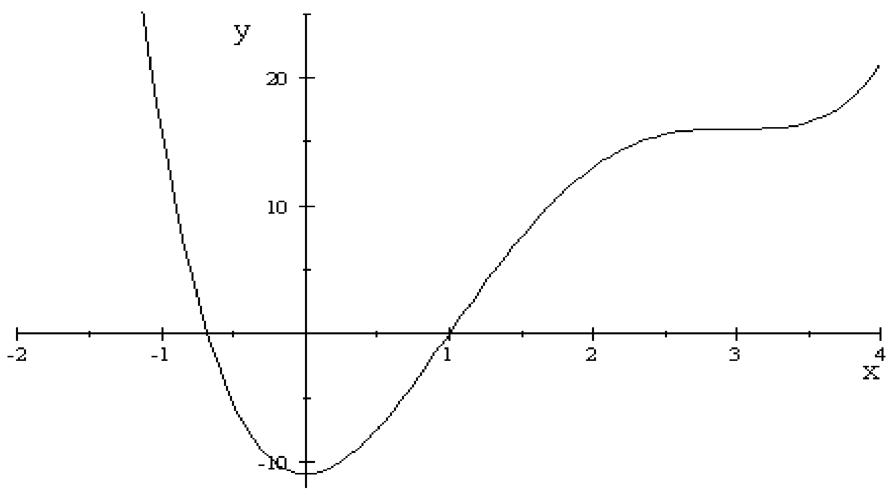
$y = f(x) = x^3 - 3x$

[2] Inflection points are the points where the function changes its concavity. They can be stationary but cannot be local minima or maxima.

Example 17 *Sketch* $f(x) = x^4 - 8x^3 + 18x^2 - 11$

1. $f'(x) = 4x^3 - 24x^2 + 36x = 4x(x-3)^2$. Its critical points are 0 and 3.
2. $f'(x)$ is positive as x goes to ∞ and negative as x goes to $-\infty$, positive in $(0,\infty)$ excluding $x = 3$ (double root)
3. Then $f(x)$decreasing in ($-\infty$,0) and increasing in $(0,\infty)$ excluding x=3, hence *x=0 is* **the global minimum.**
4. $f''(x) = 12x^2 - 48x + 36 = 12(x-1)(x-3) \rightarrow$ critical points are 1 and 3, $f''(x)$ is positive as x goes to ∞ or $-\infty$
5. $f(x)$ is convex in ($-\infty$,1) and $(3,\infty)$, it is concave in $(1,3)$.

   $x = 1$ and $x = 3$ are **the inflection points** due to concavity changes, $x = 3$ is also a **saddle point**[3] as it is a stationary $(f'(3) = 0)$ but not an extremum point.

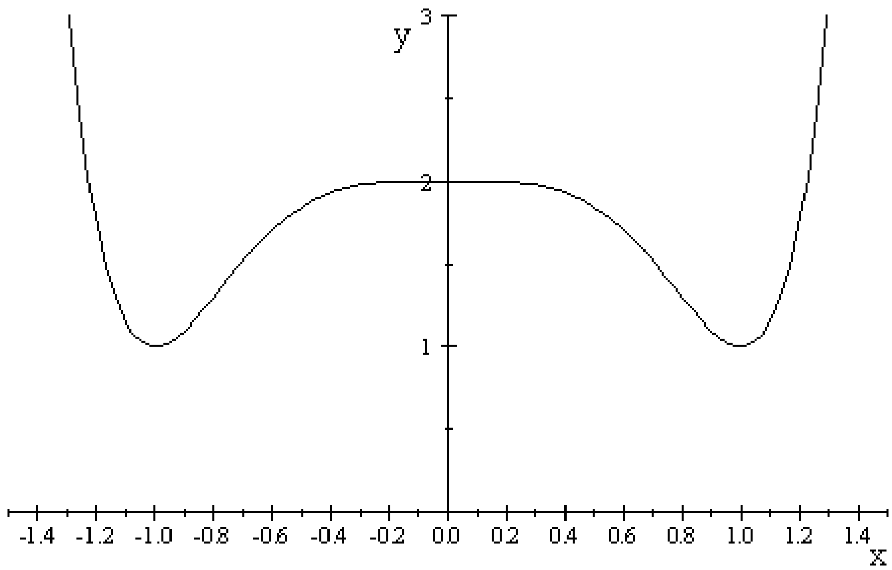


$y = f(x) = x^4 - 8x^3 + 18x^2 - 11$

Example 18 *Sketch* $f(x) = 2x^6 - 3x^4 + 2$

1. $f'(x) = 12x^5 - 12x^3 = 12x^3(x-1)(x+1)$. Its critical points are -1, 0, and 1.
2. $f'(x)$ is positive as x goes to ∞ and negative as x goes to $-\infty$, positive in $(-1,0)$, negative in $(0,1)$.
3. Then $f(x)$decreasing in ($-\infty$,-1), increasing in $(-1,0)$,decreasing in $(0,1)$, and increasing in (1,∞).Two minima at $x = -1$ and $x = 1$. Local max at $x = 0$.
4. $f''(x) = 60x^4 - 36x^2 = 12x^2(5x^2 - 3) \rightarrow$ critical points are -$\sqrt{3/5}$,double 0 and $\sqrt{3/5}$, $f''(x)$ is positive as x goes to ∞ or $-\infty$
5. $f(x)$ is convex in ($-\infty$,-$\sqrt{3/5}$) and ($\sqrt{3/5},\infty$) , it is concave in $(-\sqrt{3/5}, \sqrt{3/5})$ where $x = -\sqrt{3/5}$ and $x = \sqrt{3/5}$ are the inflection points.

[3]Saddle point is stationary point such that the curve (1D)/surface(2D) etc. in the neigbourhood of that point is not entirely on any side of the tangent space at that point. In one dimension, a saddle point is a stationary inflection point.



$$y = f(x) = 2x^6 - 3x^4 + 2$$

2.2 Graphing rational functions

- The same procedure shown in the previuos section is used also for graphing rational functions.
- In addition, we need to identify the vertical and horizontal asymtotes of the function.
- The points (x) that make the denominator of the function zero are the **vertical asymtotes.**
- If $f(x)$ becomes close to a finite number in the limit as x goes to ∞ or -∞, than that y axis point is called **horizontal asymtote.**
- For polynomials, the leading term -the monomial with the highest degree (let a_0x^k)- determines the shape of the tail of the graph whether to go ∞, -∞ or an horizontal asymtote.

  -if k is even both tails go to ∞ as $|x| \to \infty$ if $a_0 > 0$, both tails go to -∞ as $|x| \to \infty$ if $a_0 < 0$

  -if k is odd one tails go to ∞ and the other goes to -∞ as $|x| \to \infty$ depending on the sign of a_0
- $g(x) = \frac{a_0x^k+a_1x^{k-1}+...+a_{k-1}x^1+a_k}{b_0x^m+b_1x^{m-1}+...+b_{m-1}x^1+b_m} \longrightarrow l(x) = \frac{a_0x^k}{b_0x^m} = \frac{a_0}{b_0}x^{k-m}$as $|x| \to \infty$

  -if k>m $l(x)$ is a monomial, the tails of the rational funtion goes to $\pm\infty$ as stated above

  -if k<m $l(x) \to 0$ Both the tails of $g(x)$ are asymtotic to the x axis ($y = 0$) that is a horizontal asymtote for $g(x)$

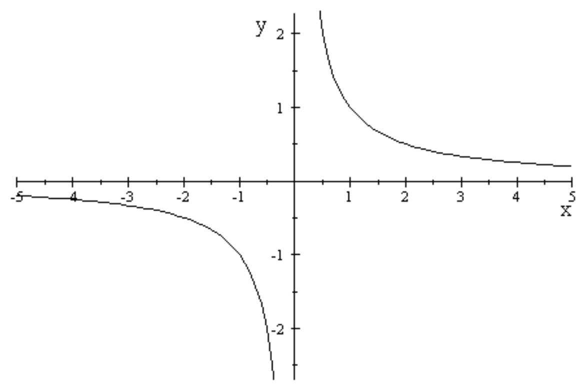
  -if k=m $l(x) \to \frac{a_0}{b_0}$ Both the tails of $g(x)$ are asymtotic to the horizontal line ($y = \frac{a_0}{b_0}$)

Example 19 *Sketch* $f(x) = \frac{1}{x}$

$f(x)$ has horizontal asymtote to the x axis.

$f'(x) = -\frac{1}{x^2}$ has a critical point at $x = 0$, so $f(x)$is decreasing in both sides of the point.

$f''(x) = \frac{2}{x^3}$ is positive as x goes to ∞ and negative as x goes to $-\infty$, hence $f(x)$ is convex on the right and concave on the left of y axis.



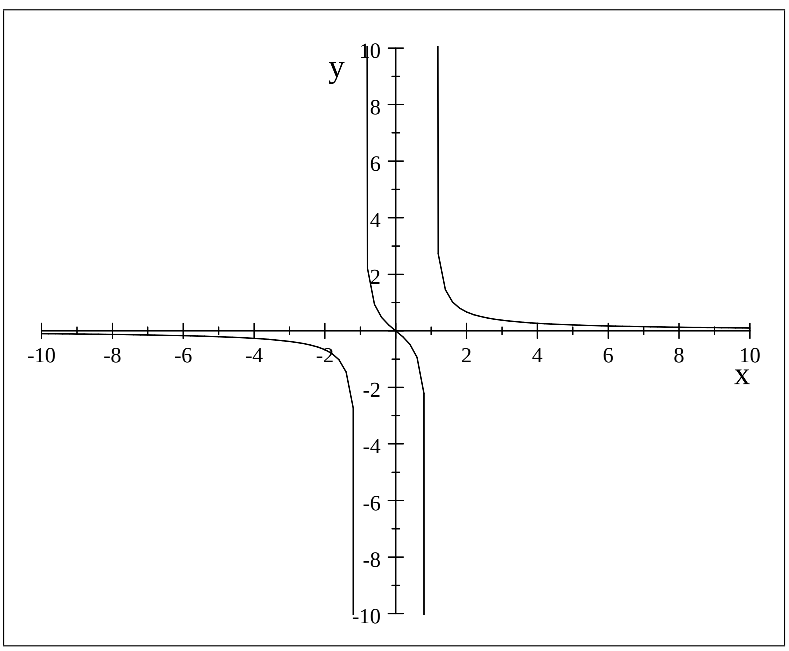
$$y = f(x) = \frac{1}{x}$$

Example 20 *Sketch* $f(x) = \frac{x}{x^2-1}$

$f(x) \to \frac{1}{x}$ as $|x| \to \infty$ so it has a horizontal asymtote to the x axis. Its denominator becomes zero for $x = -1$ and $x = 1$ which are its vertical asymtotes.

$f'(x) = \frac{-x^2-1}{(x^2-1)^2}$ has a critical points at $x = -1$ and $x = 1$, since they are double repeated f(x) is decreasing before, after and between these points.

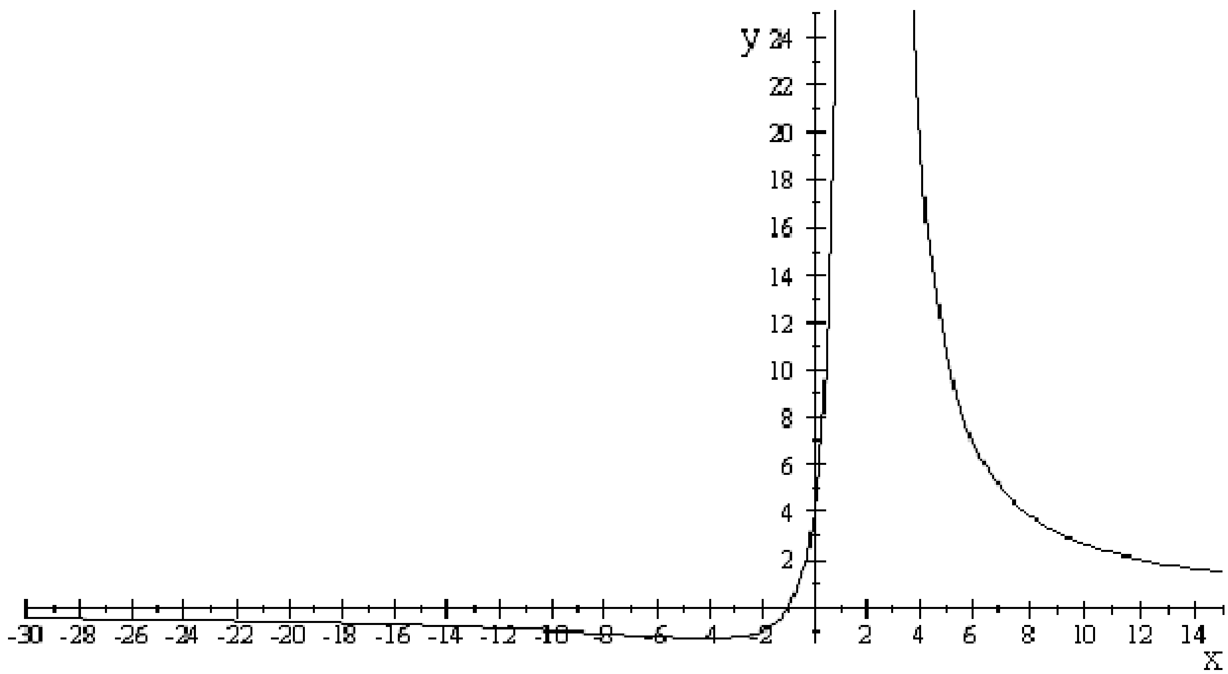
$f''(x) = 2\frac{x}{(x^2-1)^3}\left(x^2+3\right)$ has a critical points at $x = -1$, $x = 0$ and $x = 1$, it is positive as x goes to ∞ and negative as x goes to $-\infty$, hence $f(x)$ is concave $(-\infty, -1)$, convex in $(-1, 0)$,concave in $(0, 1)$ and concave in $(1, \infty)$



$$y = f(x) = \frac{x}{x^2-1}$$

Example 21 *Sketch* $f(x) = \frac{16(x+1)}{(x-2)^2}$

$f(x) \to \frac{16}{x}$ as $|x| \to \infty$ so it has a horizontal asymtote to the x axis. Its denominator becomes zero for $x = 2$ which is its only vertical asymtote.
$f'(x) = \frac{16(x-2)^2 - 16(x+1)2(x-2)}{(x-2)^4} = -\frac{16}{(x-2)^3}(x+4)$, critical points at $x = -4$ and $x = 2$, $f(x)$ is increasing between these points decreasing elsewhere.
$f''(x) = \frac{32(x+7)}{(x-2)^4}$, critical points at $x = -7$ and $x = 2$, concave in $(-\infty, -7)$,in convex in $(-7, 2)$ and $(2, \infty)$.Hence $x = -7$ is the inflection point. Since $f''(-4) > 0$ $x = -4$ is a minimum and $f(-4)$ negative at that point, whereas before that point the function always decreases until it becomes zero asmytotically. So $x = -4$ is a global minimum.



$$y = f(x) = \frac{16(x+1)}{(x-2)^2}$$

2.3 Maxima and Minima

- Finding the maximum or minimum points (extremum points) is very important in economics in order to reach optimal values of economic variables. e.g. maximizing utility or profit, minimizing cost.

