# University of Rome
# “Tor Vergata”

## Mathematics

### MSC in Economics – MSC in Finance & Banking

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1**. Website**

http://economia.uniroma2.it/master-science/economics/corso/211/

2. **Textbooks**

The textbooks used in the course are:

- “Mathematics for Economists” by C. Simon and L. Blume, Norton & Company;
- “Statistical Inference” by G. Casella and R.L. Berger, Duxbury.

3. **Entrance qualifications**

It is taken for granted that students have a basic knowledge of calculus and linear algebra. In particular they know: how to study a function in one variable, the fundamental theorem of calculus, how to evaluate a definite integral, how to study a system of linear equations, the basic geometry of three-dimensional space.

As a reference one can use the Appendices A1, A2, A3, A4 and Part I – Part II of the book by Simon-Blume.

4. **Course content**

The course has four parts: Calculus, Linear Algebra, Optimization and Probability.
The main goals of the course are the study of:

- integration in several variables (Fubini, change of variable formula, polar coordinates, …);
- linear transformations, eigenvalues, eigenvectors, projections and the spectral theorem;
- unconstrained and constrained optimization (Taylor formula in several variables, Kuhn-Tucker);
- limit theorems in probability and conditional expectation (weak law of large numbers, central limit theorem, multivariate gaussian).

The detailed program is available in the website of the course.

5. **Learning outcomes**

Upon completion of the course the student will have the mathematic background to understand the notions required in Statistics, Econometrics and in the other parts of Economics and Finance where a quantitative approach is needed.

6. **Teaching material**

In the website of the course you find all the needed teaching material. In particular you have:

- the detailed program;

- the arguments of the lectures on a day-by-day basis;

- the practice sessions;

- simulations of the written examination containing all the rules;

- a list of other textbooks and suggested readings;

…

- and much more.

7. **Additional information**

Teaching will be in English throughout the course.

# Sets, Numbers, and Proofs

The basic ingredients of mathematics are sets, numbers, and functions. This appendix begins with a presentation of the vocabulary of sets and of numbers and concludes with a discussion about mathematical proofs.

## A1.1 SETS

### Vocabulary of Sets

A **set** is any well-specified collection of elements. A set may contain finitely many or infinitely many elements, but the criterion for membership in the set must be well understood. For any set $A$, we write $a \in A$ to indicate that $a$ is a member of set $A$, and $a \notin A$ to indicate that $a$ is not in the set $A$. The most commonly used set in this book is the set $\mathbf{R}$ of all real numbers.

We sometimes encounter the set which *contains no elements*. It is called the **empty set** or **null set** and is denoted by $\varnothing$.

We will use standard notation for defining sets. For example, the set of all nonnegative numbers is written as

$$\mathbf{R}_+ \equiv \{x \in \mathbf{R} : x \geq 0\}.$$

Since every element of $\mathbf{R}_+$ is an element of $\mathbf{R}$, we say that $\mathbf{R}_+$ is a **subset** of $\mathbf{R}$, and write $\mathbf{R}_+ \subset \mathbf{R}$ or $\mathbf{R} \supset \mathbf{R}_+$. Sometimes, a set is defined simply by listing its elements: $A = \{1, 2, 3\}$, or even $\mathbf{N} = \{1, 2, 3, \ldots\}$, provided the ellipsis $(\ldots)$ is well understood.

### Operations with Sets

Given two sets $A$ and $B$, new sets can be formed through the following set operations on $A$ and $B$:

(1) $A \cup B$, spoken "$A$ union $B$," is the set of all elements that are either in $A$ or in $B$ (or in both):

$$A \cup B \equiv \{x : x \in A \text{ or } x \in B\};$$

(2) $A \cap B$, spoken "$A$ intersect $B$," is the set of all elements that are common to both $A$ and $B$:

$$A \cap B \equiv \{x : x \in A \text{ and } x \in B\};$$

(3) $A - B$, or sometimes $A \setminus B$, spoken "$A$ minus $B$," is the set of all elements of $A$ that are not in $B$:

$$A - B \equiv \{x : x \in A \text{ and } x \notin B\}.$$

If it is clear that all sets under discussion are subsets of some (universal) set $U$, $U - A$ is often written as $A^c$, and called the **complement** of $A$ (in $U$). For example, if all sets under discussion are sets of real numbers, then the complement of $\mathbf{R}_+$, $(\mathbf{R}_+)^c$, is the set of all negative numbers, $\{x \in \mathbf{R} : x < 0\}$.

## A1.2 NUMBERS

### Vocabulary

Nearly all the sets discussed in this text are sets of numbers. The most basic numbers are the **counting numbers** $\{1, 2, 3, \ldots\}$, also called the **natural numbers**. The set of natural numbers is usually denoted by $\mathbf{N}$:

$$\mathbf{N} = \{1, 2, 3, 4, \ldots\}.$$

The sum or product of two natural numbers is another natural number, but the difference of two natural numbers need not be in $\mathbf{N}$. For example, $3 - 5 \notin \mathbf{N}$. If $\mathbf{N}$ is augmented by the number zero and by the negatives of the natural numbers, the resulting set is the set of **integers**, and often denoted by

$$\mathbf{Z} = \{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\}.$$

The sum, *difference*, and product of two integers in another integer, but the *quotient* of two integers is usually not an integer. So, the next natural extension is to the set $\mathbf{Q}$ of all *quotients* of integers:

$$\mathbf{Q} \equiv \left\{\frac{a}{b} : a, b \in \mathbf{Z};\ b \neq 0\right\}.$$

This set $\mathbf{Q}$ is called the set of **rational numbers**, since it is formed by taking *ratios* of integers. The set of rational numbers has the desired property that if $a$ and $b$ are elements of $\mathbf{Q}$, then so are $a + b$, $a - b$, $a \cdot b$, and $a/b$. (We always rule out division by 0.)

