Mathematics Preparatory Course - MSc in EEBL Lecture Notes

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- Objective:

  The Mathematics preparatory course aims to review the basic concepts of calculus and linear algebra and to provide students with the necessary tools to understand the notions of Economics where a quantitative approach is needed.

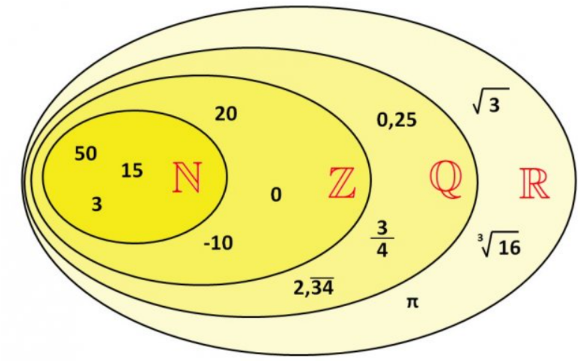
- Course content:

  The course consists in some lectures which will cover the key concepts of the following subjects:

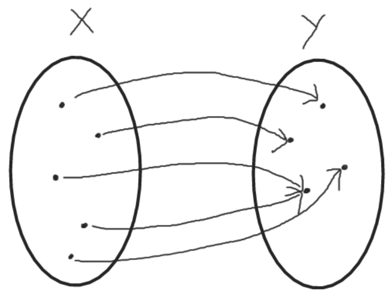
  – One variable calculus: foundations and applications
  – Exponential and logarithmic Functions
  – Linear Algebra
  – Function of several variables

1. One Variable Calculus Foundations

Definition 1.1. A real number is a value of a continuous quantity that can represent a distance along a line. It can be rational (like 3/4 or -232) or irrational (like $\sqrt{2}$ ≈1.41421356...) or transcendent number (like π ≈3.14159265…, or e, Nepero number).

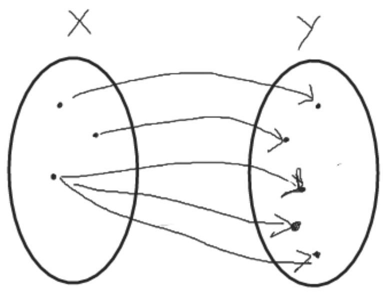


Definition 1.2. Let X and Y be two subsets of R. A **function** *f* defined on *X* with values in *Y* is a correspondence associating to each element $x \in X$ **at most** one element $y \in Y$.



$f : x \to f(x) = y$

The following diagram doesn't represent a function because there is an element of the first subset associated with more elements of the second one.



x is **independent** , while y is **dependent** variable

The **domain** is the set of numbers x at which f(x) is defined. When the domain is not specified, it is assumed that it includes all the real numbers for which the function takes meaningful values.

The **range** (or co−domain) of a function is the set of all the possible values of it.

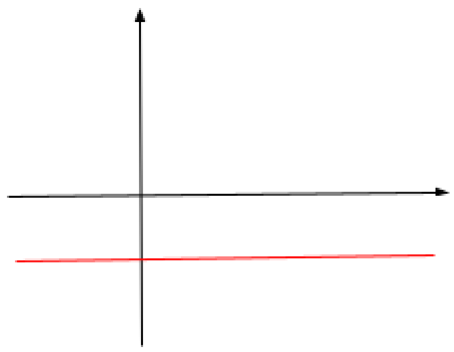
1.3. Function Types

Polynomials : Obtained by the addition of monomials; the highest exponent defines the order of the polynomial.

ex. $y = 5x^3 - 3x^2 + 2$

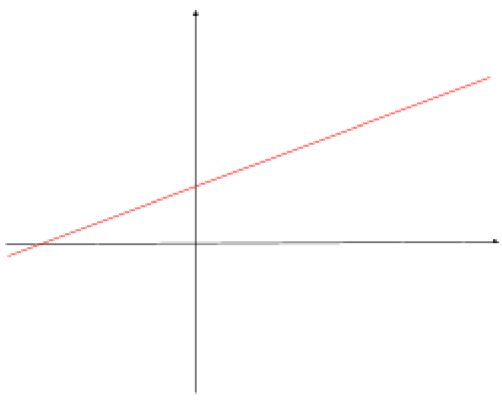
Constant function is polynomial of order zero: y = a

ex. $y = - 5$



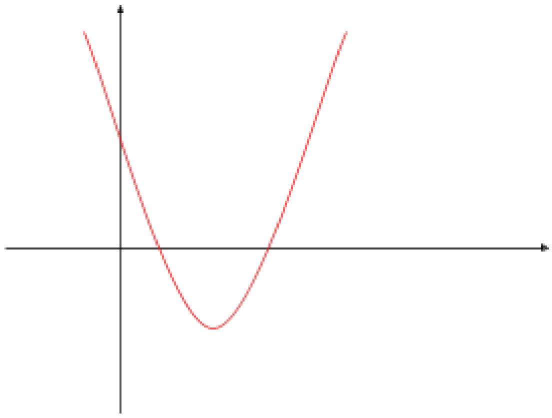
Linear functions is polynomial of order one: y = mx + n

ex. $y = \frac{1}{2}x + 4$



Quadratic function (parabola) is a polynomial of order two: y = ax^2 + bx + c

ex. $y = x^2 - 5x + 6$

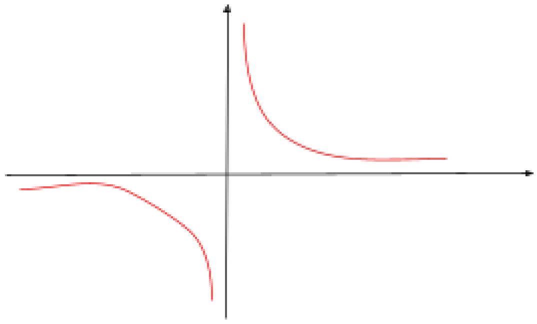


Rational functions : Ratios of polynomials f(x) =$\frac{g(x)}{h(x)}$ (h(x) isn't constant)

ex. $y = \frac{2x-1}{x^2-5}$

A simple example is Hyperbola (constant over a monomial of order one) : y = $\frac{a}{x}$

ex. $y = \frac{1}{x}$

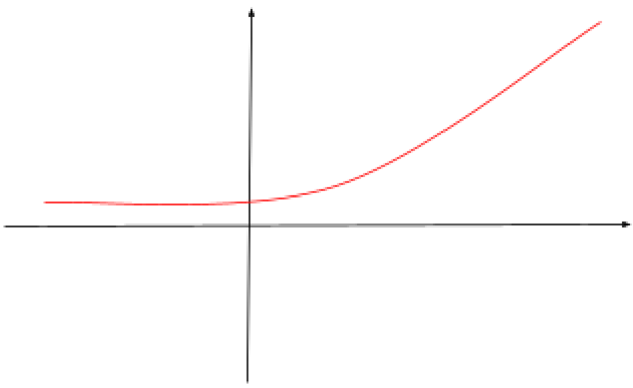


Irrational functions: is a function like y=$\sqrt[n]{\frac{g(x)}{h(x)}}$ (g and h mat be constant function, not simultaneously)