1. - An extremum point can be on the boundary or in the interior part of the domain.
   - If it is an interior extremum point then it is a critical point. For the crtical points:

   -if $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a maxima of $f(x)$

   -if $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a minima of $f(x)$

   -if $f'(x_0) = 0$ and $f''(x_0) = 0$, then x_0 can be a minima (e.g. $f(x) = x^4$) , maxima (e.g. $f(x) = -x^4$) or neither (e.g. $f(x) = x^3$) .

Example 22 *Find minima and maxima of* $f(x) = x^4 - 4x^3 + 4x^2 + 4$

$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x-1)(x-2)$, critical points are 0,1 and 2
$f''(x) = 12x^2 - 24x + 8$
$f''(0) = f''(2) = 8 > 0 \to x = 0$ and $x = 2$ are minimum points $f''(1) = -4 < 0 \to x = 1$ is a maxima

Global Maxima and Minima
In three cases global maximum or minumum points are found easily:

1. When the domain of $f(x)$ is an interval in $\mathbb{R}^1$ and it has only one critical point that is a local max (min) then that point is global max (min)

   The proof comes from the idea that if the point were not a global max (min) then there should be another critical point between two max (min).

2. If f is a C^2 function whose domain is an interval and f'' is never zero, then f has at most one critical point that is a global min $f''(x) > 0$ and a global max $f''(x) < 0$

   The proof: If $f'' > 0$ always then f' is an increasing function such that it can have a value "0" only one time and from the previous theorem it is global min.

3. A continuous functions whose domain is a closed and bounded interval [a,b] must have a global max and a global min (Weierstrass theorem).

   In this case, we find critical points and evaluate the function at these points and the boundary points. The point with largest value of f is the global maximum and the point with the smallest vlue of is the global minimum.

Example 23 *A firm produces every book with a cost of $5. Every book is sold for $10 and each day 10 books are sold currently. The firm expects to sell one additional book for each dollar decrease in price. What are the demand and profit functions and the price P that maximizes the profit?*

$X = mP + n$, $m = -1$, current $(X, P) = (10, 10)$, substituting them in the demand equation we find $X = 20 - P$ then the profit becomes: $\pi = (P - 5)(20 - P) = -P^2 + 25P - 100$
$\pi' = -2P + 25 \rightarrow P = 12.5$ is the only critical point, $\pi'' = -2 < 0$ then $P = 12.5$ is the global maximum point that maximes the profit of the firm.

2.4 APPLICATIONS TO ECONOMICS

In this section we briefly analyse the properties of some general types of functions which are used in economic applications and studies.

2.4.1 Production Function

- It relates the amount of output (x) to the amount of input $(q) : x = f(q)$
- Continuous or maybe C^2
- Increasing
- There is a level of input until which the function is convex and then it is concave

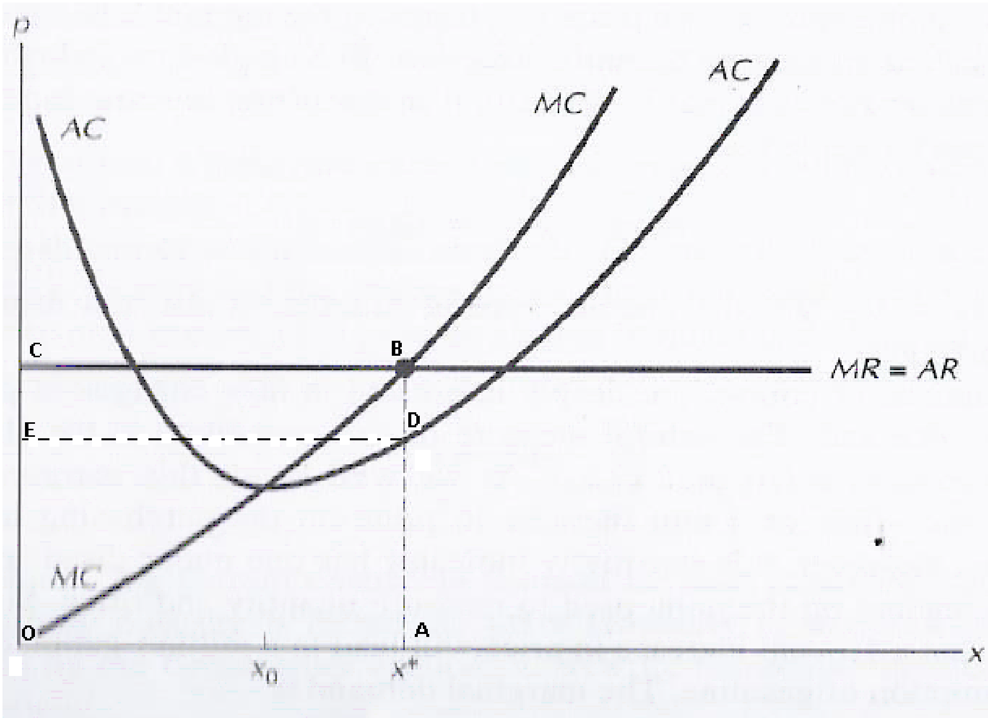
2.4.2 Cost Function

- A cost funtion assigns a cost to the level of output: $C(x)$
- $MC(x) = C'(x)$ is the marginal cost that measures the additional cost incurred from the production of one unit more when the current output is x.
- The average cost is the cost per unit produced: $AC(x) = C(x)/x$
- The essential properties of cost funtion:
  $C(x)$ is a naturally increasing function. Moreover, Let $C(x)$ is C^1 then
  -if MC>AC, AC is increasing (Doing better than average rises average)
  -if MC<AC, AC is decreasing (Doing worse than average decreases average)
  -at an interior minimum of AC, MC=AC

  The proof comes from: $AC\prime(x) = \frac{d}{dx}(\frac{C(x)}{x}) = \frac{C'(x)x - C(x)}{x^2} = \frac{C'(x) - C(x)/x}{x} = \frac{MC - AC}{x}$

2.4.3 Revenue and Profit Functions

- How much money a firm receives from selling its x unit of output: $R(x) = p(x)x$
- $MR(x) = R'(x)$ is the marginal revenue, average revenue(AR) is the unit price $p(x)$
- Profit $\pi(x) = R(x) - C(x)$ to maximize it we solve for $\pi'(x^*) = MR(x^*) - MC(x^*) = 0 \to MR(x^*) = MC(x^*)$. This means optimum output (x^*) for maximum profit occurs when $MR = MC$.
- In a model of perfect competetion (with many firms and no individual firm can control the output price by its production activity) the market price for any firm receives for its output is constant: $p(x) = p$ $R(x) = px$ $MR = AR = p$



The AC-, MC-, AR-, and MR-curves for a competitive firm.

- MC curve gives the locus of optimal price output combinations. MC curve is the firm's supply curve which relates the market price to the amount produced.
- Optimal revenue is the area of ABCO, total cost $AC(x)x$ is the area of ADEO, optimal profit is the area of BCDE
- Under perfect competetion $\pi'(x) = p - C'(x) \to \pi''(x) = -C''(x)$

  Profit maximizing $\pi''(x) \leq 0$ implies $C''(x) \geq 0$ At the optimal output the firm experiences increasing MC.

2.4.4 Demand functions and Elasticity

- $x = f(p)$ is the demand funtion wheras p=g(x) is the inverse demand function
- R(x)=px=g(x)x AR=R(x)/x=g(x)
- Economists ask often how changes in price affect changes in demand. Interpretation of marginal demand $f'(p) = \Delta x/\Delta p$ is extremely dependent, hence it is better to use how percent change in one effect the percent change in other.

- Price elasticity of demand:

$$\varepsilon = \frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}} = \frac{\frac{\Delta x}{\Delta p}}{\frac{x}{p}} = f'(p)\frac{p}{x} = \frac{f'(p)}{f(p)}p$$

- In general as $\frac{\Delta p}{p} \uparrow \ \frac{\Delta x}{x} \downarrow$ so ε is negative

- if $\frac{\Delta p}{p} \uparrow \ \frac{\Delta x}{x} \rightarrow 0$ then the good is inelastic (fuel, medical care goods)

  -A good whose $\varepsilon \in (-1, 0]$ is inelastic

  -A good whose $\varepsilon \in (-1, -\infty)$ is elastic (luxury goods, goods with many close substitutes)

  -A good whose $\varepsilon = -1$ is unit elastic

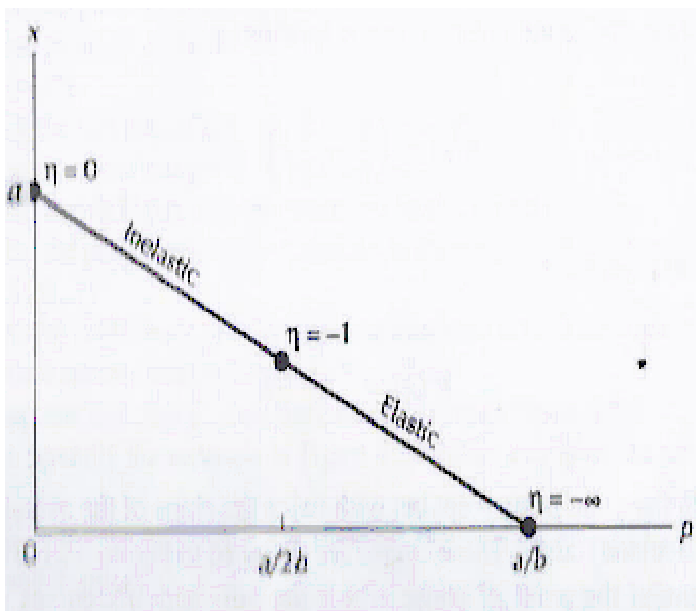
- If price of a good rises total expenditure ($E(x)$) of consumers[4] rises if the good is inelastic and it decreases if the good is elastic.

  Proof comes from $E(x) = px = pf(p) \rightarrow E'(x) = pf'(p) + 1f(p) \rightarrow \frac{E'(x)}{f(p)} = \frac{pf'(p)}{f(p)} + 1 = \varepsilon + 1$

- Example demand functions:

  Linear demand: $x = f(p) = a - bp$

  $\varepsilon = \frac{-bp}{a-bp} = \frac{1}{1-a/(bp)}$ it is zero when p=0; -1 when $p = \frac{a}{2b}$; $-\infty$ when p=$\frac{a}{b}$



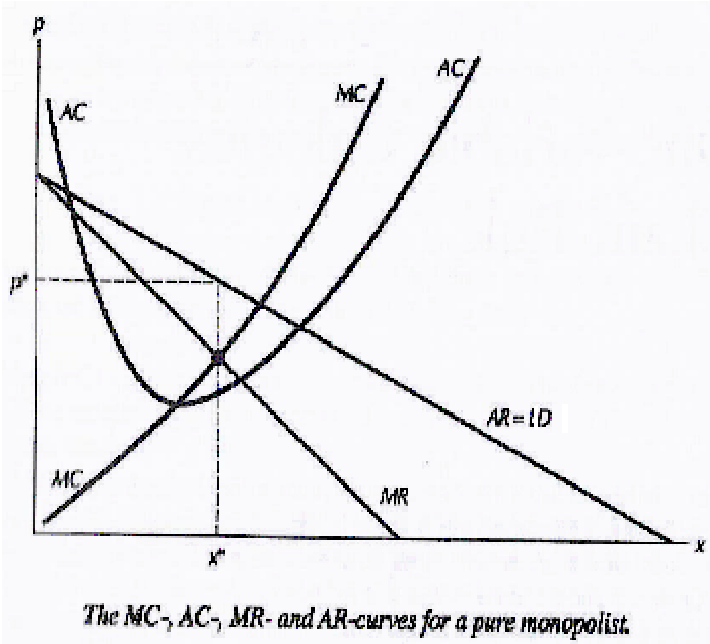
The graph of the linear demand function $x = a - bp$.

  Constant elasticity demand: $x = f(p) = kp^{-r} \rightarrow \quad \varepsilon = -\frac{krp^{-r-1}}{kp^{-r}}p = -r$

- Assume a monopolist facing a linear demand curve $a - bp$, then its inverse demand $p = AR = \frac{a-x}{b}$. If the monopolist wants to sell x unit $R(x) = \frac{a-x}{b}x \quad MR = R'(x) = \frac{a}{b} - \frac{2}{b}x$ The slope of MR is two times the slope of average revenue. Where $MR(x^*) = MC(x^*)$ the optimal output (x^*) is found for profit maximization.

[4] It is paid to the firms so that it is equivalent to the revenue of the firms.

- The price of optimal output at x^* can be read off on demand (AR) curve.