Can every number be written as the quotient of two integers? In other words, is every number a rational number? Although it is not readily apparent, some important numbers, like $\sqrt{2}$, $e$, and $\pi$, cannot be written as quotients of integers. The proof that $\sqrt{2}$ cannot be written as a quotient of integers will be presented later in this appendix. Numbers that cannot be written as ratios or quotients of integers are called **irrational numbers.**

Rational numbers can also be distinguished from irrational numbers by their decimal expansions. Numbers whose decimal expansions terminate after a finite number of digits (like 0.25 or 3.12345) or repeat the same pattern with perfect regularity from some point on (like $1.33333\cdots$ or $3.256256256\cdots$) are rational numbers. On the other hand, numbers whose decimal expansions never end and have no repeating pattern are irrational numbers. Since any number that is not rational is irrational, the set of all rational and irrational numbers is the set of all numbers **R**.

As we will see in Appendix A3, in order to solve certain polynomial equations that are needed to model oscillatory phenomena, mathematicians have expanded the set of numbers **R** to include "imaginary numbers"—numbers whose squares are negative numbers. To distinguish the set of numbers we are considering from this expanded number system, the set **R** of rational and irrational numbers is called the set of **real numbers.**

In the remainder of this appendix certain subsets of integers play an important role. This is a natural place to present their formal definitions.

**Definition** An integer $n$ is called an **even number** if there is an *integer* $m$ such that $n = 2m$. An integer that is not even is called an **odd integer.**

**Definition** A natural number $m$ is called a **prime number** if whenever $m$ can be written as the product $m = a \cdot b$ of two natural numbers, then $a = 1$ or $b = 1$. The first six prime numbers are 1, 2, 3, 5, 7, and 11.

## Properties of Addition and Multiplication

An important feature about numbers is that we can operate on them via addition and multiplication—and their inverse operations, subtraction and division—to obtain other numbers.

The operations of addition (+) and multiplication (·) on pairs of real numbers are characterized by the following properties:

(1) **(Closure)** If $a$ and $b$ are in **R**, so are $a + b$ and $a \cdot b$.
(2) **(Commutative)** For any $a, b \in \mathbf{R}$, $a + b = b + a$ and $a \cdot b = b \cdot a$.
(3) **(Associative)** If $a, b, c \in \mathbf{R}$, $(a+b)+c = a+(b+c)$ and $(a\cdot b)\cdot c = a\cdot(b\cdot c)$.
(4) **(Identity)** There is an element $0 \in \mathbf{R}$ such that $a + 0 = a$ for all $a \in \mathbf{R}$. There is an element $1 \in \mathbf{R}$ such that $a \cdot 1 = a$ for all $a \in \mathbf{R}$.
(5) **(Inverse)** For any $a \in \mathbf{R}$, there is an element $b \in \mathbf{R}$ such that $a + b = 0$; such a $b$ is usually written as $-a$. For any nonzero $a \in \mathbf{R}$, there is an element $c \in \mathbf{R}$ such that $a \cdot c = 1$; such a $c$ is usually written as $1/a$.
(6) **(Distributive)** For all $a, b, c \in \mathbf{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Each of the first five properties has an additive and a multiplicative component. The last property is the link between these two operations.

### Least Upper Bound Property

There are two more abstract properties of real numbers that arise a number of times throughout this book: the least upper bound property and the greatest lower bound property.

**Definition** Let $S$ be a subset of $\mathbf{R}$ and let $b \in \mathbf{R}$. Then, the number $b$ is called an **upper bound** for $S$ if $a \leq b$ for all $a \in S$; the number $b$ is called a **lower bound** for $S$ if $b \leq a$ for all $a \in S$.

**Definition** If $b$ is an upper bound for $S$ and no element smaller than $b$ is an upper bound for $S$, then $b$ is called a **least upper bound (lub)** for $S$. Similarly, if $b$ is a lower bound for $S$ and no number larger than $b$ is a lower bound of $S$, then $b$ is called a **greatest lower bound (glb)** for $S$.

We can now state the least upper bound and greatest lower bound properties on $\mathbf{R}$.

**(Least upper bound and greatest lower bound properties)** Let $S$ be any subset of $\mathbf{R}$. If $S$ has an upper bound, it has a least upper bound; if $S$ has a lower bound, it has a greatest lower bound.

*Example A1.1* Let $S$ be the set of all numbers of the form

$$S \equiv \{0.3, 0.33, 0.333, 0.3333, \ldots\}.$$

Of course, 0 is a lower bound and 1 is an upper bound for $S$. The least upper bound of $S$ is $1/3$. The greatest lower bound is 0.3, the first element in this increasing sequence of numbers. Notice that this set $S$ contains its glb, but not its lub.

#### EXERCISES

**A1.1** Let $A$ be the set of even integers, $B$ the set of odd integers, $C$ the set of integers from 1 to 10, and $D$ the set of nonnegative real numbers. Describe $C \cup A$, $C \cup B$, $C - B$, $A \cap D$, $B \cup D$, $A \cup B$, and $A \cap B$.