ex. $y = \sqrt[3]{\frac{x^2}{2x^2+x-1}}$

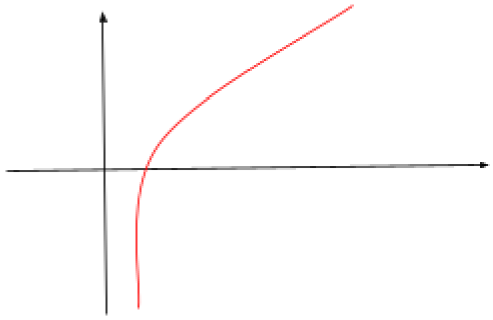
Exponential functions : f(x) = a^x

ex. $y = e^x$



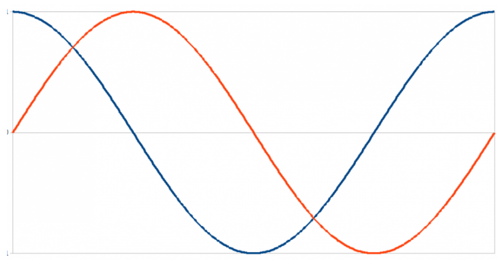
Logarithmic functions: $f(x) = \log_a x$

ex. $y = log\, x$

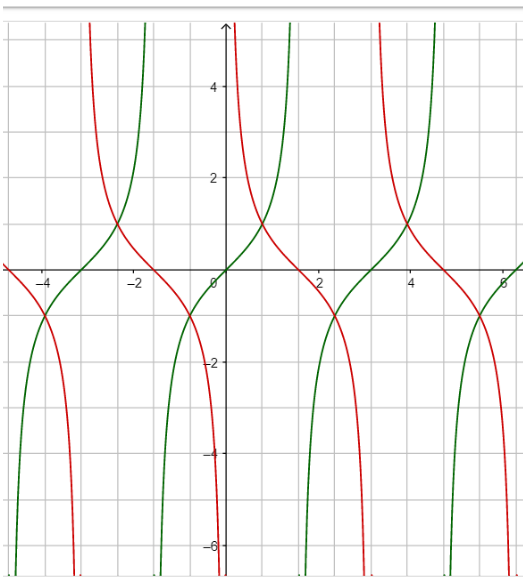


Trigonometric functions : $f(x) = \sin(x)$, $f(x) = \cos(x)$, $f(x) = \tan(x)$, $f(x) = \text{cotan}(x)$

ex. $y = sin\, x$, $y = cos\, x$



ex. $y = tan\, x$, $y = cotan\, x$



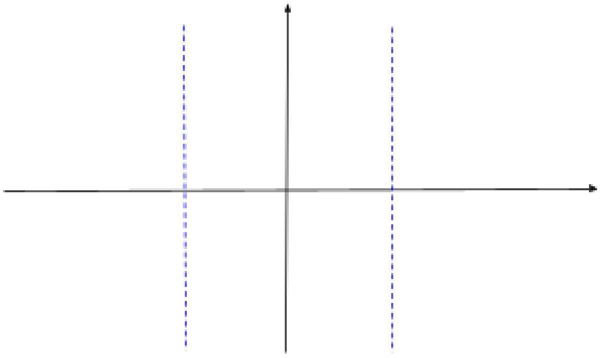
For each type of function there is a particular domain.

Polynomial functions exist in R;

Rational functions exist when the function in the denominator is non-zero;

ex.

If we have $y = \frac{2x-1}{x^2-5}$, it exists if $x^2 - 5 \neq 0$, for $x \neq \pm \sqrt{5}$. So, D=]-∞, $-\sqrt{5}$[∪]$-\sqrt{5}$, $+\sqrt{5}$ [∪] $+\sqrt{5}$, +∞[, or D= R-{$\pm\sqrt{5}$}

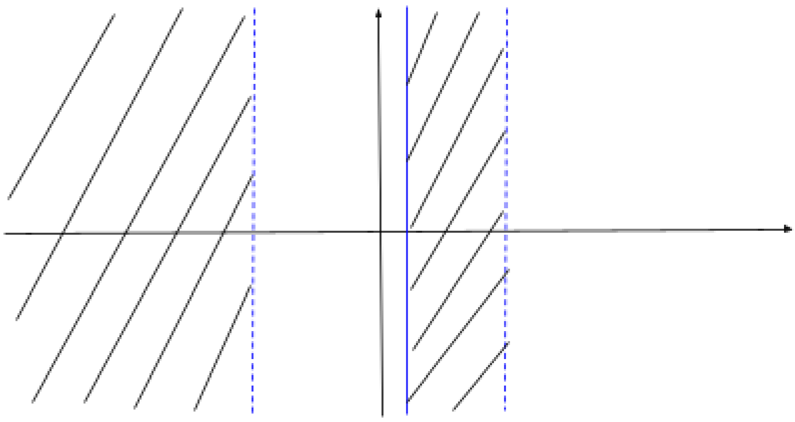


The existence of irrational functions depends on the root index n: if it is odd, the domain is reduced to that of polynomial or rational functions, if it is even it is necessary to study when $\frac{g(x)}{h(x)} \geq 0$;

ex.

$y = \sqrt[3]{x^2 + x + 1}$, the domain is R; $y = \sqrt[5]{\frac{2x}{3x-2}}$, the domain is R-{$\frac{2}{3}$}

$y = \sqrt{\frac{5x-1}{x^2-1}}$, the domain is $\{x \in R,\ x:\ -1 < x \leq \frac{1}{5} \vee x > 1\}$



Exponential functions exist in R, or where the exponent exists;

Logarithmic functions exist where the logarithmic argument is greatest than zero;

Trigonometric function exist in R, when we consider f(x) = sin(x), f(x) = cos(x), instead f(x) = tan(x) exists when x isn't $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ and f(x) = cotan(x) exists when x isn't π or 2π.

Some of the most important information of a function is contained in the **graph**.

The graph of *f* is the subset of Γ(*f*) of the Cartesian product *XxY* made of pairs (*x, f* (*x*)) when x varies in the domain of f, i.e.

$\Gamma(f) = \{(x, f(x)) \in X \times Y : x \in \text{dom} f\}$

A function may be **even** or **odd**.

Let be the function

$f : x \rightarrow Y$

Definition 1.4. The function f is **even** if

$f(-x) = f(x), \quad \forall x \in X$

The function is symmetric respect to the y axis.

ex. $y = \frac{x^2+4}{1-x^2}$ is an even function. In fact, $f(-x) = \frac{(-x)^2+4}{1-(-x)^2} = \frac{x^2+4}{1-x^2} = f(x)$.

Definition 1.5. The function f is **odd** if

$f(-x) = -f(x), \quad \forall x \in X$

The function is symmetric respect to the origin of the cartesian axis.

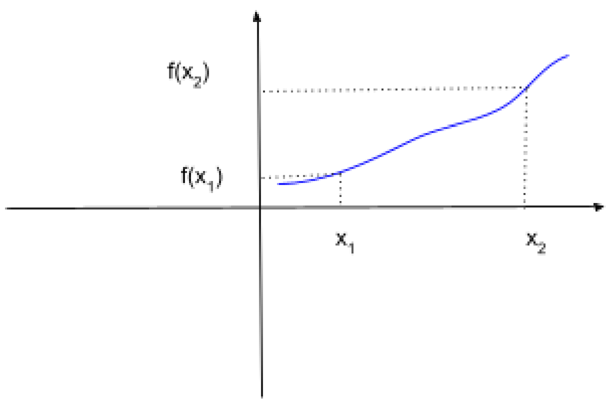
ex. $y = sin(x)$ is an odd function, because $f(-x) = sin(-x) =- sin(x) =- f(x)$;

2. Basic geometric properties of a function

The basic geometric properties of a function are whether it is increasing or decreasing and the location of its local/global maxima/minima.