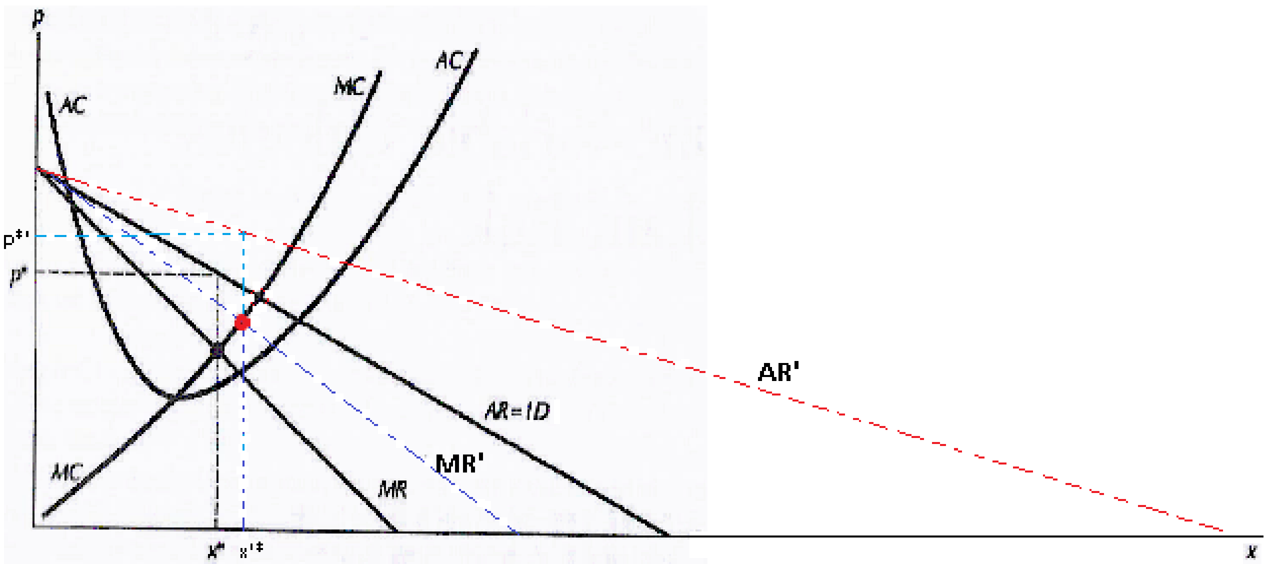


The MC-, AC-, MR- and AR-curves for a pure monopolist.

- Monopolist profit is $p^*x^* - C(x^*) = [p^* - AC(x^*)]\, x^*$
- Looking at the graph of the pure monopolist, if manufacturing costs increase MC curve rises then $x^* \downarrow$ and $p^* \uparrow$

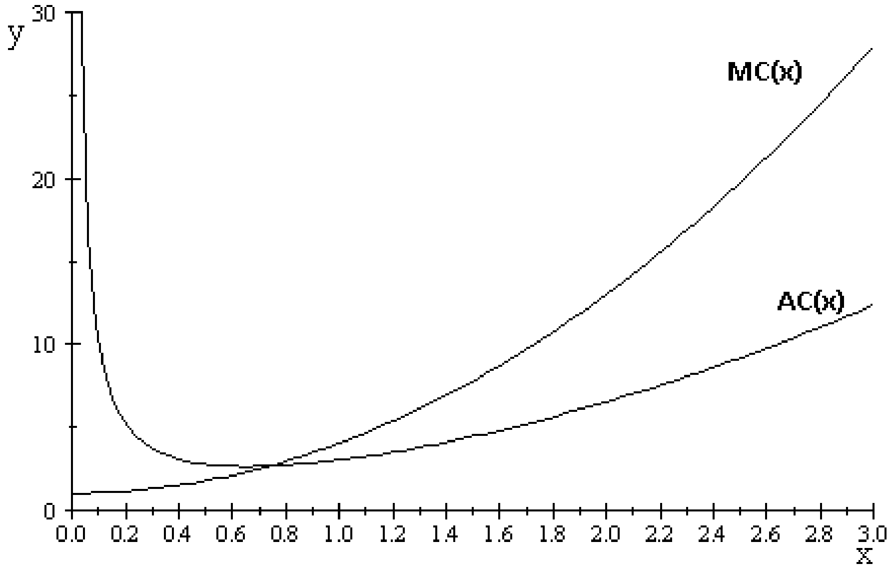
Example 24 *What happens to x^* and p^* if the inverse demand curve rises for the pure monopolist.*

Since $R(x) = p(x).x = AR(x).x$ if $p = AR$ rises then MR rises and hence $x^* \uparrow$ and $p^* \uparrow$ as seen in the following graph.



Example 25 *Show that the cost function given below has the essential properties of cost funtion?*

$C(x) = x^3 + x + 1$ $MC(x) = 3x^2 + 1 > 0$ hence $C(x)$ is an increasing function.
$AC(x) = \frac{x^3+x+1}{x} = x^2 + 1 + \frac{1}{x}$ at x=0 has a vertical asymtote
$AC'(x) = 2x - \frac{1}{x^2} = 0$ $AC(x)$ has a critical point at $2^{-\frac{1}{3}} \approx 0.79$
$AC''(x) = \frac{2}{x^3} > 0$ for x>0 so that the critical point of AC is an interior minimum where AC=MC (Check!).
To the left of that point AC>MC and to the right AC<MC.



3.One Variable Calculus Chain Rule

3.1 Chain Rule

1. - Composite functions $f(g(x))$ or $g(f(x))$ in general they are not equivalent.
     e.g. production function $y = F(L) = 5L^{\frac{2}{3}}$ then the profit $\pi(y)$ =$\pi(F(L))$

$$\pi(y) = -y^4 + 6y^2 - 5 = -\left(5L^{\frac{2}{3}}\right)^4 + 6\left(5L^{\frac{2}{3}}\right)^2 - 5 = 150L^{\frac{4}{3}} - 625L^{\frac{8}{3}} - 5$$

   - For differentiating composite functions we can use chain rule (derivative of outside times derivative of inside)

$$\frac{d}{dx}(h(g(x)) = h'(g(x))g'(x)$$

$$\frac{d}{dx}\pi(F(L)) = \pi'(F(L))F'(L) = \left[-4\left(5L^{\frac{2}{3}}\right)^3 + 12\left(5L^{\frac{2}{3}}\right)\right]\frac{10}{3}L^{\frac{-1}{3}} = -\frac{5000}{3}L^{\frac{5}{3}} + 200L^{\frac{1}{3}}$$

3.2 Inverse functions and their derivatives

- $F = \frac{9}{5}C + 32 \rightarrow C = \frac{5}{9}(F - 32)$
- In order a function to be invertible it has to be one-to-one.
- A function f defined on an interval I in $\mathbb{R}^{\ 1}$ has a well defined inverse on $f(I)$ if and only if f is monotonically increasing on all of I or monotonically decreasing on all of I .
- **The derivative of the inverse function:** Let f is C^1 on I in $\mathbb{R}^{\ 1}$, if $f'(x) \neq 0$ for all I then

  a) f is invertible on I

b) its inverse g is a is C^1 on $f(I)$

c) for all z in the domain of inverse function g ($f(x) = z$ then $g(z) = x$)

$$g'(z) = \frac{1}{f'(g(z))}$$

Example 26 $f(x) = \frac{x-1}{x+1}$ *find derivative of inverse at x=2*

$f(2) = \frac{1}{3}$ $f'(x) = \frac{2}{(x+1)^2}$ then $g'(z) = \frac{1}{f'(g(z))} \to g'(\frac{1}{3}) = \frac{1}{f'(2)} = \frac{9}{2}$
Or directly its inverse $g(y) = \frac{1+y}{1-y}$ $g'(y) = \frac{2}{(1-y)^2} \to g'(\frac{1}{3}) = \frac{9}{2}$

Example 27 $f(x) = x^2 + x + 2$ *find the derivative of inverse of* $f(x)$ *at* $f(1)$

$f(1) = 4$ $f'(x) = 2x + 1$ $f'(1) = 3$ $g'(f(1)) = \frac{1}{f'(1)} = \frac{1}{3}$ $\left(\text{Note: } g(y) = \frac{-1\pm\sqrt{1-4(2-y)}}{2} = \frac{-1\pm\sqrt{4y-7}}{2}\right)$

4 Exponential and Logarithmic Functions

- $f(t) = a^t$ where $a > 0$ is called an exponential function.

  if t is a positive integer then t means "multiply a by itself t times"

  if t=0 $f(t) = 1$ by definition

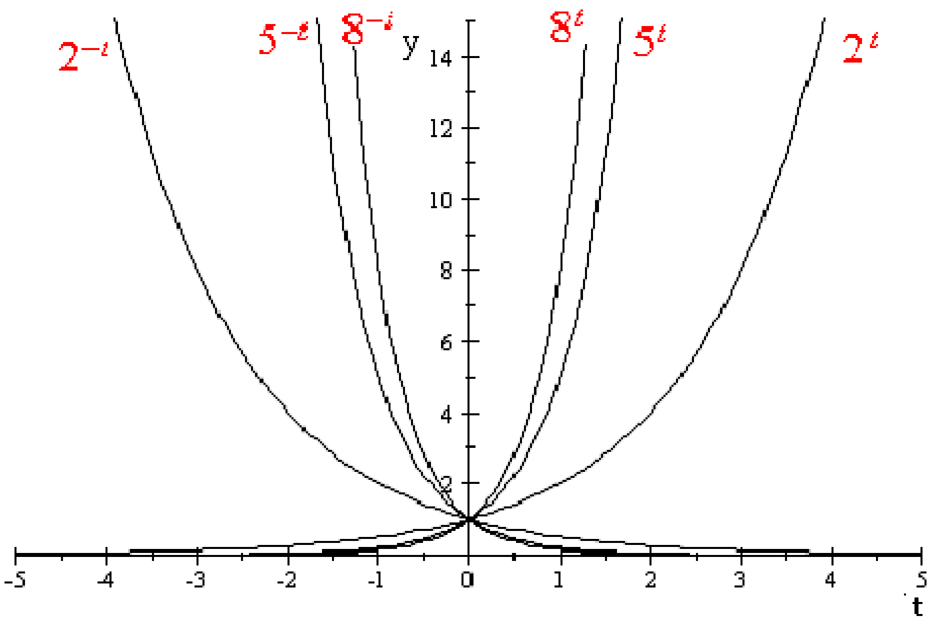
  if t=1/n $f(t) = \sqrt[n]{a}$ nth root of a

  if t=m/n $f(t) = \sqrt[n]{a^m}$ mth power of the nth root of a

  if t<0 $f(t) = a^t = \frac{1}{a^{|t|}}$ that is the reciprocal of $a^{|t|}$

  Example 28 *Graphs of* 2^{-t},5^{-t},8^{-t},2^t ,5^t *and* 8^t

  Negative exponents are the mirror image of the pozitive ones with respect to y axis. As the base increases the function becomes steeper.



- The number $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \approx 2.7$

  If one deposits A Euro in an account which pays an annual interest rate r compounded continously, then after t years the account will grow to Ae^{rt}

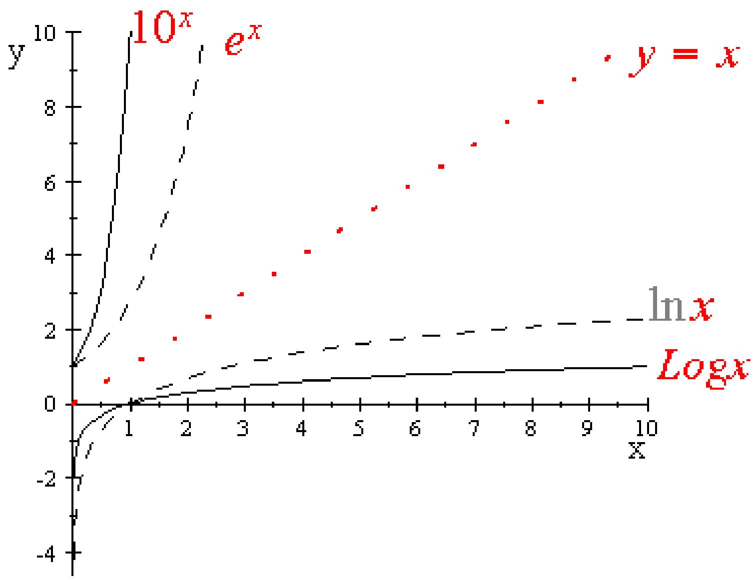
$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{r}}\right)^n \text{ let } \frac{n}{r} = m \to \lim_{n\to\infty}\left(1+\frac{1}{m}\right)^{rm} = e^r$$

- **Logarithm is the inverse of exponent.** If $a^y = z$ then the power to which one must raise a to yield z is $y = \log_a z \quad a^{\log_a z} = z \quad \log_a a^z = z$

- Base 10 logarithm=Log $\quad y = Logx \to 10^y = x \quad Log1000 = Log10^3 = 3$

- Base e logarithm=ln $\quad \ln x = y \to e^y = x$

- The graphs of e^x, $\ln x$,$10^x, Logx$ for which the logarithms are the mirro image of the exponential functions with the same base with respect to $y = x$ line. As the base increases logarithm functions becomes less steeper and closer to x axis as $x \to \infty$



The properties of exponent
1) $a^r a^s = a^{r+s}$
2) $a^{-r} = \frac{1}{a^r}$
3) $\frac{a^r}{a^s} = a^{r-s}$
4) $(a^r)^s = a^{rs}$
5) $a^0 = 1$

The properties of logarithm (assuming in base a)
1) $\log(rs) = \log r + \log s$
   let $u = \log r$ and $v = \log s \to \log(rs) = \log(a^u a^v) = u + v = \log r + \log s$
2) $\log(\frac{1}{s}) = -\log s$
   $\log(\frac{1}{a^v}) = \log a^{-v} = -v = -\log s$
3) $\log(\frac{r}{s}) = \log r - \log s$
   $\log(\frac{a^u}{a^v}) = \log(a^{u-v}) = u - v = \log r - \log s$
4) $\log r^s = s \log r$
   $\log (a^u)^s = su = s \log r$
5) $\log 1 = 0$
   $\log a^0 = 0$

Example 29 $2^{5x} = 10 \to$ *taking log of bothsides:* $5xLog2 = 1 \to x = \frac{1}{5Log2}$

Example 30 *How long A Euros deposited in saving account to double when annual interest rate r is compounded continuously?*

$2A = Ae^{rt} \rightarrow \ln 2 = rt \rightarrow t = \frac{\ln 2}{r}$ if r is 10% knowing $\ln 2 \approx 0.69 \rightarrow t = \frac{0.69}{0.10} = 6.9$ years

- Taking ln of nonlinear equations may induce them into linear equations (linearization)
  e.g. constant electicity demand function $q = kp^{\varepsilon} \rightarrow \ln q = \ln k + \varepsilon \ln p$
  In logarithmic coordoìinates demand is now a linear function whose slope is the elasticity ε