**A1.2** Find the glb and the lub (if one exists) of each of the following sets of real numbers:

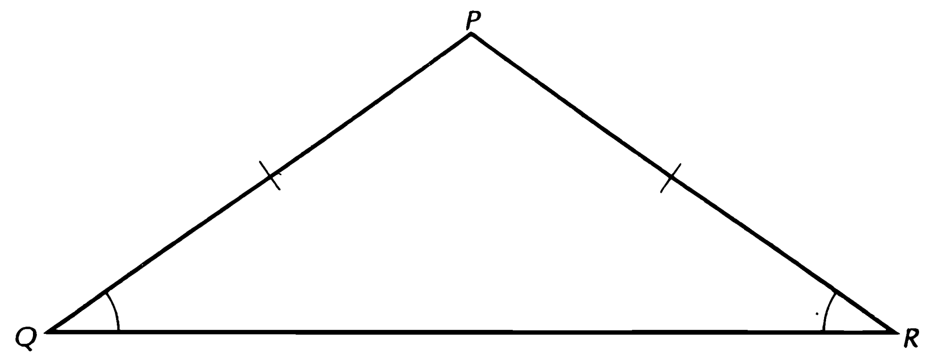
*a)* the natural numbers $\mathbf{N}$,
*b)* $\{1/1, 1/2, 1/3, \ldots, 1/n, \ldots\}$,
*c)* $\{1/2, -1/2, 2/3, -2/3, 3/4, \ldots\}$,
*d)* $\{x \in \mathbf{R} : 0 < x < 1\}$,
*e)* $\{x \in \mathbf{R} : 0 \leq x < 1\}$
*f)* $\{x \in \mathbf{R} : 0 \leq x \leq 1\}$.

## A1.3 PROOFS

One of the important roles of mathematics in the sciences is to deduce complex scientific principles from a collection of generally agreed on assumptions. For example, in classical physics one deduces from Newton's law of motion ($F = ma$) that the planets move in planar elliptical orbits about the sun. In economics one deduces from the consumer's budget equation that a 1 percent increase in a consumer's income leads to a 1 percent average increase in expenditure on the goods under study and that an increase in the price of any good leads to a decrease in the *average* consumption of all goods.

The first such scientific system one encounters is usually the Euclidean model for planar geometry that one studies in secondary school. Beginning with the undefined terms "point" and "line" and with the well-accepted system of Euclid's axioms (for example, "Given two points $P$ and $Q$, there is exactly one line that contains $P$ and $Q$;"), one uses careful techniques of mathematical logic to prove theorems about geometric objects, for example, the angles of a triangle sum to 180° or the sum of the squares of the lengths of the legs of a right triangle equals the square of the length of the hypotenuse.

The same principles of logic work in all these sciences. One starts with a clearly stated (and, hopefully, generally accepted) set of hypotheses and, usually, with some previously proven principles. Each of these hypotheses and theorems states that if some situation $A$ occurs, then situation $B$ must occur too; in short, $A$ implies $B$ ($A \Longrightarrow B$). For example, situation $A$ can be "sides $PQ$ and $PR$ of triangle $PQR$ have the same length," and situation $B$ can be "$\angle PQR = \angle PRQ$ in $\triangle PQR$." (See Figure A1.1.) Finally, one applies the principles of mathematical logic to carefully deduce new principles from the axioms and old principles.




*If the length of $PQ$ equals the length of $PR$ in $\triangle PQR$, $\angle PQR = \angle PRQ$.*

**Figure A1.1**

### Direct Proofs

The direct way of proving that $A \Longrightarrow B$ is to find a sequence of accepted axioms and theorems of the form $A_i \Longrightarrow A_{i+1}$ for $i = 1, \ldots, n$, so that $A_0 = A$ and $A_{n+1} = B$:

$$A = A_0 \Longrightarrow A_1 \Longrightarrow A_2 \Longrightarrow A_3 \Longrightarrow \cdots \Longrightarrow A_{n-1} \Longrightarrow A_n = B. \qquad (1)$$

The hard part, of course, is to find the sequence of theorems that fills in the gap from $A$ to $B$ in (1). Proofs of the form (1) are called **direct proofs**; the method is called **deductive reasoning**.

We illustrate deductive reasoning by deriving some properties of numbers from the six properties of addition and multiplication listed above. These proofs also rely on some basic properties of equality which we will accept as basic axioms; for example, for all $a, b, c, d \in \mathbf{R}$,

$$a = b \text{ and } b = c \Longrightarrow a = c, \tag{2}$$

$$a = b \Longrightarrow a + c = b + c, \tag{3}$$

$$a = b \Longrightarrow a \cdot c = b \cdot c, \tag{4}$$

$$a = b \text{ and } c = d \Longrightarrow a \cdot c = b \cdot d. \tag{5}$$

We will write out these first proofs rather carefully, that is, we will skip few steps and we will justify each step with a phrase or sentence.

**Theorem A1.1** For any $x, y, z \in \mathbf{R}$, if $x + z = y + z$, then $x = y$.

*Proof*

1. $x + z = y + z$. — Hypothesis.
2. There exists $(-z)$ such that $z + (-z) = 0$. — Additive inverse property.
3. $(x + z) + (-z) = (y + z) + (-z)$. — Rule (3).
4. $x + (z + (-z)) = y + (z + (-z))$. — Additive associative property.
5. $x + 0 = y + 0$. — Step 2.
6. $x = y$. — Additive identity property. ■

**Theorem A1.2** For any $x \in \mathbf{R}$, $x \cdot 0 = 0$.

*Proof*

1. $0 + 0 = 0$. — Additive identity property.
2. $x \cdot (0 + 0) = x \cdot 0$. — Rule (4).
3. $(x \cdot 0) + (x \cdot 0) = (x \cdot 0)$. — Distributive property.
4. $(x \cdot 0) + 0 = (x \cdot 0)$. — Additive inverse property.
5. $(x \cdot 0) + (x \cdot 0) = (x \cdot 0) + 0$ — Rule (2).
6. $x \cdot 0 = 0$. — Theorem A1.1. ■

The following proposition about even integers will be useful in proving that $\sqrt{2}$ is an irrational number.