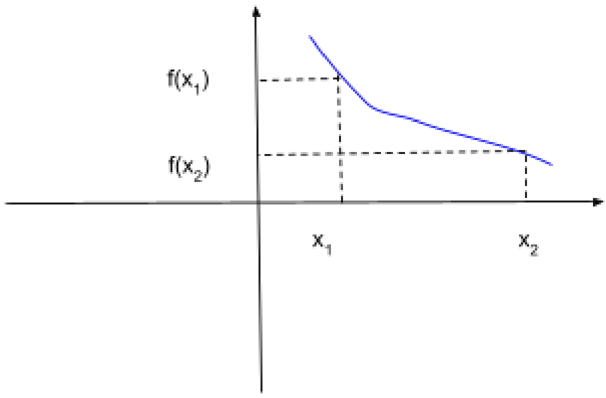
Definition 2.1. The function f is increasing if

$\forall x_1, x_2 \in dom(f) \qquad x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

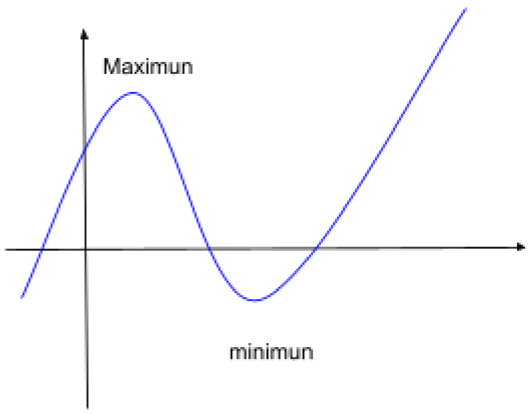


Conversely, it is decreasing if

$\forall x_1, x_2 \in dom(f) \quad x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$



The point where the function turns from decreasing to increasing is a **minimum** for the function and the point where the function turns from decreasing to increasing is a **maximum** for the function.

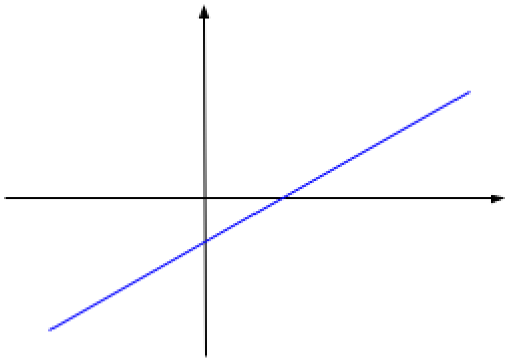


If there is no greater (smaller) value of the function in its range from that maximum (minimum) then the maximum (minimum) is called global maximum (minimum)

2.2. Linear Function

Polynomial of degree 1 are interesting function, they are also called **Linear function**
$f(x) = mx + q$

The graph of a linear function is a straight line. In order to draw this function knowing two points in the Cartesian plane is enough.



One of the main features which distinguish two different lines is the **slope** that is given by m. This function is increasing as $m > 0$ and decreasing if $m < 0$; if $m = 0$ the function degenerates to the constant function $f(x) = q$ The slope is given by the increase of y as x increases.

We can define $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

2.3. The slope of nonlinear functions

The slope of a nonlinear function of f at point (x_0, $f(x_0)$) is the slope of the tangent line to its graph at that point. It is the rate of change (marginal effect) of f with respect to x at that point. It is necessary to study limits to know the slop.

3. Limit and continuity

Limit is the value that a function approaches as the input (or index) approaches some value.
Let $X, Y \subseteq \mathbb{R}$ and $f : X \to Y$ be a function of real variables. If

$$\lim_{x \to x_0} f(x) = l$$

means that $f(x)$ can be made to be as close to l as desired, by making x sufficiently close to x_0. Or equivalently, $f(x)$ goes to l as x approaches x_0

In the most simple cases, if $f(x)$ is a real value function which goes to l as x goes to x_0, we can find l (the limit) substituting x_0 in the function.

ex. $\lim_{x \to -2} \sqrt{4x^2 + 9} = 5$

The limit of the function can be also $+\infty / -\infty$.

ex. $\lim_{x \to 0} \frac{5x+7}{x^2} = +\infty$

In order to find these limits, it is important to know the graph of the function.

In calculus, a one-sided limit is either of the two limits of a function $f(x)$ of a real variable x as x approaches a specified point either from the left or from the right.

3.1. Definition right limit: $\lim_{x \to x0^+} f(x) = l$

3.2. Definition left limit: $\lim_{x \to x0^-} f(x) = l$

The $\lim_{x \to x0} f(x)$ exists only if the right limit and the left limit exist and they are equal.

3.3. Mathematical operation

$$\lim_{x \to x0} f(x) \pm g(x) = \lim_{x \to x0} f(x) \pm \lim_{x \to x0} g(x)$$

$$\lim_{x \to x0} f(x) \cdot g(x) = \lim_{x \to x0} f(x) \cdot \lim_{x \to x0} g(x)$$

$$\lim_{x \to x0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x0} f(x)}{\lim_{x \to x0} g(x)}$$

$$\lim_{x \to x0} a^{f(x)} = a^{\lim_{x \to x0} f(x)}$$

$$\lim_{x \to x0} \log_a f(x) = \log_a \lim_{x \to x0} f(x)$$

ex.

$$\lim_{x \to 1} \frac{2x^3 - x + 5}{1 - x - x^3} = \frac{\lim_{x \to 1}(2x^3 - x + 5)}{\lim_{x \to 1}(1 - x - x^3)} =$$

So, we have

$$= \frac{2 \cdot 1 - 1 + 5}{1 - 1 - 1^3} = \frac{6}{-1} = -6$$

ex.

$$\lim_{x\to 5} \frac{x+3-x^2}{x^2-25}$$

Thi limit become

$$\lim_{x\to 5} \frac{x+3-x^2}{(x+5)(x-5)}$$

If we put 5 at the denominator, we have ∞ , but we want to know this situation better. So we can study left and right limits.

$$\lim_{x\to 5^-} \frac{x+3-x^2}{(x+5)(x-5)}$$

$$\lim_{x\to 5^+} \frac{x+3-x^2}{(x+5)(x-5)}$$

So, we have that

$$\frac{-17}{10\cdot 0^-} = \frac{-17}{0^-} = +\infty$$

$$\frac{-17}{10\cdot 0^+} = \frac{-17}{0^+} = -\infty$$

The left limit is +∞ and the right one is − ∞.

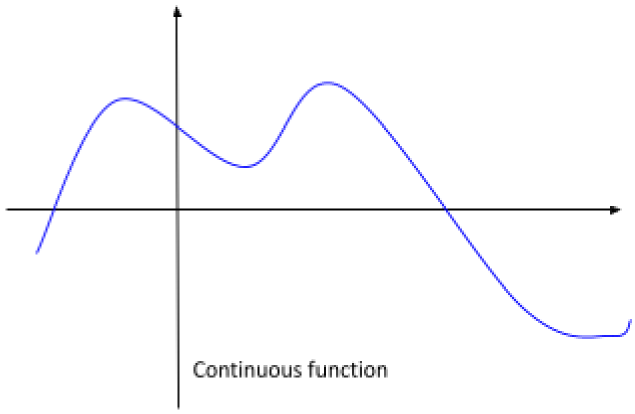
We may end up in a **indeterminate form**

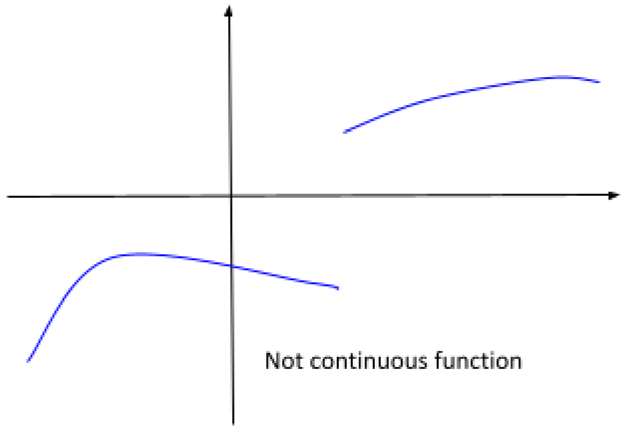
$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty$

ex. $\lim_{x\to+\infty} \frac{3x^3-5x^2+7}{2x^4-x^3-1} = \lim_{x\to+\infty} \frac{x^3(3-\frac{5}{x}+\frac{7}{x^3})}{x^4(2-\frac{1}{x}-\frac{1}{x^4})} = \lim_{x\to+\infty} \frac{3}{2x}$ because $\frac{a}{\infty}$=0, so the limit is 0.