4.1 Derivatives of exponential and logarithm:

- **a)** $(e^x)' = e^x$
  $\frac{d}{dx}(\lim\limits_{n\to\infty} \left(1+\frac{r}{n}\right)^n) = \lim\limits_{n\to\infty} n\left(1+\frac{r}{n}\right)^{n-1}\frac{1}{n} = \lim\limits_{n\to\infty}\left(1+\frac{r}{n}\right)^{n-1} = \lim\limits_{n\to\infty}\left(1+\frac{r}{n}\right)^n$ as n$\to\infty$

  b)$(\ln x)' = \frac{1}{x}$
  $(\ln x)' = \lim\limits_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} = \lim\limits_{h\to 0}\frac{1}{h}\ln(\frac{x+h}{x}) = \lim\limits_{h\to 0}\ln(1+\frac{h}{x})^{\frac{1}{h}} = \lim\limits_{h\to 0}\ln(1+\frac{1/x}{1/h})^{\frac{1}{h}} = \ln e^{1/x} = 1/x$

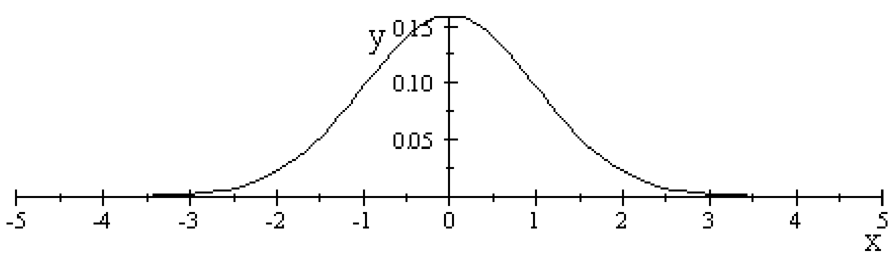
  c) $(e^{u(x)})' = e^{u(x)}.u'(x)$ obtained by chain rule

  d) $(\ln(u(x))' = \frac{u'(x)}{u(x)}$ if $u(x) > 0$ obtained by chain rule

  e) $(b^x)' = b^x.\ln b$
  $(b^x)' = (e^{x\ln b})'$ using c) $(b^x)' = e^{x\ln b}.\ln b = b^x.\ln b$

Example 31 *The sketch of standart normal density function*

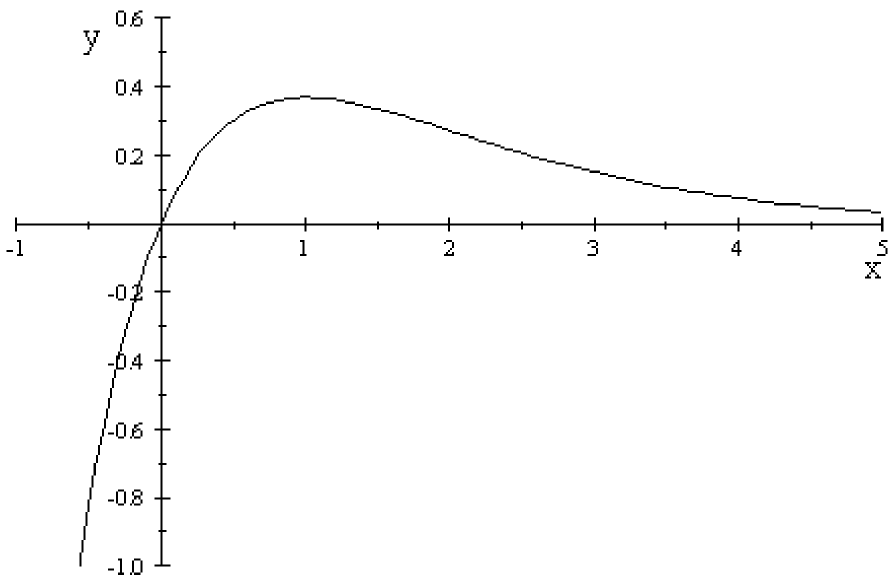
$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}e^{-\frac{x^2}{2}} \text{ goes to 0 when } |x| \rightarrow \infty\text{, hence x axis is the horizontal asymtote} \\
f'(x) &= -\frac{1}{2\pi}xe^{-\frac{x^2}{2}} \text{ the critical point is } x=0\text{, f is decrasing to the right and increasing to the left} \\
f''(x) &= \frac{1}{2\pi}(x^2-1)e^{-\frac{x^2}{2}} \text{ the critical points are -1 and 1. Convex in } (-\infty,-1)\text{, concave in } (-1,1)\text{, convex in } (1,\infty) \\
f''(0) &< 0 \text{ so that } x=0 \text{ is the maximum point.}
\end{aligned}
$$



$$f(x) = \frac{1}{2\pi}e^{-\frac{x^2}{2}}$$

Example 32 *The sketch of* xe^{-x}

$$\begin{aligned} f(x) &= xe^{-x} \text{ goes to 0 when } x \to \infty \text{, hence } +x \text{ axis is the horizontal asymtote} \\ f'(x) &= (1-x)e^{-x} \text{ the critical point is } x=1 \text{, f is decrasing to the right and increasing to the left} \\ f''(x) &= (x-2)e^{-x} \text{ the critical point is } x=2 \text{. Convex in } (-\infty,2) \text{ and concave in } (2,\infty) \\ f''(1) &< 0 \text{ so that } x=1 \text{ is the maximum point.} \end{aligned}$$



$f(x) = xe^{-x}$

4.2 Applications

4.2.1 Present Value

If we put B Euros into a saving account with annual interest rate r which is compounded continuously, then after t years it becomes $A = Be^{rt}$.Conversely in order to generate A Euros t years from now in an account compunded interest rate r continuously, we would have to invest $B = Ae^{-rt}$ Euros that is the present value (**PV**) of B Euros t years from now.
Annuity is a sequence of payments at regular intervals over a specified period. The present value of an annuity that pays A Euros at the end of the next N years with an interest rate r of continous compounding.
$PV = Ae^{-r} + Ae^{-2r} + ... + Ae^{-nr} = A(e^{-r} + e^{-2r} + ... + e^{-nr})$ Using the property of geometric series[5] :
$= A\frac{e^{-r}(1-e^{-rn})}{1-e^{-r}} = \frac{A(1-e^{-rn})}{e^{r}-1}$

Example 33 *Assuming 10% interest rate compunded continuously, what is the present value of an annuity thay pays* 500 *Euros a year?*

a)for the next 5 years: $\frac{500(1-e^{-0.5})}{e^{0.1}-1} = 1870.6$
b)forever:$\frac{500}{e^{0.1}-1} = 4754.2$

[5] $X = a + a^1 + ... + a^n = \frac{a(1-a^n)}{1-a}$ (Found by subtracting X/a from X)

- It is sometimes covenient to compute PV of an annuity using annual compounding

$$\begin{aligned} PV &= \frac{A}{1+r} + \frac{A}{(1+r)^2} + ... + \frac{A}{(1+r)^n} \text{ using again geometric series expansion} \\ &= A\frac{\frac{1}{1+r}(1-\left(\frac{1}{1+r}\right)^n)}{1-\frac{1}{1+r}} = \frac{A}{r}\left(1-\left(\frac{1}{1+r}\right)^n\right) \\ &= \frac{A}{r} \text{ as } n \to \infty \end{aligned}$$

In order to generate A Euros per year from an account paying annual interest with rate r one must deposit inti the account $\frac{A}{r}$ initially.

Example 34 *Redo the previous example with annual compunding?*

a)for the next 5 years: $\frac{500}{0.1}\left(1-\left(\frac{1}{1.1}\right)^5\right) = 1895.4$
b)forever: $\frac{500}{0.1} = 5000$

4.2.2 Optimal Holding Time

Suppose the market value of your real estate will be $V(t)$ Euros t years from now. If the interest rate remains constant (r) and continuously compunded during this period then the present value of the real estate is $V(t)e^{-rt}$.Maximizing the present value gives the optimal time to sell it.

$$\begin{aligned} \left(V(t)e^{-rt}\right)' &= V'(t)e^{-rt} - rV(t)e^{-rt} = 0 \\ \frac{V'(t)}{V(t)} &= r \text{ at the optimal selling time} \end{aligned}$$

percent growth rate of the value of the real estate =percent rate of change of money in the bank

Logarithmic Derivative: Since the logarithm turns exponentiation to into multiplication and multiplication into addition and division onto substraction, it can often simplify the computation of the derivative of a complex function.

$$(\ln(u(x)))' = \frac{u'(x)}{u(x)} \to u'(x) = u(x)\left(\ln(u(x))\right)'$$

Example 35 *Use logarithmic derivative to compute the derivative of*

$$\begin{aligned} y &= \frac{\sqrt[4]{x^2-1}}{x^2+1} \\ \ln y &= \frac{1}{4}\ln\left(x^2-1\right) - \ln\left(x^2+1\right) \\ y' &= y(\ln y)' = \frac{\sqrt[4]{x^2-1}}{x^2+1}.\left(\frac{1}{4}\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right) = \frac{\sqrt[4]{x^2-1}}{x^2+1}\left(\frac{-3x^3+5x}{2(x^2-1)(x^2+1)}\right) = \frac{-3x^3+5x}{2(x^2-1)^{\frac{3}{4}}(x^2+1)^2} \end{aligned}$$

Example 36 *Derivative of* x^x?

$$(x^x)' = x^x(x\ln x)' = x^x(\ln x + 1)$$

Example 37 *The value of a land is increasing according to the formula* $V = 2000e^{t^{\frac{1}{4}}}$.*If the interest rate is 10%, how long it should be held to max its present value?*

$\ln V = \ln 2000 + t^{\frac{1}{4}} \to (\ln V)' = \frac{V'(t)}{V(t)} = r = \frac{1}{4}t^{-\frac{3}{4}} \to r = 0.1$ then $t = 3.39$

5 Linear Algebra

In general, an equation is linear if it has the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

where the letters $a_1, a_2, .., a_n, b$ are the fixed numbers and therefore the are called parameters wheras $x_1, x_2, ..., x_n$ stand for the variables.
As a key feature of linear equations, each term contains at most one variable and that variable can have only the first power. Linear equations are easy to handle (they build on the techniques learned in high school, such as the solution of two linear equations in two unknowns such as substitution or elimination of variables and they build on simple geometry of the plane and the cube which are easy to visualize). These equations can often have exact solutions unlike nonlinear systems. With suitable assumptions or linearizations they can be good approximations of the nonlinear systems. Moreover, some of the most frequently studied models are linear.

5.1 Example of linear economic models

5.1.1 Tax benefits of charitable contributions

A firm earns before-tax profits of $100,000. It has agreed to donate 10% percent of its after-tax profits to a charity fund. It must pay a state tax of 5 percent of its profits (after the donation) and a federal tax of 40 percent of its profits (after the donation and state taxes are paid). How much does the company pay in state taxes, federal taxes, and charitable donation?

- Let S, F and C are state taxes, federal taxes, and charitable donation. After tax profits are 100,000-(S+F); so C becomes $C = 0.1(100,000 - (S + F)) \rightarrow C + 0.1S + 0.1F = 10,000$
- S is 5% of profits net of the donation then $S = 0.05(100.000 - C) \rightarrow 0.05C + S = 5,000$
- Federal taxes are 40% the profit after deducting C and S $\rightarrow F = 0.4(100.000 - C - S) \rightarrow 0.4C + 0.4S + F = 40,000$

In summary we get three linear equations, subsituting the second equation to the others we get two equations with two unknowns:

$$\begin{aligned} C + 0.1(5000 - 0.05C) + 0.1F &= 10,000 \\ 0.4C + 0.4(5000 - 0.05C) + F &= 40,000 \end{aligned}$$

Then the solution becomse C=5956 S=4702 F=35737 and after tax&contribution profit is $53605

Without donation the after tax profits become $57000 meaning that $5956 donation costs $3395 to the firm.

5.1.2 Linear Model of Production

As a simplification constant return to scale production is assumed that is the amount of output linearly proportional to the amount of input. e.g. 50 cars need 50 times the input of one car.

- In an open Leontief system of economy, the production of a good i (there are n+1 goods in the economy) can be described by a set of input output coefficients where a_{ij} denotes the input of good i needed to produce one unit of good j. The output of good i must be allocated between production activities and consumption. Good 0 is labor that is supplied by consumers so the consumption (demand) for each good i is given exogenously (this is why it is called open system) that is not solved for in the model. Each good i is used for producing other goods and consumption (c_i). Good "0" is labor that is supplied by consumers so its consumption c_0 is negative.

$$x_i = a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n + c_i$$

- Explicitly (gross output=used as input+consumption):

$$\begin{aligned}
(1-a_{11})x_1 \quad & -a_{12}x_2 - .... - a_{1n}x_n = c_1 \\
-a_{21}x_1 + (1-a_{22})x_2 & - .... - a_{2n}x_n = c_2 \\
& .... \\
-a_{n1}x_1 \quad & -a_{n2}x_2 - ... + (1-a_{nn})x_n = c_n \\
-a_{01}x_1 \quad & -a_{02}x_2 - .... - a_{0n}x_n = c_0
\end{aligned}$$

5.1.3 Markov Models of Employment

- Using transition probabilities from unemployment to employment or from employment to uneployment, these models are commonly used to understand the log run employment behaviour.
- Let x_t and y_t are the number of employed and unemployed, q and p are the probability of employment for them respectively. Assuming finding job or leaving job are independent of number of weeks worked. Then the number of employed and unemployed in the next period (say week) is given as:

$$\begin{aligned}
x_{t+1} &= qx_t + py_t \\
y_{t+1} &= (1-q)x_t + (1-p)y_t
\end{aligned}$$

- Normalizing the total number of employed and unemployed to 1, in the steady state:

$$\begin{aligned}
x &= qx + py \\
y &= (1-q)x + (1-p)y \\
x + y &= 1
\end{aligned}$$

- The first two equations are the same equations with minus sign so in fact we have two equations

$$\begin{aligned}
(q-1)x + py &= 0 \\
x + y &= 1
\end{aligned}$$

- Then $x = \frac{p}{1+p-q}$ and $y = \frac{1-q}{1+p-q}$
- Using $q = 0.998$ and $p = 0.136$ (Hall,1966) for US white males in 1966
  $x = \frac{0.136}{1+0.136-0.998} = 0.986 \quad y = 1 - x = 1.4\%$ of white males were unemployed on average in 1966.