**Theorem A1.3** Let $m$ be an *even* integer and $p$ be any integer. Then, $m \cdot p$ is an even integer.

*Proof*

| | |
|---|---|
| 1. $m$ is an even integer. | Given. |
| 2. There is an integer $q$ such that $m = 2 \cdot q$. | Definition of even integer. |
| 3. $m \cdot p = (2 \cdot q) \cdot p$. | Rule (4) above. |
| 4. $m \cdot p = 2 \cdot (q \cdot p)$. | Associative property of multiplication. |
| 5. $m \cdot p$ is even. | Definition of even integer. ■ |

## Converse and Contrapositive

Given a proposition $A \Longrightarrow B$. we now discuss two closely related propositions: the converse and the contrapositive.

**Definition** Consider a proposition $\mathcal{P}$ of the form $A \Longrightarrow B$: if hypothesis $A$ holds, conclusion $B$ holds. The **converse** of $\mathcal{P}$ is the statement $B \Longrightarrow A$, which reverses the hypothesis and conclusion of $\mathcal{P}$.

If statement $\mathcal{P}$ is true, its converse need not be true. For example, suppose $\mathcal{P}$ is the proposition: if ($A$) a person lives in Detroit, ($B$) that person lives in Michigan. Proposition $\mathcal{P}$ is true, but its converse — if a person lives in Michigan, that person lives in Detroit — is not true.

As another example, suppose $A$ is the situation "$n$ is a prime number greater than 2" and $B$ is the situation "$n$ is an odd number." It is true that $A$ implies $B$, but, as the integer $9 = 3 \cdot 3$ illustrates, it not true that $B$ implies $A$.

**Definition** If the proposition $A \Longrightarrow B$ and its converse $B \Longrightarrow A$ are both true, we say that $A$ holds **if and only if** $B$ holds or that $A$ is **equivalent** to $B$. The equivalence of $A$ and $B$ is written as $A \Longleftrightarrow B$.

For example, if $A$ is the statement that "$n$ is an even prime" and $B$ is the statement "$n = 2$," both $A \Longrightarrow B$ and $B \Longrightarrow A$ are true.

There is a proposition formed from proposition $\mathcal{P}$ that is true when $\mathcal{P}$ is true: the contrapositive of $\mathcal{P}$. We have been writing $A \Longrightarrow B$ to denote the proposition that when situation $A$ holds, so does situation $B$. Write $\sim A$ for the statement "it is not true that $A$ holds."

**Definition** The proposition $\sim B \Longrightarrow \sim A$ is called the **contrapositive** of the proposition $A \Longrightarrow B$.

For example, the contrapositive of the proposition "If a person is President of the United States, he or she must be at least 35 years old," is "If a person is not yet 35 years old, he or she is not President of the United States." Earlier, we discussed the proposition that if ($A$) $n$ is a prime integer different than 2, then ($B$) $n$ is an odd integer. The contrapositive states that if ($\sim B$) $x$ is not an odd integer, then ($\sim A$) $n$ is not a prime different than 2; or restated, if $n$ is even, then $n$ either equals 2 or is not a prime.

The contrapositive of Theorem A1.3 will be useful later. One way of stating it is as follows.

**Theorem A1.4** Suppose that $a$, $b$, and $c$ are integers with $a \cdot b = c$. If $c$ is odd, then $a$ and $b$ are odd too.

The following result can be considered a **corollary** of Theorem A1.4 in that it is a special case of Theorem A1.4 or follows almost without proof from Theorem A1.4.

**Theorem A1.5** Let $a$ be an integer. If $a^2$ is odd, so is $a$.

## Indirect Proofs

Since a proposition $A \implies B$ is true if and only if its contrapositive is true, one way to prove $A \implies B$ is to prove $\sim B \implies \sim A$. This idea can be extended: one way to prove that $B$ is true is to consider all alternatives to $B$. If every such alternative to $B$ leads to a contradiction — of $A$ itself, of an axiom of the system, or of a previously proven proposition — then $B$ must be true. This line of reasoning is called **indirect proof** or **proof by contradiction**, or sometimes **reductio ad absurdum**.

For example, suppose you left a professional (U.S.) football game early with the score tied: Detroit 10, Chicago 10. When you arrive home, you learn that the final score was Detroit 12, Chicago 10. You know that there are only four ways to score points in a professional football game: 1) by a 7-point touchdown and successful place kick; 2) by a 6-point touchdown and unsuccessful place kick, 3) by a 3-point field goal place kick, and 4) by a 2-point safety in which the scoring team tackles its opponent behind its own touchdown line. Knowing the 10–10 and 12–10 scores enables you to eliminate the first three possibilities and to conclude (by indirect reasoning) that Detroit won by a safety.

Many of the results in this book can be more easily proved indirectly than directly. To carry out an indirect mathematical proof of $A \implies B$, one assumes at some point that situation $B$ does not hold and then applies rigorous inductive arguments until a contradiction is reached. The assumption in the proof that $B$ does not hold is sometimes called the "working hypothesis."

We illustrate proof by contradiction by proving that $\sqrt{2}$ is an irrational number. First, we state without proof two principles that can be derived from our basic axioms, but whose derivation we omit to save time and space. The first proposition is the converse of Theorem A1.5.

**Theorem A1.6** Let $a$ be an integer. If $a^2$ is even, so is $a$.

**Theorem A1.7** Suppose $a = p/q$ is a rational number with $p$ and $q$ integers. Then, $p$ and $q$ can be chosen so that both are not even integers.