3.4. Continuity

We could say if a function is continuous just by looking at the graph. A function is continuous if its graph has no breaks (no jumps for the same point) in its domain.





Let $f : X \longrightarrow Y$ and $x_0 \in X$. $f(x)$ is continuous in x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

3.5. Differential

Steepness is a key concept in Economics. We are often interested in evaluating what is the effect of an increase of the independent variable to the dependent variable. As we have already seen, regarding linear function the steepness is easily given by the slope coefficient.

Let $f : X \longrightarrow Y$ and $x_0 \in X$. If x increases by Δx, e.g. from x_0 to $x_0 + \Delta x$, also the function

$f(x)$ will increase as follows

$$\Delta f(x) = f(x_0 + \Delta x) - f(x_0)$$

interesting studying Difference quotient

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

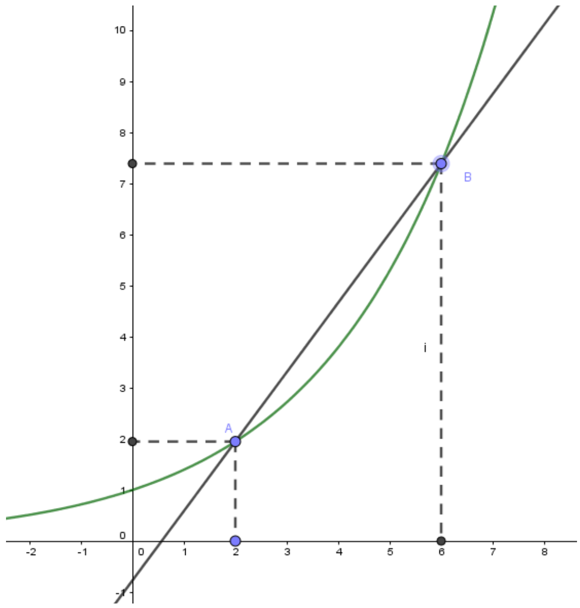
In particularly, when Δx is very little

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

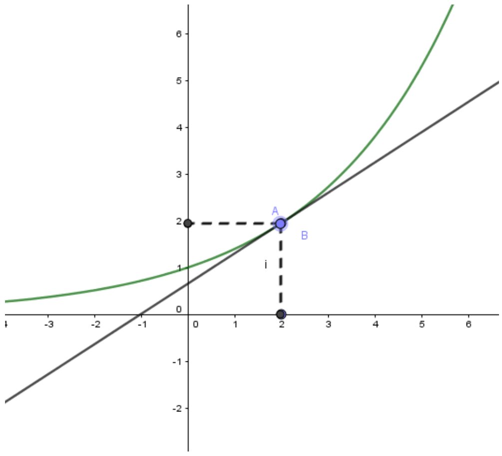
The limit of the difference quotient as Δx approaches zero, if it exists, should represent the slope of the tangent line to $(x_0, f(x_0))$. This limit is defined to be the derivative of the function f at x_0:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Geometrically:



Initially, the line who passes for A and B is secant to the curve. When $\Delta x \to 0$, the point B goes over A and the line that was secant becomes tangent to the curve in A.



So, the derivative in the x_A is the angular coefficient of the line tangent to the curve in A.

If the limit of the difference quotient does not exist $f(x)$ is said to be not not differential.

If a function is differentiable then it continous the opposite it is not true.

A function is differentiable at every point in its domain we say that it is a differentiable (smooth) function.

Rule of derivative

- $(x^k)' = kx^{k-1}$
- $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$
- $(kf)'(x_0) = kf'(x_0)$
- $(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$
- $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x0) \cdot g(x0) - f(x0) \cdot g'(x0)}{g(x)^2}$
- $f(g(x_0))' = f'(g(x_0)) \cdot g'(x_0)$

Function	Derivative
$f(x) = k$	$f'(x) = 0$
$f(x) = x$	$f'(x) = 1$
$f(x) = x^{\alpha}, \alpha \in \mathrm{R}$	$f'(x) = \alpha x^{\alpha-1}$
$f(x) = \alpha^x$	$f'(x) = \alpha^x \ln(\alpha)$
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = \log_{\alpha}(x)$	$f'(x) = \frac{1}{x \log a}$
$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$
$f(x) = sin(x)$	$f'(x) = cos(x)$
$f(x) = cos(x)$	$f'(x) = -sin(x)$
$f(x) = cot(x)$	$f'(x) = -\frac{1}{sen^2 x}$
$f(x) = tan(x)$	$f'(x) = \frac{1}{cos^2 x}$

$f(x) = arcsin(x)$	$f'(x) = \frac{1}{\sqrt{1-x^2}}$
$f(x) = arccos(x)$	$f'(x) = -\frac{1}{\sqrt{1-x^2}}$
$f(x) = arctan(x)$	$f'(x) = \frac{1}{1+x^2}$
$f(x) = arccot(x)$	$f'(x) = -\frac{1}{1+x^2}$

3.6. Higher order derivative

If $f(x)$ be a differentiable function, and $f'(x)$ its derivative. If $f'(x)$ is again differentiable, $D[f'(x)] = f''(x)$ is its second derivative. If we can do the same with the second derivative we will have the third derivative, and so on and so forth. Generally, we can talk about **higher order derivatives**, defined as $f^{(n)}(x)$.

3.7. Max and min

Let $f : X \rightarrow \mathrm{R}$ be a function, and x_0 a point in the domain in X of $f(x)$. We can state that

- x_0 is a minimum if $\exists\, I_{x_0}$ s.t.

$f(x) > f(x_0)\ \ \forall x \in I_{x_0} \backslash \{x_0\}$

- x_0 is a maximum if $\exists\, I_{x_0}$ s.t.

$f(x) < f(x_0)\ \ \forall x \in I_{x_0} \backslash \{x_0\}$

- If x_0 is a max/min, then $f'(x_0) = 0$.
- If $f'(x) > 0\ \ \forall x \in (a, b)$, then $f(x)$ is increasing in (a, b)
- If $f'(x) < 0\ \ \forall x \in (a, b)$, then $f(x)$ is decreasing in (a, b)

Example of a function study.

It is given the function $y = \frac{1}{3}x^3 - 2x^2 + 3x$.

It is a polynomial function, so its domain is R.

We can continue to study this function. Where is this function positive?