5.1.4 IS-LM Analysis

IS (Investments, Savings) and LM(Liquidity, Money) analysis is a linear model of a closed economy with total national income (Y) total national spending (Consumption, Investment and Government expenditures)

$$Y = C + I + G$$

- For IS analysis, on the consumer side spending is proportional to total income $C = bY \quad (0 < b < 1)$ where b is called marginal propensity to consume, while s=1-b is the marginal propensity to save.
- On the firms side, either they invest to keep their money in the bank with an interest rate (r) so investment is a decreasing function of r.

$$I = I^0 - ar$$

- Putting these together gives the IS schedule:

$$Y = bY + I^0 - ar + G$$

- Or we write it as

$$sY + ar = I^0 + G$$

- IS equation describes the real side of the economy by summarizing consumption, investment and saving decisions.

On the other hand, the LM equation is determined by the money market equilibrium condition that money supply (M_s) equals money demand (M_d). M_d has two components: the transactions or precautionary demand (M_{dt}) and the speculative demand (M_{ds}).The transactions demand derives from the fact that most transactions are denominated in money. Thus, as national income rises, so does the demand for funds. So that $M_{dt} = mY$

The speculative demand comes from the portfolio management problem faced by an investor in the economy. The investor must decide whether to hold bonds or money. Money is more liquid but returns no interest, while bonds pay at rate r. It is usually argued that the speculative demand for money varies inversely with the interest rate (directly with the price of bonds). The simplest such relationship is the linear one:

$$M_{ds} = M^0 - hr$$

Equating the supply to the demand:

$$M_s = mY + M^0 - hr$$

Hence we can write IS-LM system of equations as:

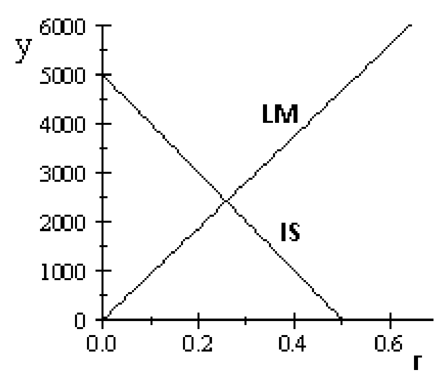
$$\begin{aligned} sY + ar &= I^0 + G \\ mY - hr &= M_s - M^0 \end{aligned}$$

where the solution (Y, r) depend upon the policy parameters M_s, and G and on the behavioral parameters a, h, I^0, m, M^0 and s.

Example 38 *Consider the above model with no fiscal policy ($G = 0$). Suppose that $M_s = M^0$: that is. the intercept of the LM curve is 0. Suppose that $I^0 = 1000$,s = 0.2, h = 1500, a = 2000, and ,m = 0.16. Write out the explicit IS-LM syslem of equations. Solve them for the equilibrium GNP Y and the interest rate r.*

$$\begin{aligned} sY + ar &= I^0 + G \\ 0.2Y + 2000r &= 1000 \\ mY - hr &= 0 \\ 0.16Y - 1500r &= 0 \end{aligned}$$

The solution is r=0.26 and Y=2419



5.1.5 Investment and Arbitrage

For the investment decisions there will be different possible states of nature and for each of them the the portfolios give diffferent returns.

- If a portfolio provides same return in every state of nature it is called riskless. For A assets and S states this equalities of **riskless (or risk-free) asset** can be shown as:

$$\sum_{i=1}^{A} R_{1i}x_i = \sum_{i=1}^{A} R_{2i}x_i = ... = \sum_{i=1}^{A} R_{si}x_i$$

  where x_i is the portfolio weight for asset i, R_{si} is the return (future value/ current value).

- A nonzero A-tuple$(x_1, ..., x_A)$ is called **an arbitrage portfolio** if $x_1 + ... + x_A = 0$ instead of 1. In such a portfolio, the money received from the short sales is used in purchase of the long positions, so that the portfolio costs nothing.

- A portfolio is called dupplicable if there is a different portfolio with exactly the same return in every state.

$$\sum_{i=1}^{A} R_{si}x_i = \sum_{i=1}^{A} R_{si}w_i \text{ for each s=1,...S}$$

- A state s* is called **insurable** if there is a portfolio $(x_1,x_2,...x_A)$ which has a positive return if state s* occurs and zero return if any other state occurs:

$$\begin{aligned} \sum_{i=1}^{A} R_{s^*i}x_i &> 0 \\ \sum_{i=1}^{A} R_{si}x_i &= 0 \text{ for all } s \neq s^* \end{aligned}$$

- It is sometimes convenient to assign a price to each of the s states of nature, then the state return relations can be written as:

$$\begin{aligned} p_1R_{11} + p_2R_{21} + ... + p_sR_{s1} &= 1 \\ p_1R_{12} + p_2R_{22} + ... + p_sR_{s2} &= 1 \\ &. \\ &. \\ &. \\ p_1R_{1A} + p_2R_{2A} + ... + p_sR_{sA} &= 1 \end{aligned}$$

Example 39

Suppose that there are two assets and three possible states. If state 1 occurs, asset 1 returns $R_{11} = 1$ and asset 2 returns $R_{12} = 3$. If state 2 occurs, $R_{21} = 2$ and $R_{22} = 2$. If state 3 occurs, $R_{31} = 3$ and $R_{32} = 1$. If both assets have the same current value and if the investor buys $n_1 = 3$ shares of asset 1 and $n_2 = 1$ share of asset 2, the corresponding portfolio is $(\frac{3}{4}, \frac{1}{4})$ and the returns are:

$$\begin{aligned} R_{11}\frac{3}{4} + R_{12}\frac{1}{4} &= \frac{3}{2} \text{ in state 1} \\ R_{21}\frac{3}{4} + R_{22}\frac{1}{4} &= 2 \text{ in state 2} \\ R_{31}\frac{3}{4} + R_{32}\frac{1}{4} &= \frac{5}{2} \text{ in state 3} \end{aligned}$$

Note that portfolio $(\frac{1}{2}, \frac{1}{2})$ yields a return of 2 in all three states, hence it is a risk-free portfolio.

5.2 Systems of Linear Equations

There are essentially three ways of solving systems of linear equations: substitution,elimination of variables, and
matrix methods.

5.2.1 Substitution and elimination of variables methods

- **Substitution** is simply made by writing one variable in terms of other(s) using an equation and substituting this relation into the other equation(s).

  Example 40 *An example for linear production model (look section 5.1.2 for detail) of 3 goods* (x_1, x_2, x_3) *given their production input-output proportions and exogenous consumption amounts (130,74,95) can be written as:*

$$\begin{aligned} x_1 &= 0x_1 + 0.4x_2 + 0.3x_3 + 130 \rightarrow x_1 - 0.4x_2 - 0.3x_3 = 130 \\ x_2 &= 0.2x_1 + 0.12x_2 + 0.14x_3 + 74 \rightarrow -0.2x_1 + 0.88x_2 - 0.14x_3 = 74 \\ x_3 &= 0.5x_1 + 0.2x_2 + 0.05x_3 + 95 \rightarrow -0.5x_1 - 0.2x_2 + 0.95x_3 = 95 \end{aligned}$$

substituting $x_1 = 0.4x_2 + 0.3x_3 + 130$ into the other equations :

$$\begin{aligned} -0.2\left(0.4x_2 + 0.3x_3 + 130\right) + 0.88x_2 - 0.14x_3 &= 74 \\ -0.5\left(0.4x_2 + 0.3x_3 + 130\right) - 0.2x_2 + 0.95x_3 &= 95 \end{aligned}$$

Simplifying them we get:

$$\begin{aligned} 0.8x_2 - 0.2x_3 &= 100 \\ -0.4x_2 + 0.8x_3 &= 160 \end{aligned}$$

substituting $x_2 = \frac{100+0.2x_3}{0.8} = 125 + 0.25x_3$ into the other equation :

$$\begin{aligned} -0.4\left(125 + 0.25x_3\right) + 0.8x_3 &= 160 \\ x_3 &= 300 \\ x_2 &= \frac{100 + 0.2x_3}{0.8} = 200 \\ x_1 &= 0.4x_2 + 0.3x_3 + 130 = 300 \end{aligned}$$

- **Elimination of variables** is generally more conducive to the theoretical analysis. It is done by multiplying equations and adding them up such that eliminating unknown(s) to solve the equation with less unknowns. This is called Gauss elimination.

Example 41 *We do the previous example with elimination of variables. Multiplying the first one by 0.2 and adding it to second to eliminate* x_1 *; multiplying the first one by 0.5 and adding it to the third to eliminate* x_1*:*

$$\begin{array}{rl} & 0.2(x_1 - 0.4x_2 - 0.3x_3) = 0.2 * 130 \\ + & -0.2x_1 + 0.88x_2 - 0.14x_3 = 74 \\ \hline & 0.8x_2 - 0.2x_3 = 100 \end{array}$$

$$\begin{array}{rl} & 0.5\left(x_1 - 0.4x_2 - 0.3x_3\right) = 0.5 * 130 \\ + & -0.5x_1 - 0.2x_2 + 0.95x_3 = 95 \\ \hline & -0.4x_2 + 0.8x_3 = 160 \end{array}$$

Then multiplying the above found by 0.5 and adding it to the this:

$0.5(0.8x_2 - 0.2x_3) = 0.5 * 100$
$+ \quad -0.4x_2 + 0.8x_3 = 160$
$0.7x_3 = 210$
means our system transforms into

$$\begin{aligned} x_1 - 0.4x_2 - 0.3x_3 &= 130 \\ 0.8x_2 - 0.2x_3 &= 100 \\ 0.7x_3 &= 210 \end{aligned}$$

$x_3 = 300$ by back subtitution into the others we find $x_2 = 200$ and $x_1 = 300$

As variant of Gauss elimination, the first nonzero coefficient is transformed to 1 instead of back subtitution6

$$\begin{aligned} x_1 - 0.4x_2 - 0.3x_3 &= 130 \\ x_2 - 0.25x_3 &= 125 \\ x_3 &= 300 \end{aligned}$$

Then adding to the second equation 0.25 times the third we find $x_2 = 200$, then adding 0.3 times the third and 0.4 times the second to the first equation we find $x_1 = 300$.

6. Matrix Algebra

- We can write a linear system of equations in matrix form.

$$\begin{bmatrix} a_{11} & . & . & . & a_{1n} \\ . & & & & . \\ . & & a_{ij} & & . \\ . & & & & . \\ a_{k1} & . & . & . & a_{kn} \end{bmatrix} . \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ . \\ . \\ . \\ b_k \end{bmatrix}$$

- In compact form we can write as $Ax = b$ Where the matrix A is the coefficient matrix, x vector of n variables, b vector of k constants.
- For example:

$$\begin{aligned} 0.8x_2 - 0.2x_3 &= 100 \\ -0.4x_2 + 0.8x_3 &= 160 \end{aligned}$$

$$\begin{bmatrix} 0.8 & -0.2 \\ -0.4 & 0.8 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 160 \end{bmatrix}$$

- Where the matrix $A = \begin{bmatrix} 0.8 & -0.2 \\ -0.4 & 0.8 \end{bmatrix}$ is the coefficient matrix, $x = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$ vector of variables, $b = \begin{bmatrix} 100 \\ 160 \end{bmatrix}$ vector of constants
- The size of a matrix is n x k where n is the number of rows and k is the number of columns; for example, a 2 x 3 is a matrix with 2 rows and 3 columns; a n x n matrix is called

  square matrix (same number of rows and columns)

- A row of a matrix is said to have k leading zeros if the first k elements of the row are zeros and the (k+1)th element of the row is not zero. A matrix is in **row Echelon form** if each row has more leading zeros than the row preceeeding it.
$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 2 & 3 \\ 0 & 6 \\ 0 & 0 \end{bmatrix}$ are in row Echelon form, $\begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 7 \\ 9 & 0 \\ 0 & 2 \end{bmatrix}$ are not in row Echelon form.

- The **rank** of a matrix is the maximal number of linearly independent columns of A. Rank can be found by writing the matrix using some row operations (interchanging two rows of a matrix, adding to rows, multiplying each element of a row with a scalar) in **row Echelon** form. Number of nonzero rows in its row Echelon form, gives the rank of the matrix.

Example 42 *Find the rank of the following matrix? (Denote r_1 as first row, r_2 as second row and r_3 as the third row to show row opeations)*

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \to 2r_1 + r_2 : \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix} \to -3r_1 + r_3 : \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \to r_2 + r_3 : \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

so its rank is 2

- A system of linear equations with coefficient matrix **A** and nonzero vector of constants **b** will have a solution for every **b** if and only if **rank A=number of rows A**
- A system of linear equations with coefficient matrix **A** and nonzero vector of constants **b** will have at most one solution for every **b** if and only if **rank A=number of columns A**
- A coefficient matrix is nonsingular, that is the corresponding system has one and only one solution for every **b** if and only if **rank A=number of columns A=number of rows A.** This rank condition can easily be checked by the determinant of the square matrix (it will be covered later).
- A homogenous system of linear equations (**b**=zero vector) has more unknowns (columns) than linearly independent equations (rank), must have inifinitely many distinct solutions.