The proof of Theorem A1.7 is based on the fact that if 2 divides both $p$ and $q$, then a 2 can be factored out of the denominator and numerator of the fraction $p/q$.

**Theorem A1.8** $\sqrt{2}$ is an irrational number.

*Proof*

| | |
|---|---|
| 1. $\sqrt{2}$ is either rational or irrational. | Definition of irrational number. |
| 2. Suppose that $\sqrt{2}$ is rational. | Working hypothesis. |
| 3. $\sqrt{2} = p/q$, where $p$ and $q$ are *not both even*. | A1.7. |
| 4. $\sqrt{2} \cdot \sqrt{2} = (p/q) \cdot (p/q)$. | Property of equality. |
| 5. $2 = (p^2/q^2)$. | Definition of square root and rule for multiplication of fractions. |
| 6. $2 \cdot q^2 = p^2$. | Rule (4) above. |
| 7. $p^2$ is even. | Definition of even. |
| 8. $p$ is even. | Theorem A1.6. |
| 9. $p = 2 \cdot m$ for some integer $m$. | Definition of even. |
| 10. $p \cdot p = 2m \cdot 2m$. | Rule (5) above. |
| 11. $p^2 = 2 \cdot (2 \cdot m^2)$. | Definition of square; associative and commutative laws of multiplication. |
| 12. $2 \cdot q^2 = 2 \cdot (2 \cdot m^2)$. | Rule (2) applied to Steps 6 and 11. |
| 13. There exists $1/2$ such that $(1/2) \cdot 2 = 1$. | Multiplicative inverse. |
| 14. $(1/2) \cdot 2 \cdot q^2 = (1/2) \cdot 2 \cdot (2 \cdot m^2)$. | Rule (4) above. |
| 15. $q^2 = 2 \cdot m^2$. | Multiplicative associative and inverse properties. |
| 16. $q^2$ is even. | Definition of even number. |
| 17. $q$ is even. | Theorem A1.6. |
| 18. $p$ and $q$ are both even. | Steps 8 and 17. |
| 19. Contradiction to Step 3. | |
| 20. $\sqrt{2}$ is irrational. | Working hypothesis is false. ■ |

## Mathematical Induction

There is a third method of mathematical proof that differs significantly from proofs by deduction and proofs by contradiction: **proof by induction**. Inductive proofs can only be used for propositions about the integers or propositions indexed by the integers, but they are powerful tools in such situations.

To get a flavor for how inductive arguments work, suppose that a hundred men line up in a straight line, one behind the other, and that each whispers his name to the man behind him. Suppose that we know only two things about this line of men: 1) the first man's name is David, and 2) directly behind every man whose name is David is another man whose name is David. We can conclude that all hundred men are named David. For, we know by statement 1 that the first man is David. We conclude from statement 2 that the second man's name is David too. Applying statement 2 to the second man, we conclude that the third man's name is David, too. Continuing this boot strap argument step by step, we conclude that every man in the line is named David.

If there were an *infinite* number of men in the line and we knew that statements 1 and 2 held, we could still conclude that all the men in the line are named David.

The principle of induction works just this way. Suppose that we are considering a sequence of statements indexed by the natural numbers, so that the first statement is $P(1)$, the second statement is $P(2)$, and the $n$th statement is $P(n)$. Suppose that we can verify two facts about this sequence of statements:

(1) statement $P(1)$ is true;
(2) whenever any statement $P(k)$ is true for some $k$, then $P(k+1)$ is also true.

By the same logic as with the line of Davids, we conclude that all of the statements are true. In an inductive proof, step 2) is called the **inductive step**. The hypothesis that some general statement $P(k)$ is true is called the **inductive hypothesis**.

Let's work out some examples of proofs by induction. We first prove that the sum of the first $n$ natural numbers $1 + 2 + \cdots + n$ is $\frac{1}{2}n(n+1)$.

**Theorem A1.9** The sum of the first $n$ natural numbers $1 + 2 + 3 + \cdots + n$ equals $\frac{1}{2}n(n+1)$.

*Proof* For any natural number $n$, let $P(n)$ be the statement

$$P(n): \qquad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \tag{6}$$

Let's check $P(1)$. If we let $n = 1$ on the right-hand side of (6), we find $1(1+1)/2$, which does indeed equal the left-hand side when $n = 1$.

Now, we make the inductive hypothesis by assuming that statement $P(k)$ is true for some integer $k$:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}. \tag{7}$$

Adding $(k+1)$ to both sides of (7) preserves equality.

$$
\begin{aligned}
1+2+3+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1)\\
&= \left(\frac{k}{2}+1\right)(k+1)\\
&= \frac{(k+1)(k+2)}{2}.
\end{aligned}
$$

But this last expression is exactly statement $P(k+1)$. We have shown that $P(1)$ is true and that $[P(k) \text{ true} \Longrightarrow P(k+1) \text{ true}]$ for any $k$. We conclude, by the principle of mathematical induction, that $P(n)$ holds for all $n$. ■

Let's look at one more example, one that uses induction to verify a formula for the sum of the first $n$ odd natural numbers.