Let's study the sign of our function: $\frac{1}{3}x^3 - 2x^2 + 3x \geq 0$

$x\left(\frac{1}{3}x^2 - 2x + 3\right) \geq 0$

We have two factors:

$f_1 = x$

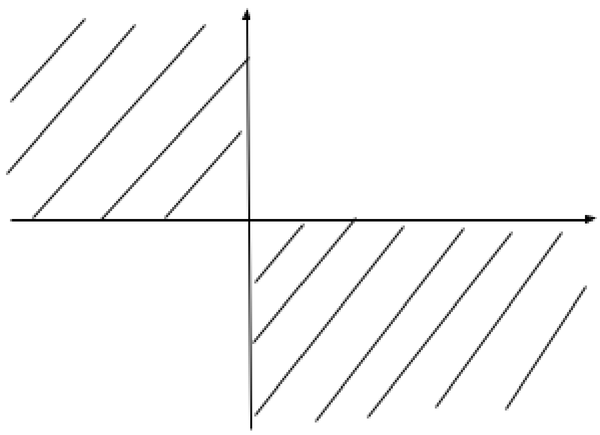
$f_2 = \frac{1}{3}x^2 - 2x + 3$

$f_1 \geq 0$ if $x \geq 0$

$f_2 \geq 0$ if $\frac{1}{3}x^2 - 2x + 3 \geq 0$. We can study second degree inequality.

$\Delta = 0$, so $f_2 \geq 0$ always.

The function is positive for $x \geq 0$.

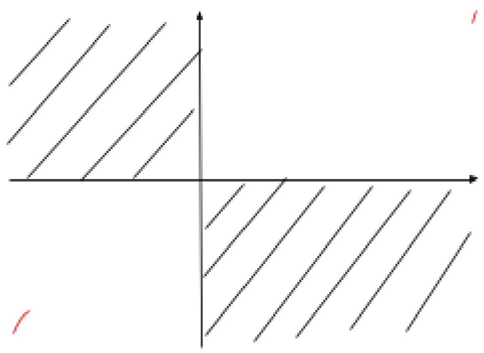


The function meets both the axis x and y in the Origin.

Now, we'll calculate limits for $x \to \pm\infty$, because its domain is R.

$$\lim_{x \to +\infty} \frac{1}{3}x^3 - 2x^2 + 3x = \lim_{x \to +\infty} x^3\left(\frac{1}{3} - \frac{2}{x} + \frac{3}{x^2}\right) = +\infty$$

$$\lim_{x \to -\infty} \frac{1}{3}x^3 - 2x^2 + 3x = \lim_{x \to -\infty} x^3\left(\frac{1}{3} - \frac{2}{x} + \frac{3}{x^2}\right) = -\infty$$

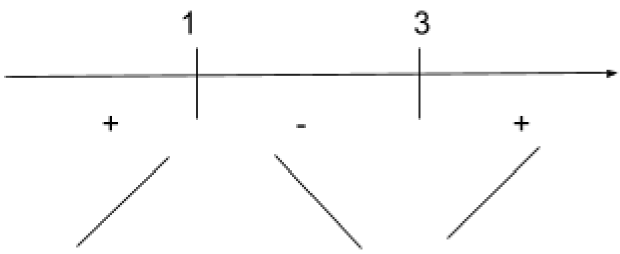


Now, we'll calculate the first derivative of our function.

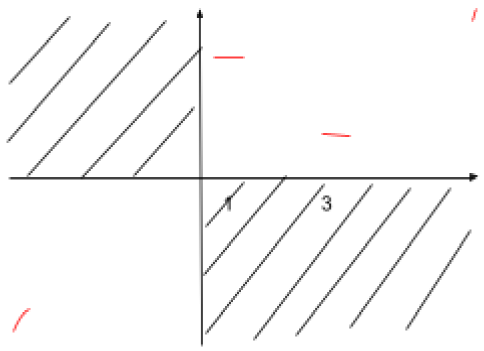
$f'(x) = x^2 - 4x + 3.$

We'll study the sign of the new function to see if it has maximum or minimum points.

$x^2 - 4x + 3 \geq 0$ for $x \leq 1 \vee x \geq 3$, that we are sign on the following graph:



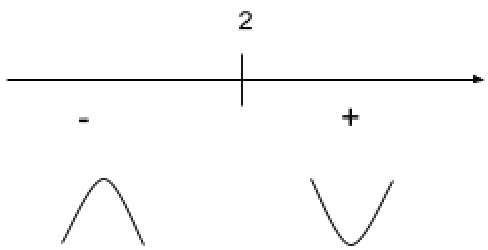
In $x = 1$ there is a maximum point, in $x = 3$ a minimum.



If we'll study $f''(x)$, we'll have information about the concavity function.

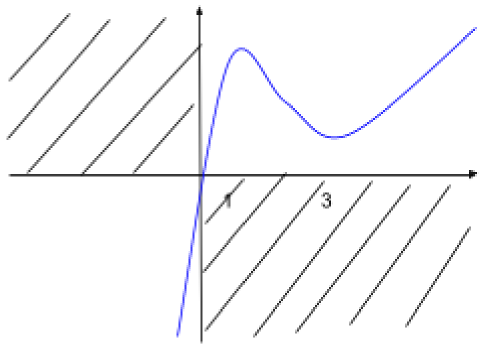
$f''(x) = 2x - 4$, then we'll study its sign.

$2x - 4 \geq 0$ if $x \geq 2$.



For $x < 2$, the concavity is downwards, for $x > 2$ upwards.

The probable graph of our function is the following:


1
3

4. Integral

The **indefinite integral** represents a class of functions (antiderivative) whose derivative is the integrand.

Definition 4.1. Let $f(x)$ be a function in (a, b). If a function $F(x)$ exists and it is continuous in $[a, b]$ and it is also differentiable in (a, b) such that

$$F'(x) = f(x) \quad \forall x \in (a, b)$$

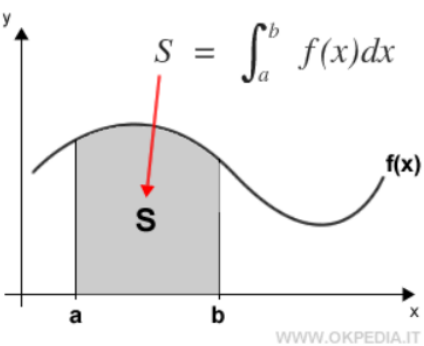
$F(x)$ is said to be the antiderivative of $f(x)$.

If $F(x)$ is an antiderivative function of $f(x)$, also $G(x) = F(x)+c$, $c \in \mathrm{R}$ is an antiderivative of $f(x)$, given that

$$G'(x) = D[F(x) + c] = F'(x) = f(x)$$

Definition 4.2. Let $f(x)$ be a function in (a, b) and let it have antiderivatives. All the antiderivatives of $f(x)$ are defined as $\int f(x)\,dx = F(x)$, $f(x)$ is said to be the integrand.

The **definite integral** is $\int_a^b f(x)dx = F(b) - F(a)$, it is used to calculate the area between a curve and x axis.