6.1 Addition or subtraction

- In order to perform algebraic operations, matrices must meet some requirements about their size. For addition or subtraction, they have to be the same size. Each element of the matrix added to or subtracted from the element in the same position. **A±B**

$$\begin{bmatrix} a_{11} & . & . & . & a_{1n} \\ . & & & & . \\ . & & a_{ij} & & . \\ . & & & & . \\ a_{k1} & . & . & . & a_{kn} \end{bmatrix} \pm \begin{bmatrix} b_{11} & . & . & . & b_{1n} \\ . & & & & . \\ . & & b_{ij} & & . \\ . & & & & . \\ b_{k1} & . & . & . & b_{kn} \end{bmatrix} = \begin{bmatrix} a_{11} \pm b_{11} & . & . & . & a_{1n} \pm b_{1n} \\ . & & & & . \\ . & & a_{ij} \pm b_{ij} & & . \\ . & & & & . \\ a_{k1} \pm b_{k1} & . & . & . & a_{kn} \pm b_{kn} \end{bmatrix}$$

6.2 Scalar multiplication

There is no size requirement for scalar multiplication. Each element is multipled by the scalar. **rA**

$$r \begin{bmatrix} a_{11} & . & . & . & a_{1n} \\ . & & & & . \\ . & & a_{ij} & & . \\ . & & & & . \\ a_{k1} & . & . & . & a_{kn} \end{bmatrix} = \begin{bmatrix} ra_{11} & . & . & . & ra_{1n} \\ . & & & & . \\ . & & ra_{ij} & & . \\ . & & & & . \\ ra_{k1} & . & . & . & ra_{kn} \end{bmatrix}$$

6.3 Matrix multiplication

- We can define the matrix product **AB** if and only if

  number of columns of **A**=number of rows of **B**

- If **A** is k x m matrix and **B** is m x n matrix **AB** becomes k x n :
  $(kxm)\,(mxn) = (kxn)$
- $IA = A$ where $I = (kxk)$ Identity matrix (with all ones on the diagonal and other terms 0)
- To obtain the (i,j)th entry of **AB,** multiply the ith row of **A** and the jth column of **B** as the following:

$$\begin{bmatrix} a_{i1} & a_{i2} & . & . & . & a_{im} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \\ \\ b_{mj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_2j + ... + a_{im}b_{mj}$$

- For example:Note that **AX** can be done but **XA** cannot (2 x2 3x 2)

$$\mathbf{AX} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} X & Y \\ Z & T \end{bmatrix} = \begin{bmatrix} Xa + Zb & Tb + Ya \\ Xc + Zd & Td + Yc \\ Xe + Zf & Ye + Tf \end{bmatrix}$$

Example 43 *Perform the follwing mulltiplication and check the result.*

$$\begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 8 & 11 \\ -2 & -3 & -1 & -4 \\ 0 & -1 & 2 & 1 \\ 2 & 2 & 3 & 5 \end{bmatrix}$$

6.5 Laws od Matrix Algebra

- Associative: **(A+B)+C=A+(B+C)** (**AB**)**C**=**A**(**BC**)
- Commutative for addition: **A+B=B+A** not for multipication **AB≠BA**
- Distributive Laws: **A(B+C)=AB+AC (A+B)C=AC+BC**
- Transpose (writing rows as columns and columns as rows). If **A** is k x n its transpose $\mathbf{A}^{\mathrm{T}}$ (or **A'**) is n x k

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}^T = \quad a_{11} \quad a_{21}$$

-$(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$ $(\mathbf{A}-\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} - \mathbf{B}^{\mathrm{T}}$ $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$ $(\mathrm{r}\mathbf{A})^{\mathrm{T}} = r\mathbf{A}^{\mathrm{T}}$
-$(\mathbf{AB})^T = B^T A^T$

6.6 Special Kinds of Matrices

A k x n matrix is a

- **Square matrix** if k=n that is equal number of rows and columns
- **Column matrix** if n=1
- **Row matrix** if k=1
- **Diagonal matrix** if k=n and $a_{ij} = 0$ for $i \neq j$, a square matrix with nondiagonal elements are 0. If the diagonal elements are all 1 then it is called an identity matrix.
- **Upper-Triangular Matrix** if $a_{ij} = 0$ for $i > j$ the entries below the dioganal is zero

  e.g. $\begin{bmatrix} x & z \\ 0 & y \end{bmatrix}$, $\mathbf{L^T} = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

- **Lower-Triangular Matrix** if $a_{ij} = 0$ for $i < j$ the entries above the dioganal is zero

  e.g. $\begin{bmatrix} x & 0 \\ z & y \end{bmatrix}$, $\mathbf{L} = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 1 \end{bmatrix}$

- **Symmetric Matrix** if $A^T = A$ that is $a_{ij} = a_{jk}$

  e.g. $\begin{bmatrix} x & z \\ z & y \end{bmatrix}$, $\begin{bmatrix} 3 & 6 & 4 \\ 6 & 2 & 5 \\ 4 & 5 & 1 \end{bmatrix}$, $\begin{bmatrix} 9 & 12 & 15 \\ 12 & 20 & 32 \\ 15 & 32 & 62 \end{bmatrix}$

  NOTE: Any symmetric positive definite matrix [6] **M** can be written as **M=LL**^{T} which is called **Cholesky decomposition.**

  $$\begin{bmatrix} 9 & 12 & 15 \\ 12 & 20 & 32 \\ 15 & 32 & 62 \end{bmatrix} = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} g_{11} & g_{21} & g_{31} \\ 0 & g_{22} & g_{32} \\ 0 & 0 & g_{33} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

  Found by equating both sides $g_{11}^2 = 9 \rightarrow g_{11} = 3$ $g_{11}g_{21} = 12 \rightarrow g_{21} = 4$ $g_{11}g_{31} = 15 \rightarrow g_{31} = 5$ etc...

- **Idempotent Matrix** if **B.B=B**

  e.g. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 5 & -5 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 5 & -5 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 4 & -4 \end{bmatrix}$

- **Permutation matrix** if it is a square matrix of 0s and 1s in which each row and each column contains exactly one 1.

  e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- **Nonsingular matrix** if it is a square matrix whose rank equals the number of rows(or columns). When such a matrix is a coefficient matrix in system of linear equations, the system has one and only one solution.

6.8 Determinant of Matrices

- Determinant is defined for square matrices. Determinant of an n x n matrix is the n-dimensional volume scaling factor of the linear transformation produced by the matrix.
- For a 2 x 2 matrix **A** its determinant is found by **Leibniz** rule:

  det $\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$ which is a scalar.

[6] A symmetric matrix M is positive definite if the scalar z'M z is strictly positive for every nonzero column vector z

- For the determinant of 3 x 3, 4x 4 or higher size matrix, again **Leibniz** rule is used (multiplying the elements of a selected row or column a_{ij} with $(-1)^{i+j}$ times the ijth cofactor C_{ij} (determinant of the submatrix obtained by deleting row i and column j from A) and adding them up).

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Note: signs come from the $(-1)^{i+j}$ where ij is the position of the first multiplicative terms $(11, 12, 13, 14)$ for the above

$$\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{22}(-1)^{1+1} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + a_{23}(-1)^{1+2} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} + a_{24}(-1)^{1+3} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}$$
$$= a_{22}a_{33}a_{44} - a_{22}a_{34}a_{43} - a_{23}a_{32}a_{44} + a_{23}a_{42}a_{34} + a_{32}a_{24}a_{43} - a_{24}a_{33}a_{42}$$

Doing for all 3 x 3 matrices and substituting the results the 4 x 4 expansion we find the determinant.

Example 44 *Find the the following determinant of matrix using diffferent row or columns for the first multiplicative terms?*

$$\begin{vmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{vmatrix} = 6 \begin{vmatrix} 5 & 0 \\ 0 & 7 \end{vmatrix} + 2 \begin{vmatrix} -2 & 0 \\ 2 & 7 \end{vmatrix} + 2 \begin{vmatrix} -2 & 5 \\ 2 & 0 \end{vmatrix} = 6 * 35 - 2 * 14 - 2 * 10 = 162$$

or

$$\begin{vmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ \mathbf{2} & \mathbf{0} & \mathbf{7} \end{vmatrix} = \mathbf{2} \begin{vmatrix} -2 & 2 \\ 5 & 0 \end{vmatrix} + \mathbf{7} \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} = -2 * 10 + 7 * 26 = 162$$

or

$$\begin{vmatrix} 6 & -2 & \mathbf{2} \\ -2 & 5 & \mathbf{0} \\ 2 & 0 & \mathbf{7} \end{vmatrix} = \mathbf{2} \begin{vmatrix} -2 & 5 \\ 2 & 0 \end{vmatrix} + \mathbf{7} \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} = -2 * 10 + 7 * 26 = 162$$

- det($\mathbf{A}^T$)=det**A** , det(**AB**)=(det**A**)(det**B**) but det(**A**+**B**)≠(det**A**)+(det**B**) in general
- A square matrix is nonsingular if and only if its determinant is nonzero.

6.7 Inverse of Matrices

- For a linear system of equations $Ax = b$ we want to find the vector of unknown variables as $x = A^{-1}b$, hence we need to find the inverse of the coeeffient matrix (A^{-1}) to solve the linear system easily with matrix method.
- The inverse of a matrix, A^{-1} exists only if the matrix is a square matrix. Not every square matrix has an inverse. If it has an inverse the matrix is called nonsingular, otherwise it is called singular.
- For a nonsingular matrix nxn matrix A, $AA^{-1} = I$ identity matrix
- For any nxn matrix **A**, let C_{ij} denote the ijth cofactor of A,that is, $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting row i and column j from A. The transpose of the cofactor matrix is called the **adjoint** of **A.** Then the inverse is found as:

$$A^{-1} = \frac{1}{\det A} adj A$$

Example 45 *Find the inverse of the following matrix* **A**.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow C_{11} = \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} = -7 \quad C_{12} = -\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = -4 \quad C_{13} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5$$

$$\det A = -7 * 1 - 4 * -1 + 5 * 1 = 2$$

$$C_{21} = -\begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} = 5 \quad C_{22} = +\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2 \qquad C_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = -3$$

$$C_{31} = \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} = -1 \quad C_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0 \qquad C_{33} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1$$

Then $\mathbf{C} = \begin{bmatrix} -7 & -4 & 5 \\ 5 & 2 & -3 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow adj A = C^T = \begin{bmatrix} -7 & 5 & -1 \\ -4 & 2 & 0 \\ 5 & -3 & 1 \end{bmatrix} \rightarrow A^{-1} = \frac{1}{\det A} adj A = \frac{1}{2} \begin{bmatrix} -7 & 5 & -1 \\ -4 & 2 & 0 \\ 5 & -3 & 1 \end{bmatrix}$

- 2x2 case is the most known case which is also derived from the Leibniz rule

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \rightarrow A^{-1} = \frac{1}{\det A} adj A = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

6.8 Eigen Values and Eigen Vectors of a Matrix

An eigenvector or characteristic vector of a linear transformation is a non-zero vector that changes by only a scalar factor when that linear transformation is applied to it.Eigen values and vectors are very useful in many applications such as solving difference equations and studying stationary points higher dimensional functions.

λ is an eigen value for an nxn matrix **A** if and only if there is a vector $v \neq 0$ and

$$\mathbf{A}v = \lambda v$$

Then the vector v is called the eigen vector of the matrix **A**.

Eigen values of a matrix is found by $\mathbf{A}v - \lambda v = (\mathbf{A} - \lambda I)\, v = 0$ where I is the identity matrix.
So we want want for $v \neq 0$ $(\mathbf{A} - \lambda I)\, v = 0$ that means also $|\mathbf{A} - \lambda I| = 0$, from this determinant we find the characteristic polynomial of the matrix $P(\lambda)$ whose roots gives the eigen values of the matrix. Substituting each eigen value to $\mathbf{A}v = \lambda v$ we find the eigen vectors of the matrix for each eigen value.

Example 46 *Find the eigen values and the eigen vectors of the following matrix* **A**.

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} |\mathbf{A} - \lambda I| &= \begin{vmatrix} 3-\lambda & 2 & 0 \\ -1 & 0-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(-1)^{3+3} \begin{vmatrix} 3-\lambda & 2 \\ -1 & 0-\lambda \end{vmatrix} = 0 \\ &= (1-\lambda)(-\lambda(3-\lambda)+2) = (1-\lambda)\left(\lambda^2 - 3\lambda + 2\right) = (1-\lambda)^2 (2-\lambda) = 0 \end{aligned}$$

Hence the eigen values are $\boldsymbol{\lambda}_1 = \mathbf{1}$ and $\boldsymbol{\lambda}_2 = \mathbf{2}$. Lets use as a general corresponding eigen vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$
\begin{aligned}
\text{for } \boldsymbol{\lambda}_1 &= \mathbf{1} \rightarrow \begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 * \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
3x + 2y &= x \rightarrow y = -x \\
-x &= y \\
z &= z \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -y \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\text{if } y &= 0 \ z = 1 \ \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ if } y = 1 \ z = 0 \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\text{for } \boldsymbol{\lambda}_1 &= \mathbf{2} \rightarrow \begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 * \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
3x + 2y &= 2x \rightarrow x = -2y \\
-x &= 2y \\
z &= 2z \rightarrow z = 0 \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \\
\text{if } y &= 1 \ \ \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

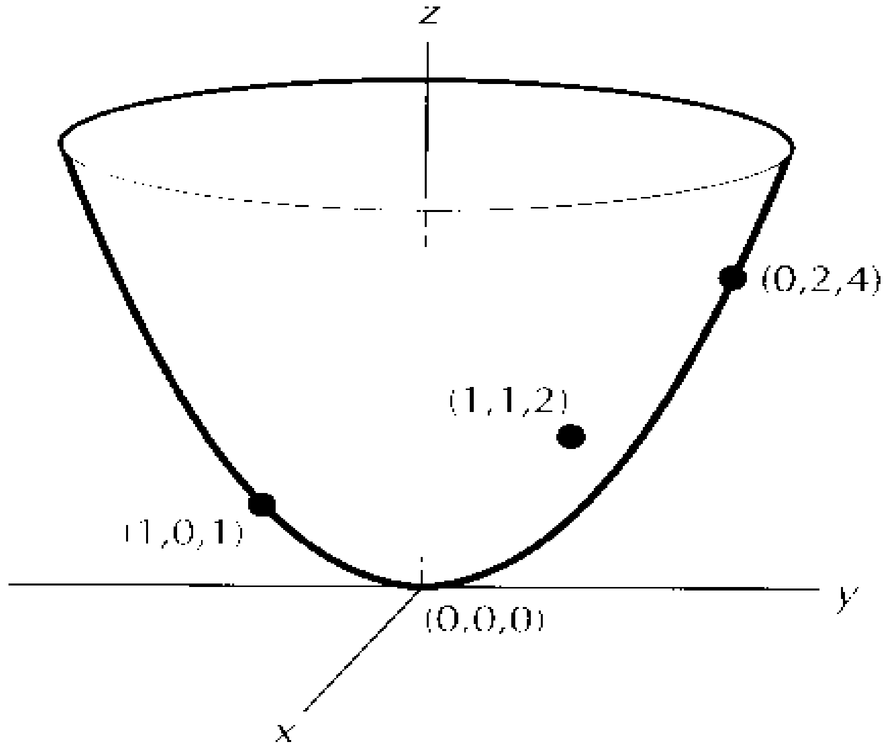
7 Functions of Several Variables

7.1 Geometric representations of functions

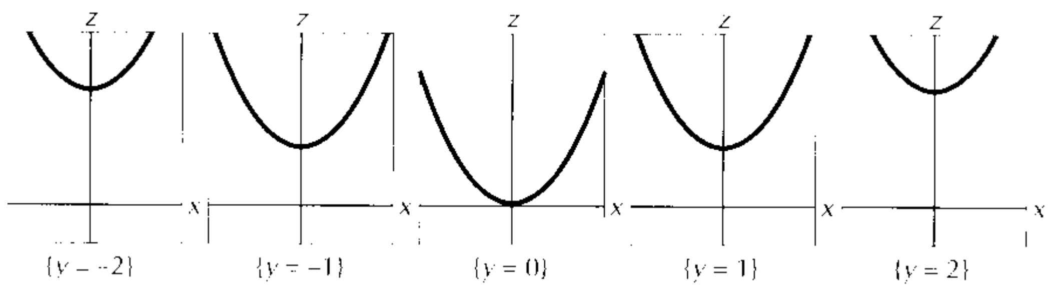
When there are more than one variable in our functions, they can be understood by taking the resulting output or one of the variables constant and combining the graphs of them by changing constant term.