**Theorem A1.10** The sum of the first $n$ odd natural numbers is $n^2$:

$$1+3+5+7+\cdots+(2n-1)=n^2. \tag{8}$$

*Proof* Formula (8) is easily seen to be true for $n = 1$. So, we carry out the inductive step. Assume (8) holds for some positive integer $k$:

$$1+3+5+7+\cdots+(2k-1)=k^2. \tag{9}$$

The next odd number to be added to the left-hand side of (9) is $2(k+1)-1 = 2k+1$. We preserve equality if we add it to both sides of (9).

$$1+3+\cdots+(2k-1)+\big(2(k+1)-1\big)=k^2+(2k+1). \tag{10}$$

But the right-hand side of (10) clearly factors as $(k+1)^2$. We conclude that

$$1+3+\cdots+(2k-1)+\big(2(k+1)-1\big)=(k+1)^2,$$

which is precisely statement (9) with $k+1$ replacing $k$. By the principle of mathematical induction, we conclude that (9) holds for all natural numbers $n$. ■

## EXERCISES

**A1.3** Write out careful proofs of the following properties of set operations:
*a)* $(A \cap B)^c = A^c \cup B^c$;
*b)* $(A \cup B)^c = A^c \cap B^c$;
*c)* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

**A1.4** Show that $\sqrt{3}$ is an irrational number.

**A1.5** Use mathematical induction to prove the squared version of (6): $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.

**A1.6** Let $a_n = 1/[n(n+1)]$. Compute $a_1, a_1 + a_2, a_1 + a_2 + a_3$, and $a_1 + a_2 + a_3 + a_4$. Guess $a_1 + a_2 + \cdots + a_n$ for any natural number $n$. Use mathematical induction to verify your guess.

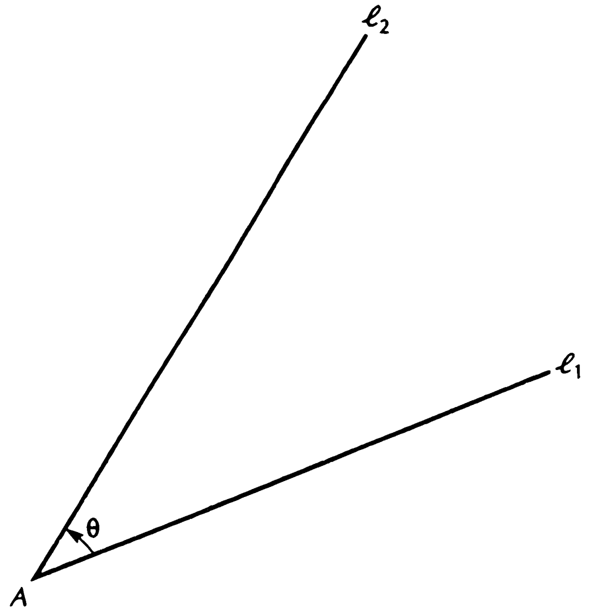
**A1.7** Use mathematical induction to prove that $n < 2^n$ for all natural numbers $n$.

# Trigonometric Functions

Most of the specific functions encountered in elementary mathematical approaches to economics are polynomials, quotients of polynomials (rational functions), or exponentials. As the independent variable $x$ goes to infinity, the graph of each of these three types of functions either goes to infinity (very quickly for exponential functions) or approaches a finite horizontal asymptote. None of these functional forms can model the regular periodic patterns that play an important role in the social, biological, and physical sciences: business cycles and agricultural seasons, heart rhythms, and hormone level fluctuations, and tides and planetary motions. The basic functions for studying regular periodic behavior are the *trigonometric functions,* especially the sine, cosine, and tangent functions.

## A2.1 DEFINITIONS OF THE TRIG FUNCTIONS

The domain of the trigonometric functions is more naturally the set of all geometric angles than some set of real numbers. We start with that approach and measure angles in *degrees* for the time being.




*Angle $\theta$ is formed by rotating ray $\ell_1$ to $\ell_2$.*

**Figure A2.1**

Consider the acute angle $\theta$ in Figure A2.1 formed by the rays $\ell_1$ and $\ell_2$ from the point $A$. Think of $\theta$ as being swept out by a *counterclockwise* rotation about the point $A$ from ray $\ell_1$ to ray $\ell_2$. In this case, we will call $\ell_1$ the initial ray and $\ell_2$ the terminal ray of angle $\theta$. Pick a point $B$ on the terminal ray $\ell_2$ and draw a perpendicular line $n$ from $B$ to the initial ray $\ell_1$, intersecting $\ell_1$ at the point $C$ as in Figure A2.2. Triangle $ABC$ is a right triangle with *hypotenuse* $AB$, *opposite leg* $BC$, and *adjacent leg* $AC$, relative to the angle $\theta$.

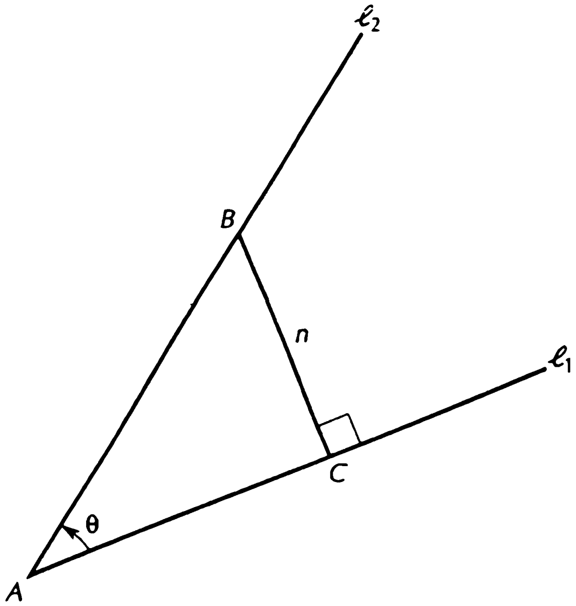



**Figure A2.2** *Computing the sine, cosine, and tangent of $\theta$.*

The six trigonometric functions are defined by taking ratios of the lengths of these three sides of triangle $ABC$:

$$\text{sine } \theta = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{\|BC\|}{\|AB\|},$$

$$\text{cosine } \theta = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{\|AC\|}{\|AB\|},$$

$$\text{tangent } \theta = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{\|BC\|}{\|AC\|},$$

$$\text{cotangent } \theta = \frac{\text{adjacent leg}}{\text{opposite leg}} = \frac{\|AC\|}{\|BC\|},$$

$$\text{secant } \theta = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{\|AB\|}{\|AC\|},$$

$$\text{cosecant } \theta = \frac{\text{hypotenuse}}{\text{opposite leg}} = \frac{\|AB\|}{\|BC\|},$$