As the derivative, the integral is a **linear operator**

- $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)\, dx$
- $\int kf(x)dx = k\int f(x)dx$

Rules of Integration

$\int 0 \cdot dx = c$	
$\int dx = x + c$	$\int k \cdot f(x) = k \cdot \int f(x)dx$
$\int x^n dx = \frac{x^{n+1}}{n+1} + c, (n \neq -1)$	$\int [f(x)]^n \cdot f'(x)dx = \frac{1}{n+1}[f(x)]^{n+1} + c$
$\int \frac{dx}{2\sqrt{x}} dx = \sqrt{x} + c$	$\int \frac{f'(x)}{2\sqrt{f(x)}} dx = \sqrt{f(x)} + c$
$\int sin x\, dx = -\cos x + c$	$\int \operatorname{sen} f(x) \cdot f'(x)dx = -\cos f(x) + c$
$\int \cos x\, dx = \operatorname{sen} x + c$	$\int \cos f(x) \cdot f'(x)dx = \operatorname{sen} f(x) + c$
$\int \frac{1}{\cos^2 x} dx = tgx + c$	$\int \frac{f'(x)}{\cos^2 f(x)} dx = tg\, f(x) + c$
$\int \frac{1}{\operatorname{sen}^2 x} dx = -ctgx + c$	$\int \frac{f'(x)}{\operatorname{sen}^2 f(x)} dx = -ctg\, f(x) + c$
$\int \frac{dx}{\sqrt{1-x^2}} = arcsin x + c$	$\int \frac{f'(x)}{\sqrt{1-[f(x)]^2}} dx = arcsin\, f(x) + c$
$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + c$	$\int \frac{f'(x)}{1+[f(x)]^2} dx = \operatorname{arctg} f(x) + c$
$\int \frac{dx}{x} = \ln\lvert x \rvert + c$	$\int \frac{f'(x)}{f(x)} dx = \ln\lvert f(x) \rvert + c$
$\int e^x dx = e^x + c$	$\int e^{f(x)} f'(x)dx = e^{f(x)} + c$
$\int a^x dx = \frac{a^x}{\ln a} + c$	$\int a^{f(x)} f'(x)dx = \frac{a^{f(x)}}{\ln a} + c$

ex. $\int (3x^2 - 4e^x + \frac{1}{2x})dx = 3\int x^2 dx - 4\int e^x dx + \frac{1}{2}\int \frac{1}{x}\, dx = 3 \cdot \frac{x^3}{3} - 4\, e^x + \frac{1}{2}\, ln\, x$

So, the solution is $x^3 - 4\, e^x + \frac{1}{2}\, ln\, x + c$.

The **integration by part** rule is presented here:

$\int f'(x)g(x)\, dx = f(x)g(x) - \int f(x)g'(x)\, dx + c,\ c \in R$

ex. We resolve the following integral, with $f'(x) = sinx$ e $g(x) = x$

$$\int xsinx\,dx = -xcosx - \int 1 \cdot (-cosx)\,dx = -xcosx + \int cosx\,dx = -xcosx + sinx + c$$

The is another integration method, with a substitution, but we don’t consider.

5. Linear system and matrix algebra

The analysis of many economic models reduces to the study of systems of equations. There are essentially three ways of solving systems of linear equations: substitution, elimination of variables, and matrix methods.

- Substitution of variables methods: substitution is simply made by writing one variable in terms of other(s) using an equation and substituting this relation into the other equation(s);

- Elimination of variables is generally more conducive to the theoretical analysis. It is done by multiplying equations and adding them up such that eliminating unknown(s) to solve the equation with less unknowns. This is called Gauss elimination.

We will look at the simplest possible system of equations - linear systems, using matrix methods.

Definition 5.1. Generally, an equation is said to be linear if it has the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b \qquad (55)$$

where $a_1, ..., a_n$ are **parameters** and $x_1, ..., x_n$ are **variables**.

We can solve linear systems of equations using **matrices**.

We can generate $\mathbb{R}^n$ by n-ary Cartesian product of $\mathbb{R}$. Therefore, we can represent $\mathbb{R}^2$ and $\mathbb{R}^3$ though arrows which connect the origin of the cartesian plane to the coordinates given by the pair (x_1, x_2) or (x_1, x_2, x_3)

Definition 5.2. Let $\underline{v}_1 = (x_1, ..., x_n)$ and $\underline{v}_2 = (y_1, ..., y_n)$ be two vectors, their are said to be equal if $x_i = y_i$, $\forall\, i = 1, 2, .., N$.

Definition 5.3. Let $\underline{v}_1 = (x_1, ..., x_n)$ and $\underline{v}_2 = (y_1, ..., y_n)$ be two vectors, $\underline{v}_3 = (x_1 + y_1, ..., x_n +$

$y_n)$ is the sum of former vectors

ex. In $\mathbb{R}^3$, $\underline{v}_1 = (1,\ 2,\ -1)$ and $\underline{v}_2 = (-1,\ 0,\ 3)$. $\underline{v}_3 = (0,\ 2,\ 2)$

We can multiply a vector by a scalar, and we will obtain $c\underline{v}_1 = (cx_1, ..., cx_n)$

ex. If we have $\underline{v}_3 = (0,\ 2,\ 2)$, it is $2\underline{v}_4$, with $\underline{v}_4 = (0,\ 1,\ 1)$

Given m vectors $\underline{v}_1, \underline{v}_2, ..., \underline{v}_m$ of $\mathbf{R}^n$ and m scalars $c_1, c_2, ..., c_m$ in **R**, a **linear combination** of the vectors $\underline{v}_1, \underline{v}_2, ..., \underline{v}_m$ with coefficients $c_1, c_2, ..., c_m$ the vector given by

$$v_1c_1+v_2c_2+...+v_mc_m = \sum_{i=1}^{m} v_ic$$

Definition 5.4. A sequence of vectors $(\underline{v}_1, \underline{v}_2, . . . , \underline{v}_k)$ from a vector space V is said to be linearly dependent, if there exist scalars $a_1, a_2, . . . , a_k$, not all zero, such that

$a_1\underline{v}_1 + a_2\underline{v}_2 + \cdot \quad \cdot \quad \cdot + a_k\underline{v}_k = 0$

ex. the vector $\underline{v}_1, \underline{v}_2, \underline{v}_3$ of the last example are linearly dependent because for the scalars 1, 1, -1 we have that $\underline{v}_1+\underline{v}_2-\underline{v}_3=0$

On the other hand, two vectors are said to be linearly independent if there is no linear combination which gives as a result the null vector except for the one with null coefficients.

5.5. Matrix algebra

A **matrix** is a rectangular array of numbers. The size of a matrix is indicated by the number of its rows and number of its columns. A matrix with m rows and n columns is called a mxn matrix. The element in row i and column j is called the $(i, j)th$ entry, and it is often written as a_{ij}.

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

A matrix with the number of columns equal to the number of rows is called **square matrix**.
ex. in R^2

$$A = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$$

in R^3

$$B = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix}$$

5.6. Operations

Let A and B two matrices mxn, their **sum** is the matrix *C* whose elements are $c_{ij} = a_{ij} + b_{ij}$ $\forall\, i,$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & b_{ij} & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & a_{ij} + b_{ij} & \vdots \\ a_{k1} + b_{k1} & \cdots & a_{kn} + b_{kn} \end{pmatrix}.$$

ex.

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 2 & -2 & -1 \\ 3 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix} =$$

$$\begin{pmatrix} 3 & 1 & -1 \\ 3 & -1 & 1 \\ -1 & 3 & 2 \end{pmatrix}$$

Matrices may be multiplied by scalars. This operation is called **scalar multiplication**.

More generally, the product of the matrix *A* and the number *r*, denoted *rA*, is the matrix whose elements are ra_{ij} $\forall\, i, j$

$$r \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} = \begin{pmatrix} ra_{11} & \cdots & ra_{1n} \\ \vdots & ra_{ij} & \vdots \\ ra_{k1} & \cdots & ra_{kn} \end{pmatrix}.$$

ex.