7.1.1 Graphs of Functions of two variables

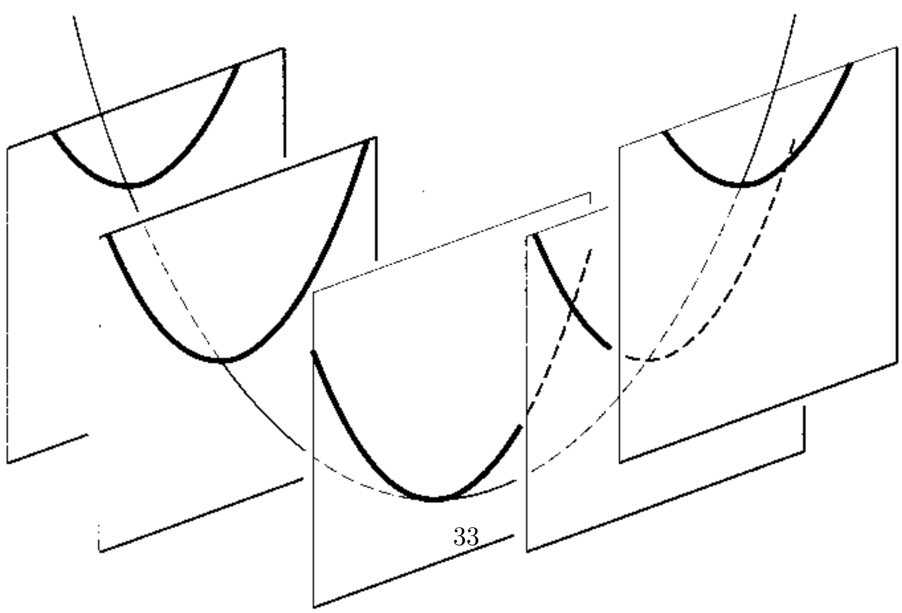
- The graph of a circle can be written as $x^2 + y^2 = r^2$ where r is a constant radius. If we want to graph $f(x, y) = z = x^2 + y^2$, then we start to think from a constant radius say zero that means a point in the origin since x and y also becomes zero for f(x)=0. Increasing f(x) we get bigger radius circles and adding them up we obtain the following first graph. If we do not know the graph of the circle we may also simply think to assume y=0, then $z = x^2$ is a parabola where y=0 that means if we slice the graph at y=0 we have a parabola $z = x^2$ on xz plane, for y=-1 and y=1 we get the usual parabola pushed up one unit and for y=-2 andy=2 it is pushed up 4 units. Putting slices together we have the graph of the function. If we do the same for x we also get a parabolas also zy plane and same graph.



The graph of $f(x, y) = x^2 + y^2$.

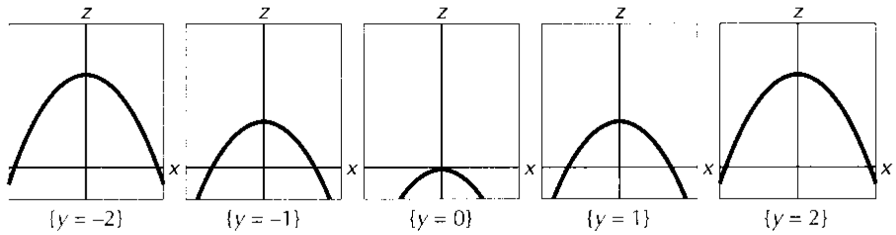


Slices of $z = x^2 + y^2$ in the $\{y = b\}$-planes.

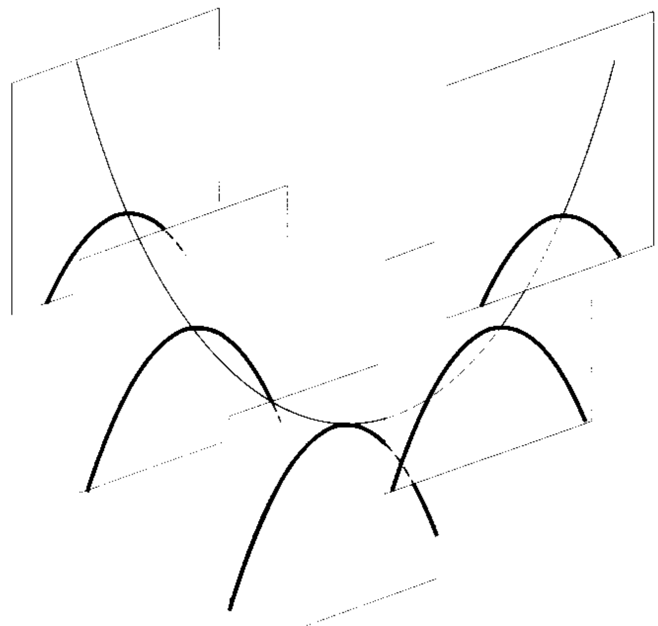


Putting the slices together.

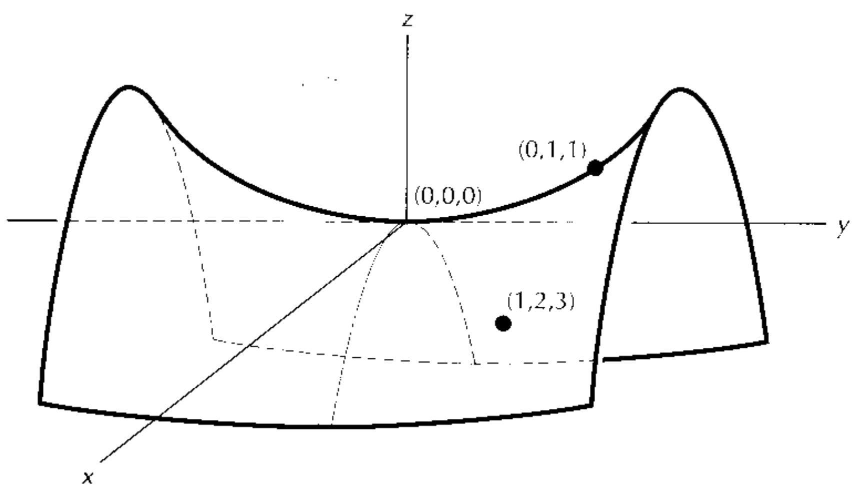
- The graph of $z = y^2 - x^2$: Restricting y=0 we get the concave parabola $z = -x^2$, for y=1 and y=-1 we find one unit pushed version of the previous one $z = 1 - x^2$ and for y=-2 and y=2 we obtain the four unit pushed slice. Putting the slices together we graph the function.



Restrictions of $z = y^2 - x^2$ to the planes $\{y = b\}$.



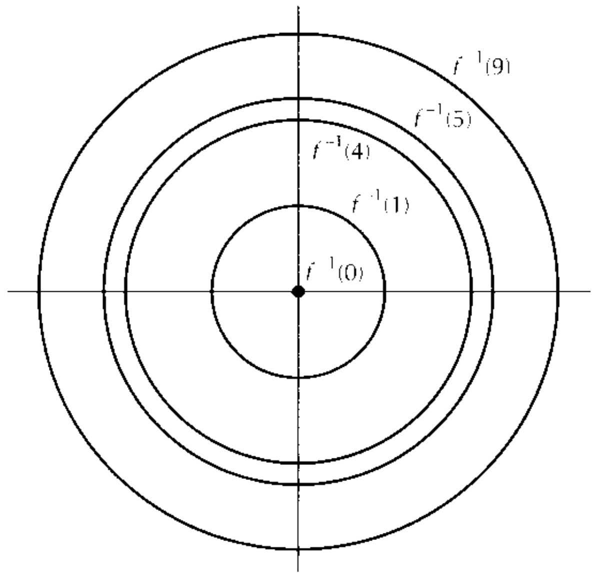
Putting the slices together.



The graph of $f(x, y) = y^2 - x^2$.

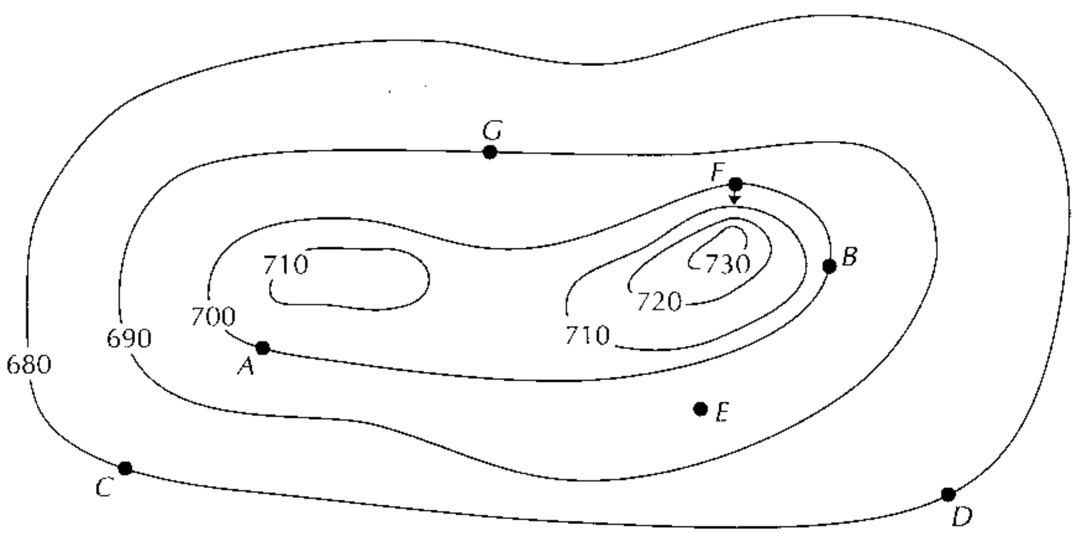
7.1.2 Level Curves

Graphing the points that result the same function value is a more easier method to visualize 3D functions. Again looking on $f(x, y) = z = x^2 + y^2$,if we graph the points where the function is constant we get a circle, if we plot all the possible circles increasing the function value in a 2D space, we get the following graph. Note if we pull from the increasing parts (outside) we get the same 3d graph.



Level curves of $z = x^2 + y^2$.

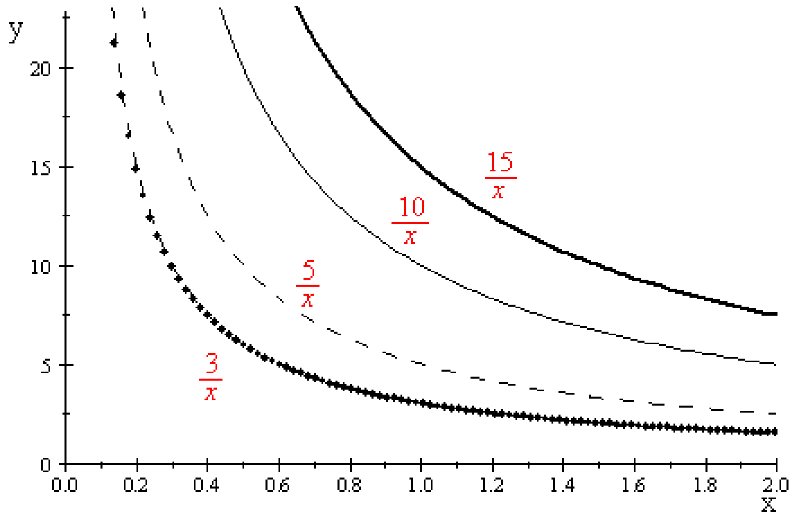
The level curves are useful to show isotherms which show the places with the same temperature and also hiking maps to see which part of the mountain is steep or flat. As seen in the following figure, at the point F the curves are close to each other such that there is close rises in altitude. So the point F is a very steep part of the mountain. On the other hand, point G seems rather flat and easy to climb.



A hiking map with its level curves.

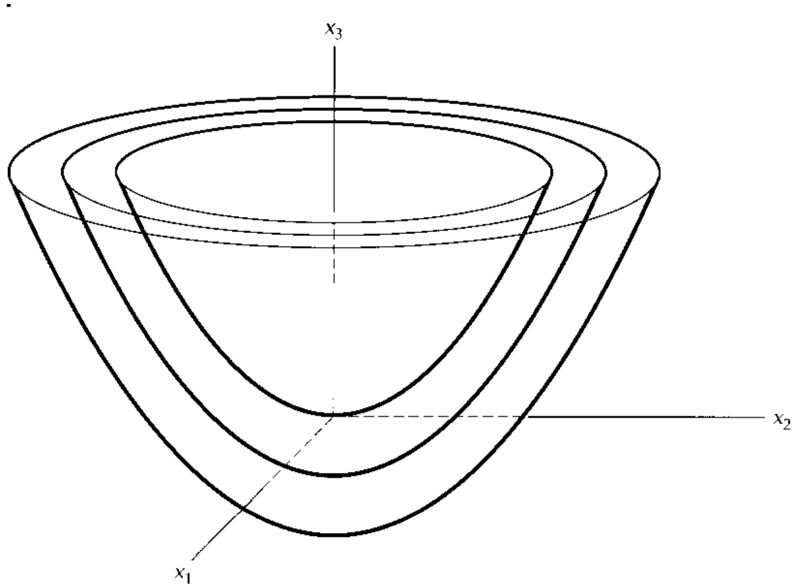
7.1.3 Planar level sets in Economics

Economists use level sets to study the two fundemental functions of microeconomics-production function and utility function. Lets graph a simple version of Cobb-Dougless production function ($Q = kx^{\alpha}y^{\beta}$) with $k = \alpha = \beta = 1$ that is $Q = xy$ where x is capital and y is labor. Taking production constant, we obtain the level sets which are called isoquants for the production function. For a constant Q (say 3) $y = \frac{3}{x}$, for $Q = 5 \rightarrow y = \frac{5}{x}$ and $Q = 10 \rightarrow y = \frac{10}{x}$ and $Q = 15 \rightarrow y = \frac{15}{x}$. As the isoquants increase the production increases. For different k, α and β similar level curves can be obtained.The same curves can also be used to understand the level curves of the utility function which are called indifference curves and they increase as the utility increases.