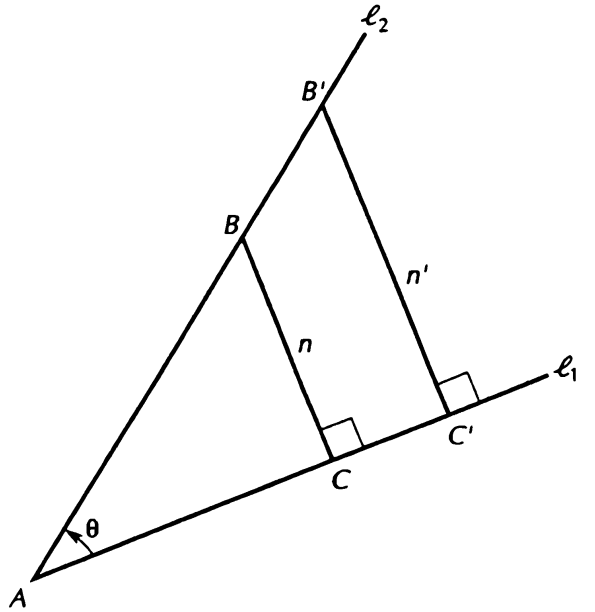
where $||AC||$ denotes the *length* of line segment $AC$. These six functions represent all the possible ratios of the three sides of triangle $ABC$. The first three trigonometric functions — sine, cosine, and tangent — are the most important of the six. As the following theorem asserts, the last four can be expressed in terms of the first two.

**Theorem A2.1** Every trig function can be expressed as a quotient of sine and cosine. In particular,

$$\text{tangent } \theta = \frac{\text{sine } \theta}{\text{cosine } \theta}, \qquad \text{cotangent } \theta = \frac{\text{cosine } \theta}{\text{sine } \theta},$$

$$\text{secant } \theta = \frac{1}{\text{cosine } \theta}, \quad \text{and} \quad \text{cosecant } \theta = \frac{1}{\text{sine } \theta}.$$

The above definitions are independent of how far the point $B$ is from $A$ on the second ray $\ell_2$. To see this, suppose that $B'$ is another point on $\ell_2$. Let $n'$ be the perpendicular line from $B'$ to $\ell_1$, meeting $\ell_1$ at the point $C'$, as indicated in Figure 2.3. Since $n$ and $n'$ are both perpendicular to $\ell_1$, they are parallel to each other. Therefore, the corresponding sides of triangles $ABC$ and $AB'C'$ are parallel to each other. By a fundamental result of the Euclidean geometry one studies in high school, triangles $ABC$ and $AB'C'$ are similar to each other; they have equal sets of angles, and therefore the ratios of their corresponding sides are equal. Therefore, the six trig functions have the same values whether one uses triangle $ABC$ or triangle $AB'C'$.




*The values of the trig functions are independent of the size of the defining triangle.* **Figure A2.3**

For angles that do not lie between 0° and 90°, there is a sign convention necessary for a consistent definition of the trigonometric functions. The simplest way to describe this convention is to choose the point $B$ in Figure A2.2 so that it is always *1 unit* away from the tip of the angle at $A$. Since the orientation of the angle under study should not make a difference, we can, without any loss of generality, always take the point $A$ at the tip of the angle to be the origin of some $xy$-coordinate system, and its initial ray $\ell_1$ to be the positive $x$-axis. To carry out this approach, draw the circle $S$ of radius 1, the *unit circle*. For any angle $\theta$, think of $\theta$ as being swept out counterclockwise by rays from the origin starting with the positive $x$-axis. As before, call the terminal ray $\ell_2$. Choose the point $B$ on $\ell_2$ 1 unit from the origin, that is, on the unit circle $S$, as indicated in Figure A2.4.

