Exercise sheet 2
Solutions

Statistics Pre-course 2023

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Exercise 1

You flip a coin three times.

(a) Represent the sample space and determine its cardinality.
Answer:

First Toss	Second Toss	Third Toss
T	T	T
T	T	H
T	H	T
T	H	H
H	T	T
H	T	H
H	H	T
H	H	H

There are 2^3 possible outcomes.

(b) Compute probability of the following events: (i) obtain at least 2 heads; (ii) obtain at least one tail.
Answer:
(i) 4/8
(ii) 7/8

Exercise 2

Two fair dice are rolled.

(a) Construct the sample space and determine its cardinality.
Answer:

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are $6^2 = 36$ possible outcomes.

(b) Find the probability of rolling a sum of 7.
Answer: $6/36 = 1/6 \approx 0.1\bar{6}$.

(c) Find the probability of getting a total of at least 10.
Answer: $6/36 = 1/6 \approx 0.1\bar{6}$.

(d) Find the probability of getting an odd number as sum.
Answer: $18/36 = 0.5$.

Exercise 3

A sample of 1000 subjects is interviewed to assess whether they are happy in their marriages or not. The following table summarizes the survey:

Gender	Very Happy	Pretty Happy	Not Too Happy	Total
Male	184	235	41	**460**
Female	232	265	43	**540**

(a) Compute the probability that a married adult is Very Happy (VH).
Answer: $\mathbb{P}(\text{VH}) = (184 + 232)/1000 = 0.42$

(b) Compute the probability that a married adult is very happy given: (i) that their gender is male; (ii) that their gender is female.
Answer:
(i) $\mathbb{P}(\text{VH} \mid \text{M}) = \mathbb{P}(\text{VH} \cap \text{M})/\mathbb{P}(\text{M}) = (184/1000)/(460/1000) = 0.4$
(ii) $\mathbb{P}(\text{VH} \mid \text{F}) = \mathbb{P}(\text{VH} \cap \text{F})/\mathbb{P}(\text{F}) = (232/1000)/(540/1000) = 0.43$

(c) Are the events being Very Happy and Male independent?
Answer: No, since both $\mathbb{P}(\text{VH} \mid \text{M})$ and $\mathbb{P}(\text{VH} \mid \text{F})$ are different from $\mathbb{P}(\text{VH})$

Exercise 4

A population is composed of 2000 American children. It has been asked them questions about their participation in organized and team sports. Among them, 52.7% were boys and 47.3% were girls. Of the boys, 73% currently participate in sports. Of the girls, 63% currently participate in sports. Gender and sports participation are not independent events.

(a) Compute the probability of randomly selecting an individual from this population who is a girl and currently participate in sports.
Answer: $\mathbb{P}(\mathrm{F} \cap \mathrm{S}) = \mathbb{P}(\mathrm{S} \mid \mathrm{F}) \times \mathbb{P}(\mathrm{F}) = 0.63 \times 0.473 = 0.30$

(b) What is the probability that a randomly selected individual is a boy who does not currently participate in sports?
Answer: $\mathbb{P}(\mathrm{M} \cap \mathrm{S}^c) = \mathbb{P}(\mathrm{S}^c \mid \mathrm{M}) \times \mathbb{P}(\mathrm{M}) = (1 - 0.73) \times 0.527 = 0.142$

Exercise 5

A wheat farmer living in Pennsylvania finds that his annual profit is \$ 80 if the summer weather is typical, \$ 50 if the weather is unusually dry, and \$ 20 if there is a severe storm that destroys much of his crop. Weather bureau records indicate that the probability is 0.70 of typical weather, 0.20 of unusually dry weather, and 0.10 of a severe storm. Let X be the farmer's profit.

(a) Construct the probability distribution of X.
Answer:

x_i	$p(x_i)$
80	0.70
50	0.20
20	0.10

(b) Find the expected value $E(X)$.
Answer: $E(X) = 80 \times 0.70 + 50 \times 0.20 + 20 \times 0.10 = 68$.

(c) Find the variance $\mathcal{V}(X)$.
Answer:

x_i	$p(x_i)$	$x_i - E(X)$	$p(x_i) * [x_i - E(X)]^2$
80	0.70	12	100.8
50	0.20	-18	64.8
20	0.10	-48	230.4

Exercise 6

Let X be a continuous random variable with probability density function given by

$$f_X(x) = \frac{1}{2} e^{-|x|} \quad \forall\, x \in \mathbb{R}$$

If $Y = X^2$ find $F_Y(y)$.
Answer: First, note that the support $\mathcal{Y}$ of the random variable $Y = X^2$ is $\mathcal{Y} = [0, \infty)$. Then:

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) = \mathbb{P}(X^2 leqy) \\ &= \mathbb{P}(X \leq \sqrt{y}) = \mathbb{P}(-\sqrt{y} < X < \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} dx \;\; \frac{1}{2} e^{-|x|} = \int_0^{\sqrt{y}} \;\; \frac{1}{2} e^{-x} \, dx \\ &= 1 - e^{-\sqrt{y}} \;\; \forall \, y \geq 0 \end{aligned}$$

Exercise 7

Each newborn baby has a probability of approximately 0.49 of being female and 0.51 of being male. For a family of four children, let X = number of children who are girls.

(a) Identify and describe the random variable X.
Answer: This a Binomial random variable. Repeated trials (4 children), each with the same probability of success (0.49) and trials are independent.

(b) Is this a parametric family? Characterise the probability law.
Answer: Yes. The probability law is characterised by two parameters $n = 4$ and $p = 0.49$.

(c) Compute mean and variance of X.
Answer: $E(X) = np = 4 * 0.49 = 1.96$; $\mathcal{V}(X) = np(1-p) = 0.996$

(d) Find the probability that the family has 2 boys and 2 girls.
Answer: $\mathbb{P}(X = 2) = \frac{4!}{2!2!}(0.49)^2(0.51)^2 = 0.3747$.

Exercise 8

A mid-term exam in a statistics course has four multiple-choice questions, each with five possible answers. A passing grade is three or more correct answers to the four questions. Victoria has not studied for the exam. She has no idea of the correct answer to any of the questions and decides to guess at random for each.

(a) Find the probability that she answers all four questions correctly.
Answer: We have $n = 4$ and $p = 0.2$ so that
$\mathbb{P}(X = 4)\frac{4!}{4!0!}(0.2)^4(0.8)^0 = 0.0016$.

(b) Find the probability that she passes the exam.
Answer:

$$\begin{aligned} \mathbb{P}(X \geq 3) &= \mathbb{P}(X = 3) + \mathbb{P}(X = 4) = \\ &= \frac{4!}{3!1!}(0.2)^3(0.8)^1 + \frac{4!}{4!0!}(0.2)^4(0.8)^0 = 0.0256 + 0.0016 = 0.0272. \end{aligned}$$