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & -2 & 1 \\ 0 & 3 & -4 \end{pmatrix} \qquad 3A = \begin{pmatrix} 6 & 0 & 3 \\ -3 & -6 & 3 \\ 0 & 9 & -12 \end{pmatrix}$$

We can define the **matrix product** *AB* if number of columns of A = number of rows of B

To obtain the *(i, j)th* entry of AB, multiply the *ith* row of A and the *jth* column of B as follows

$$[a \ \ b \ \ c] \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} =$$

$$= [a \cdot d + b \cdot e + c \cdot f]$$

ex.

$$(-1, 3, 2)\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} = -1 \cdot 2 + 3 \cdot (-1) + 2(-3)$$

$$= -2 - 3 - 6 = -11$$

If A is a kxm and B is mxn, then the product $C = AB$ will be kxn. Usually, $AB \neq BA$.

ex.

$$A = \begin{pmatrix} -3 & 1 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} -3 \cdot 1 + 1 \cdot (-1) & -3 \cdot 0 + 1 \cdot 2 \\ 0 \cdot 1 + 2 \cdot (-1) & 0 \cdot 0 + 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ -2 & 4 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 1 \cdot (-3) + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot 2 \\ -1 \cdot (-3) + 2 \cdot 0 & -1 \cdot 1 + 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 3 & 3 \end{pmatrix}$$

The nxn matrix

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

with $a_{ii} = 1$, $\forall i$ and $a_{ij} = 0$, $\forall i \,/\!= j$, has the following property $AI = A$, for any mxn matrix A. I is called **identity matrix**.

Definition 5.7. The **transpose** of a kxn matrix A is the nxk matrix obtained by interchanging the rows and the columns.

$A = (a_{mn})$ $\qquad A^T = (a_{nm})$

ex.

$$A = \begin{pmatrix} 5 & 1 & 0 \\ 2 & -2 & 1 \\ 0 & 3 & -1 \end{pmatrix} \quad A^T = \begin{pmatrix} 5 & 2 & 0 \\ 1 & -2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

Definition 5.8. A **Triangular matrix** is a square matrix containing elements different from zero only above/below the main diagonal. In the first case, we have a Upper triangular matrix,

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & a_{2,n} \\ 0 & 0 & a_{3,3} & a_{3,n} \\ 0 & 0 & 0 & a_{m,n} \end{pmatrix}$$

in the second a Lower triangular matrix.

$$\begin{pmatrix} a_{1,1} & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{m,1} & a_{m,2} & a_{m,3} & a_{m,n} \end{pmatrix}$$

Definition 5.9. Only square matrix is associated a **determinant**, $|A|$. From a geometric point of view, it represents the area (volume) of the parallelogram generated by the vectors of the matrix.

Given *A,* 2*x*2 matrix, we have that

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

ex.

$$A = \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} \qquad \det A = 2 \cdot 2 - (1)(-3) = 4 + 3 = 7$$

In the case of a *nxn* matrix with *n* > 3, in order to find the determinant, we can use the **minor** of the matrix. To calculate the determinant of a square matrix, with n>3, we'll consider a row i, or a column j: $det\ A\ =\ \sum_{i,j=1}^{n} a_{ij} (-\ 1)^{i+j} det\ Aij$, where Aij is the matrix obtained from A deleting the i-th row and the j-th column.

ex.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{respect the first row}$$

$$\det A = 1 \cdot (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 2 \cdot (-1)^{1+2} \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} + 3 \cdot (-1)^{1+3} \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 1 \cdot 1 - 2 \cdot (-1) + 3(-1) =$$

$$= 1 + 2 - 3 = 0$$

Respect another row or column, the result is the same.

Definition 5.10. A minor of a matrix A is the determinant of some smaller square matrix, cut down from A by removing one or more of its rows and columns. Minors obtained by removing just one row and one column from square matrices (first minors) are required for calculating matrix cofactors, which in turn are useful for computing both the determinant and inverse of square matrices.

The cofactor is:

$A_{ik} = (-1)^{i+k} M_{ik}$

where M_{ik} is a minor of the matrix.

Therefore, using the **Laplace theorem** we can obtain the determinant of a matrix *nxn* as the sum of the product, of any row or column, by their cofactor.

we note three algebraic proprietes of the determinant function, giver two matrix *A* e *B:*

- $\det A = \det A^T$
- $\det AB = \det A \cdot \det B$
- $\det (A+B) \neq \det A + \det B.$

The **rank** of A is the largest order of any non-zero minor in A.

Definition 5.11. Let *A* be a *nxn* matrix. The *nxn* matrix A^{-1} is an **inverse** for A if $AA^{-1} =$

$A^{-1}A = I_n$

A matrix can have at most one inverse

Not every matrix is invertible. In order to be invertible, $\det |A| \neq 0$

We have that

$A^{-1} = \frac{1}{\det |A|} \cdot B^T,$

where *B* is the added matrix of A.

B has the algebraic complements A_{ij} at the place a_{ij}.

The algebraic complements A_{ij} is:

$A_{ij} = (-1)^{i+j} \cdot \det \overline{A}_{ij}$

$\overline{A}_{ij}$ is obtained from matrix A deleting the i-th row and the j-th column.

ex.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \qquad \det A = 2$$

$$A^{-1} = \frac{1}{2} B^T$$

$$B = \begin{pmatrix} -1 & -3 & 2 \\ 1 & 1 & 0 \\ -1 & -1 & 2 \end{pmatrix} \qquad B^T = \begin{pmatrix} -1 & 1 & -1 \\ -3 & 1 & -1 \\ 2 & 0 & 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \end{pmatrix}$$

In fact, $A^{-1} \cdot A = I$

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5.12. Linear system of equations

As we have stated above, we can use matrix to solve linear systems of equations.

Recall: Generally, an equation is said to be linear if it has the form

$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$

where $a_1, \ldots, a_n$ are **parameters** and $x_1, \ldots, x_n$ are **variables**.

The solution of the linear equation is given by $(x_1, \ldots, x_n)$ which is substituted into the equation solving it.
If there are several linear equations which have to be true all together we talk about **Linear System of equations**

The solution of the system is given by $(x_1, x_2, \ldots, x_n)$ which solves all the equations contemporaneously.

The system above can be expressed much more compactly using matrix notations.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + + a_{mn}x_n = b_m.$$

Let *A* denote the coefficient matrix of the system:

$$A := \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

Also, let x

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and b

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then, the system of equations can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or as

$$A\mathbf{x} = \mathbf{b}$$

Then, if A is *nonsingular* (det$|A|$ = 0), we can solve the system as $\mathbf{x} = A^{-1}\mathbf{b}$. To solve a linear system of simultaneous equations we can also use **Cramer's rule**. If the matrix A is nonsingular, the linear system of system of n linear equations and n unknowns. Then the theorem states that in this case the system has a unique solution, whose individual values for the unknowns are given by:

$$x_i = \frac{det\,|Ai|}{det\,|A|} \qquad i = 1, \dots, n$$

where A_i is the matrix formed by replacing the i-th column of A by the column vector b.

ex.

$$\begin{cases} 3x - 2y = 1 \\ x + y - 2z = 0 \\ -x - y = -1 \end{cases} \qquad \begin{pmatrix} 3 & -2 & 0 \\ 1 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix} \qquad \det A = -10$$

$$A_x = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix} \quad A_y = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & -2 \\ -1 & -1 & 0 \end{pmatrix} \quad A_z = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

$$x = \frac{\det A_x}{\det A} = \frac{3}{5} \qquad y = \frac{\det A_y}{\det A} = \frac{2}{5} \qquad z = \frac{\det A_z}{\det A} = \frac{1}{2}$$

In Economics, we may be interested in system of the following form
$Ax = \lambda x$
where A is a square matrix.
It is equal to write:
$(A - \lambda I)x = 0$