7.1.4 3D level sets

The same approach is used for 3D level sets by fixing the z value and graphing the function with fixed z using the technique we described in section 7.1.1. For example the graph of $z = x_3 - x_1^2 - x_2^2$ is the same as the graph in section 7.1.1 if z=0 and increasing z the value decreases as seen in the following graph.



Some level sets of $z = x_3 - x_1^2 - x_2^2$.

7.2 Stationary Points of Functions with Several Variables

Stationary points are found by equating the first partial derivatives of a function to zero with respect to each variables. Whether they are a local minimum, local maximum or a saddle point are checked similar to checking the second derivative of a one variable function, by checking a matrix of second derivatives (called **Hessian** matrix). The method can be used for functions with more variables also, but for simplicity we only cover two and three variables cases here. For a function $f(x,y)$ or $g(x,y,z)$ with stationary point P_0 (found by $\nabla f(P_0) = 0$ or $\nabla g(P_0) = 0$) Hessian matrix is defined as

for f(x,y): $H(P_0) = \begin{bmatrix} f_{xx}(P_0) & f_{xy}(P_0) \\ f_{yx}(P_0) & f_{yy}(P_0) \end{bmatrix}$ or $g(x,y,z)$ $H(P_0) = \begin{bmatrix} g_{xx}(P_0) & g_{xy}(P_0) & g_{xz}(P_0) \\ g_{yx}(P_0) & g_{yy}(P_0) & g_{yz}(P_0) \\ g_{zx}(P_0) & g_{zy}(P_0) & g_{zy}(P_0) \end{bmatrix}$

$$\begin{aligned}
\text{if } H(P_0) &> 0 \text{ namely all Eigen values}(\lambda_i)>0 \text{ then } \mathbf{P_0} \textbf{ is a local minimum} \\
\text{if } H(P_0) &< 0 \text{ namely all Eigen values}(\lambda_i)<0 \text{ then } \mathbf{P_0} \textbf{ is a local maximum} \\
\text{if some } \lambda_i &> 0 \text{ and some } \lambda_j < 0 \text{ then } \mathbf{P_0} \textbf{ is a saddle point} \\
\text{if } H(P_0) &\leq 0 \text{ or } H(P_0) \geq 0 \text{ at least one Eigen value=0 } \textbf{inconclusive}
\end{aligned}$$

In general Hessian matrices are symmetric namely $f_{yx}(P_0) = f_{xy}(P_0), g_{xy}(P_0) = g_{yx}(P_0), g_{zx}(P_0) = g_{xz}(P_0), g_{yz}(P_0) = g_{zy}(P_0)$

Example 47 *Study the stationary points of*

$$\begin{aligned}
g(x,y) &= e^{x^2} + xy - y^2 + 3 \\
g_y(x,y) &= x - 2y = 0 \rightarrow y = \frac{x}{2} \\
g_x(x,y) &= 2xe^{x^2} + y = 0 \text{ using } y = \frac{x}{2} \rightarrow x(2e^{x^2} + \frac{1}{2}) = 0 \rightarrow x = y = 0 \rightarrow P_0\ (0,0) \\
g_{xx}(x,y) &= 2e^{x^2} + 2x\left(2xe^{x^2}\right) = 2e^{x^2}(1+2x^2) \rightarrow g_{xx}(0,0) = 2 \\
g_{yy}(x,y) &= -2 \\
g_{xy}(x,y) &= g_{yx}(x,y) = 1 \\
H(P_0) &= \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 - 5 = 0 \rightarrow \lambda = \pm\sqrt{5} \text{ then } P_0 \text{ is a saddle point}
\end{aligned}$$

Example 48 *Study the stationary points of*

$$
\begin{aligned}
g(x,y,z) &= x^2+y^4+y^2+z^3-2xz \\
g_x(x,y,z) &= 2x-2z=0 \rightarrow \mathbf{x}=\mathbf{z} \\
g_y(x,y,z) &= 4y^3+2y=2y(2y^2+1)=0 \rightarrow \mathbf{y}=\mathbf{0} \\
g_z(x,y,z) &= 3z^2-2x \equiv z(3z-2)=0 \rightarrow \mathbf{z}=\mathbf{0} \text{ or } \rightarrow \mathbf{z}=\mathbf{2/3} \text{ then two stationary points} \\
P_1 &= (0,0,0)\ P_2=(2/3,0,2/3) \\
g_{xx}(x,y,z) &= 2 \\
g_{yy}(x,y,z) &= 12y^2+2 \rightarrow g_{yy}(0,0,0)=2 \quad g_{yy}(2/3,0,2/3)=2 \\
g_{zz}(x,y,z) &= 6z \rightarrow g_{zz}(0,0,0)=0\ g_{zz}(2/3,0,2/3)=4 \\
g_{xy}(x,y,z) &= g_{yx}(x,y)=0 \\
g_{xz}(x,y,z) &= g_{zx}(x,y)=-2 \\
g_{yz}(x,y,z) &= g_{zy}(x,y)=0 \\
H(P_1) &= \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ -2 & 0 & -\lambda \end{vmatrix} = (2-\lambda)(-1)^{2+2} \begin{vmatrix} 2-\lambda & -2 \\ -2 & -\lambda \end{vmatrix} = 0 \\
(2-\lambda)(\lambda^2-2\lambda-4) &= 0 \rightarrow \lambda_1=2\ \lambda_{2,3}=\frac{2\pm\sqrt{4+16}}{2}=1\pm\sqrt{5} \text{ hence } P_1 \text{ is a saddle point} \\
H(P_2) &= \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \rightarrow \begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ -2 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(-1)^{2+2} \begin{vmatrix} 2-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0 \\
(2-\lambda)(\lambda^2-6\lambda+4) &= 0 \rightarrow \lambda_1=2\ \lambda_{2,3}=\frac{6\pm\sqrt{36-16}}{2}=3\pm\sqrt{5} \text{ hence } P_2 \text{ is local minimum}
\end{aligned}
$$

Reference

Mathematics for Economists, C. Simon-L. Blume (1994) . W.W. Norton & Company, Inc. ISBN 0-393-95733-O

Appendix (for further summary information)

Common Derivative and Integrals

Derivatives

Basic Properties/Formulas/Rules

$\frac{d}{dx}\left(cf(x)\right) = cf'(x)$, c is any constant.

$\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, n is any number.

$\frac{d}{dx}(c) = 0$, c is any constant.

$(f\,g)' = f'\,g + f\,g'$ – **(Product Rule)**

$\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$ – **(Quotient Rule)**

$\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)g'(x)$ **(Chain Rule)**

$\frac{d}{dx}\left(\mathbf{e}^{g(x)}\right) = g'(x)\,\mathbf{e}^{g(x)}$

$\frac{d}{dx}\left(\ln g(x)\right) = \frac{g'(x)}{g(x)}$

Common Derivatives

Polynomials

$\frac{d}{dx}(c) = 0$ $\frac{d}{dx}(x) = 1$ $\frac{d}{dx}(cx) = c$ $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\frac{d}{dx}\left(cx^n\right) = ncx^{n-1}$

Trig Functions

$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$ $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

Inverse Trig Functions

$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$

Exponential/Logarithm Functions

$\frac{d}{dx}\left(a^x\right) = a^x \ln(a)$ $\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x$

$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x},\ x > 0$ $\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x},\ x \neq 0$ $\frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x \ln a},\ x > 0$

Hyperbolic Trig Functions

$\frac{d}{dx}(\sinh x) = \cosh x$ $\frac{d}{dx}(\cosh x) = \sinh x$ $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$ $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$ $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Integrals

Basic Properties/Formulas/Rules

$\int cf(x)\,dx = c\int f(x)\,dx$, c is a constant. $\int f(x)\pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$ where $F(x)=\int f(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant. $\int_a^b f(x)\pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^a f(x)\,dx = 0$ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ $\int_a^b c\,dx = c(b-a)$

If $f(x)\ge 0$ on $a\le x\le b$ then $\int_a^b f(x)\,dx \ge 0$

If $f(x)\ge g(x)$ on $a\le x\le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

Common Integrals

Polynomials

$\int dx = x + c$ $\int k\,dx = k\,x + c$ $\int x^n dx = \frac{1}{n+1}x^{n+1}+c,\ n\ne -1$

$\int \frac{1}{x}\,dx = \ln|x| + c$ $\int x^{-1}\,dx = \ln|x| + c$ $\int x^{-n} dx = \frac{1}{-n+1}x^{-n+1}+c,\ n\ne 1$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$ $\int x^{\frac{p}{q}}dx = \frac{1}{\frac{p}{q}+1}x^{\frac{p}{q}+1}+c = \frac{q}{p+q}x^{\frac{p+q}{q}}+c$

Trig Functions

$\int \cos u\,du = \sin u + c$ $\int \sin u\,du = -\cos u + c$ $\int \sec^2 u\,du = \tan u + c$

$\int \sec u\tan u\,du = \sec u + c$ $\int \csc u\cot u\,du = -\csc u + c$ $\int \csc^2 u\,du = -\cot u + c$

$\int \tan u\,du = \ln|\sec u| + c$ $\int \cot u\,du = \ln|\sin u| + c$

$\int \sec u\,du = \ln|\sec u + \tan u| + c$ $\int \sec^3 u\,du = \frac{1}{2}\left(\sec u\tan u + \ln|\sec u + \tan u|\right) + c$

$\int \csc u\,du = \ln|\csc u - \cot u| + c$ $\int \csc^3 u\,du = \frac{1}{2}\left(-\csc u\cot u + \ln|\csc u - \cot u|\right) + c$

Exponential/Logarithm Functions

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$ $\int a^u\,du = \frac{a^u}{\ln a} + c$ $\int \ln u\,du = u\ln(u) - u + c$

$\int \mathbf{e}^{au}\sin(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\sin(bu) - b\cos(bu)\right) + c$ $\int u\mathbf{e}^u du = (u-1)\mathbf{e}^u + c$

$\int \mathbf{e}^{au}\cos(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\cos(bu) + b\sin(bu)\right) + c$ $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + c$

Inverse Trig Functions

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)+c \qquad \int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2}+c$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c \qquad \int \tan^{-1}u\,du = u\tan^{-1}u - \frac{1}{2}\ln\left(1+u^2\right)+c$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right)+c \qquad \int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2}+c$$

Hyperbolic Trig Functions

$$\int \sinh u\,du = \cosh u + c \qquad \int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + c \qquad \int \operatorname{sech}^2 u\,du = \tanh u + c$$

$$\int \cosh u\,du = \sinh u + c \qquad \int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + c \qquad \int \operatorname{csch}^2 u\,du = -\coth u + c$$

$$\int \tanh u\,du = \ln\left(\cosh u\right)+c \qquad \int \operatorname{sech} u\,du = \tan^{-1}\left|\sinh u\right|+c$$

Miscellaneous

$$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right|+c \qquad \int \frac{1}{u^2-a^2}\,du = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+c$$

$$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\right|+c$$

$$\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2}-\frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|+c$$

$$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)+c$$

$$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2}+\frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right)+c$$

Standard Integration Techniques

Note that all but the first one of these tend to be taught in a Calculus II class.

u Substitution

Given $\int_a^b f\left(g(x)\right)g'(x)\,dx$ then the substitution $u=g(x)$ will convert this into the integral, $\int_a^b f\left(g(x)\right)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$.

Integration by Parts

The standard formulas for integration by parts are,

$$\int u\,dv = uv - \int v\,du \qquad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Choose u and dv and then compute du by differentiating u and compute v by using the fact that $v=\int dv$.

Trig Substitutions

If the integral contains the following root use the given substitution and formula.

$$\sqrt{a^2 - b^2x^2} \quad \Rightarrow \quad x = \frac{a}{b}\sin\theta \quad \text{and} \quad \cos^2\theta = 1 - \sin^2\theta$$

$$\sqrt{b^2x^2 - a^2} \quad \Rightarrow \quad x = \frac{a}{b}\sec\theta \quad \text{and} \quad \tan^2\theta = \sec^2\theta - 1$$

$$\sqrt{a^2 + b^2x^2} \quad \Rightarrow \quad x = \frac{a}{b}\tan\theta \quad \text{and} \quad \sec^2\theta = 1 + \tan^2\theta$$

Partial Fractions

If integrating $\int \frac{P(x)}{Q(x)}\,dx$ where the degree (largest exponent) of $P(x)$ is smaller than the degree of $Q(x)$ then factor the denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Products and (some) Quotients of Trig Functions

$\int \sin^n x \cos^m x\,dx$

1. **If n is odd.** Strip one sine out and convert the remaining sines to cosines using $\sin^2 x = 1 - \cos^2 x$, then use the substitution $u = \cos x$
2. **If m is odd.** Strip one cosine out and convert the remaining cosines to sines using $\cos^2 x = 1 - \sin^2 x$, then use the substitution $u = \sin x$
3. **If n and m are both odd.** Use either 1. or 2.
4. **If n and m are both even.** Use double angle formula for sine and/or half angle formulas to reduce the integral into a form that can be integrated.

$\int \tan^n x \sec^m x\,dx$

1. **If n is odd.** Strip one tangent and one secant out and convert the remaining tangents to secants using $\tan^2 x = \sec^2 x - 1$, then use the substitution $u = \sec x$
2. **If m is even.** Strip two secants out and convert the remaining secants to tangents using $\sec^2 x = 1 + \tan^2 x$, then use the substitution $u = \tan x$
3. **If n is odd and m is even.** Use either 1. or 2.
4. **If n is even and m is odd.** Each integral will be dealt with differently.

Convert Example : $\cos^6 x = \left(\cos^2 x\right)^3 = \left(1 - \sin^2 x\right)^3$