Definition 5.13. The values λ that solve $det(A - \lambda I) = 0$ are called **eigenvalues**.

ex.

$$A = \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 3-\lambda & \sqrt{3} \\ \sqrt{3} & 1-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)(1-\lambda) - 3 = \cancel{3} - \lambda + \lambda^2 - 3\lambda - \cancel{3}$$
$$= \lambda^2 - 4\lambda = \lambda(\lambda - 4) = 0$$
$$\lambda_1 = 0 \qquad \lambda_2 = 4$$

Definition 5.14. While the non-trivial vectors, **x**, obtained as the solution of $(A - \lambda I)x = 0$ is called **eigenvector**.

ex.

for $\lambda_1 = 0$

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad \begin{cases} 3x_1 + \sqrt{3}\,x_2 = 0 \\ \sqrt{3}\,x_1 + x_2 = 0 \end{cases} \Rightarrow \underline{x} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

for $\lambda_2 = 4$

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad \begin{cases} -x_1 + \sqrt{3}\,x_2 = 0 \\ \sqrt{3}\,x_1 - 3x_2 = 0 \end{cases} \Rightarrow \underline{x} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

6. Function of several variables

As we have seen before, in Economics is interested looking what is the effect of a change in one variable with respect to another one. However, in most real cases, variables depend on several variables. We may be interested in $f : \mathrm{R}^n \rightarrow \mathrm{R}$
We can graph function up to R^3.

There is another way to visualize function from R^2 to R^1 which requires only two dimensional sketching: the **level curves**. They are given by

$L_c = (x, y) : f(x, y) = c$

It is like to slice the function in many pieces.

6.1. Calculus of several variable

When we deal with function of several variable, we are often interested in the partial variation - the variation brought about by the change in only one variable. We will talk, therefore, about **partial derivative**.

Definition 6.2. Let $f : \mathrm{R}^n \rightarrow \mathrm{R}$. Then for each variable x_i at each point $\mathbf{x}^0$ in the domain of

$$\frac{\partial}{\partial xi}\left(x_1^{\ 0}, \ldots, x_n^{\ 0}\right) = \lim_{h \to 0} \frac{f(x_1^{\ 0}, \ldots, x_i^{\ 0}+h, \ldots, x_n^{\ 0}) - f(x_1^{\ 0}, \ldots, x_i^{\ 0}, \ldots, x_n^{\ 0})}{h}$$

From a practical point of view, we can apply the same rules we apply to one variable function; we take the derivative with respect to one variable treating all the other variables as constant.

6.3. Higher-order derivatives

The partial derivative $\frac{\partial f}{\partial x_i}$ is itself a function, as we have seen in the case of the second derivative. When we take the first derivative with respect to a variable and the second one with respect to another variable we talk about **mixed partial derivatives.**

$$\frac{\partial^2 f}{\partial x_i x_j}$$

with i $\neq$ j.

We have that $\frac{\partial^2 f}{\partial x_i x_j}$ = $\frac{\partial^2 f}{\partial x_j x_i}$

=

If a function has n variables, then, it will have n^2 second order partial derivatives. It is common to arrange these n^2 partial derivatives into an nxn matrix whose (i, j)th entry is the $\frac{\partial^2 f}{\partial x_j x_i}$ (**x***). This matrix is called **Hessian matrix**:

$$\mathrm{H} f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\[2.2ex] \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\[2.2ex] \vdots & \vdots & \ddots & \vdots \\[2.2ex] \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

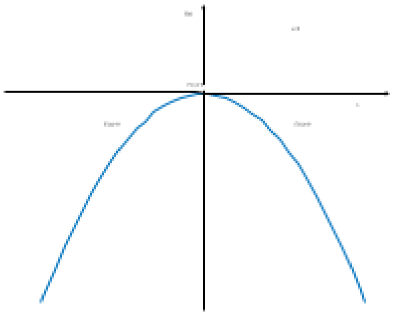
with $Hf_{ij} = \frac{\partial^2 f}{\partial x_i x_j}$.

The hessian matrix is often symmetric, that is, the elements $a_{ij} = a_{ji}$.

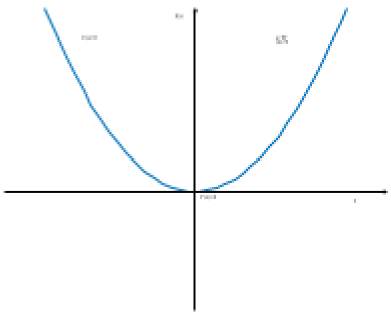
7. Optimization

We want to find the point (x, y) s.t. $f'(x) = 0$, this point is called **stationary point** (**critical point**).
A critical point may correspond to:

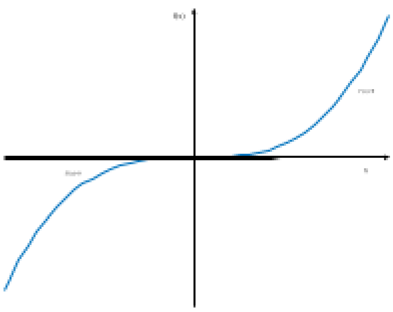
- **Maximum**



- **Minimum**



- **Saddle point**



Once we have found the stationary point with $f'(x) = 0$ (first order condition), we can check what kind of stationary point is looking at the second derivative (second order conditions)

- if $f''(x) < 0 \Rightarrow$ max
- if $f''(x) > 0 \Rightarrow$ min
- if $f''(x) = 0 \Rightarrow$ No clue. It can be max, min or saddle point in $x^\wedge$

we have a local max if $f(x^\wedge) \geq f(x)$, $\forall x$ in the of $x^\wedge$

7.1. Optimization with several variables

The **gradient vector** is the vector whose components are the partial derivatives of f.

$$\nabla f(p) = \left(\frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \ldots, \frac{\partial f}{\partial x_n}(p)\right)$$

The **Hessian Matrix** is the *nxn* matrix whose (*i, j*)th entry is the $\frac{\partial^2 f}{\partial x_j x_i}$ (**x***).

$$\mathrm{H} f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

In order to have a stationary point $\nabla f(p) = 0$ In order to attain the second order conditions:

- If the Hessian is positive definite (equivalently, has all eigenvalues positive) at a, then f attains a local minimum at a
- If the Hessian is negative definite (equivalently, has all eigenvalues negative) at a, then f attains a local maximum at a
- If the Hessian has both positive and negative eigenvalues then a is a saddle point for f (and in fact this is true even if a is degenerate).

In those cases not listed above, the test is inconclusive.

7.2. Constrained optimization

Let *f* and *h* be C^1 functions of two variables. Suppose that $\mathbf{x}^{\wedge} = (x^{\wedge}, x^{\wedge})$ is a solution of the problem

$$\text{maximize } f\ (x_1, x_2)$$

$$\text{subject to } h\ (x_1, x_2) = c$$

Suppose that (x_1^*, x_2^*) is not a critical point of *h*. Then, there is a real number μ^* s.t. (x_1^*, x_2^*, μ^*) is a critical point of the Lagrangian function

$L\ (x_1, x_2, \mu) \equiv f\ (x_1, x_2) - \mu\ [h\ (x_1, x_2) - c]$

In other words at (x_1^*, x_2^*, μ^*)

$\frac{\partial L}{\partial x_1} = 0$

$\frac{\partial L}{\partial x_2} = 0$

$\frac{\partial L}{\partial \mu} = 0$

Reference

Simon, Carl P., and Lawrence Blume. Mathematics for economists. Vol. 7. New York: Norton, 1994