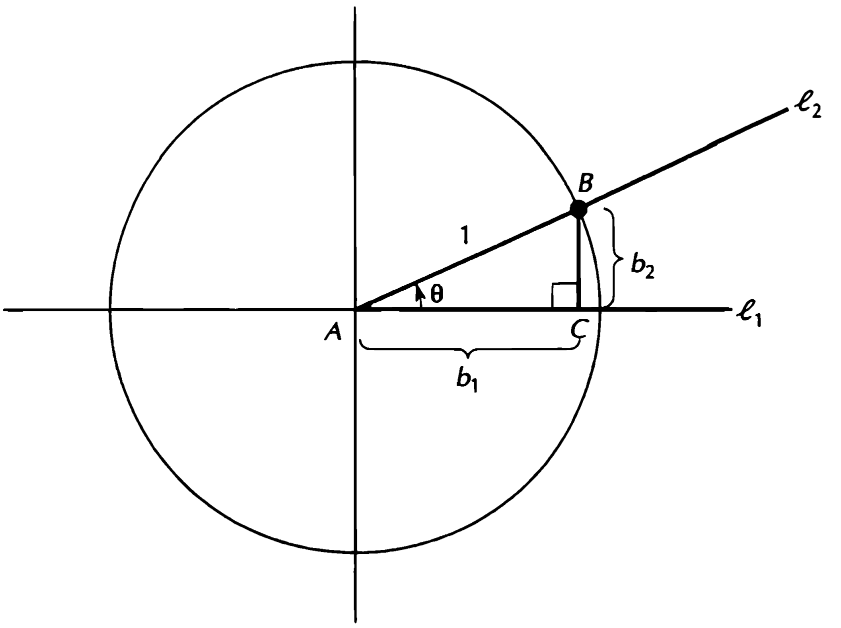



**Figure A2.4** *Measuring θ as a counterclockwise rotation on the unit circle S from the positive x-axis.*

Suppose that the coordinates of the point $B$ are $(b_1, b_2)$ in this coordinate system. Then, the perpendicular segment from $B$ to the $x$-axis $\ell_1$ meets $\ell_1$ at the point $C = (b_1, 0)$; furthermore,

$$\|AC\| = b_1 \quad \text{and} \quad \|BC\| = b_2.$$

Since $\|AB\| = 1$ in this approach, the above definitions of the trigonometric functions become

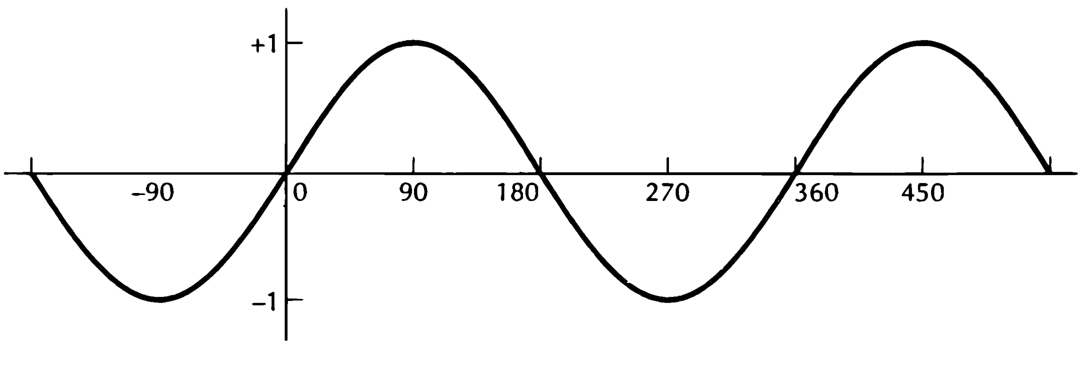
$$\begin{aligned} \sin\theta &= \frac{b_2}{1} = b_2, & \tan\theta &= \frac{b_2}{b_1}, & \sec\theta &= \frac{1}{b_1}, \\ \cos\theta &= \frac{b_1}{1} = b_1, & \cot\theta &= \frac{b_1}{b_2}, & \csc\theta &= \frac{1}{b_2}. \end{aligned} \tag{1}$$

Note the use of the standard abbreviations for the six trig functions.

We now extend formally the definitions of these functions to all angles. For *any* angle $\phi$, sweep out $\phi$ counterclockwise on the unit circle beginning with the positive $x$-axis, and let $B(b_1, b_2)$ be the point where the terminal ray intersects the unit circle $S$. Then, the six trigonometric functions are defined by the six expressions in (1); $b_1$ and $b_2$ can be positive or negative depending on the size of $\phi$. For example, since $b_2$ is negative when $B$ lies in the third quadrant (where $x_1$ and $x_2$ are negative) and in the fourth quadrant (where $x_1$ is positive and $x_2$ is negative), $\sin\phi$ is negative when $\phi$ lies between 180° and 360°. Similarly, the tangent and cotangent functions are negative precisely when $B$ lies in the second or fourth quadrants, that is, when $90° < \phi < 180°$ or $270° < \phi < 360°$.

## A2.2 GRAPHING TRIG FUNCTIONS

If we use the definitions in (1) to graph the sine function, we need only keep track of $b_2$ as $\theta$ goes from 0° to 360°. At $\theta = 0°$, $b_2 = 0$ and $\sin 0 = 0$. As $\theta$ rises from 0° to 90°, $b_2$ rises from 0 to 1. As $\theta$ passes from 90° to 180° to 270°, $b_2$ decreases from $+1$ to 0 to $-1$. As $\theta$ goes from 270° to 360°, $b_2$ increases from $-1$ back to 0. As $\theta$ increases beyond 360°, $B$ moves once again around the unit circle. If we move $B$ *clockwise* around the unit circle $S$, we sweep out *negative angles* (by definition), but we continue to use the formulas in (1) to define their sine, cosine, etc. We conclude that the graph of $\theta \mapsto \sin\theta$ is the curve in Figure A2.5.




*The graph of* $\theta \mapsto \sin\theta$.

**Figure A2.5**

Similarly, by keeping track of $b_1$ as $B$ moves around the unit circle, we generate the graph of the cosine function pictured in Figure A2.6. By keeping track of $b_2/b_1$ as $B$ moves around the unit circle, we generate the graph of the tangent function in Figure A2.7. Note that as $\theta$ moves toward 90°, $B$ approaches (0, 1) and $\tan\theta = b_2/b_1$ approaches $+\infty$.

Why do we need more than one function to keep track of angles? For one thing, as Figures A2.5 and A2.6 indicate, for every number $y$ between $-1$ and $+1$, there are *two* angles $\theta_1$ and $\theta_2$ between 0° and 360° such that $\sin\theta_1 = \sin\theta_2$. However, for these two angles, $\cos\theta_1$ and $\cos\theta_2$ have different signs. For example, $\sin 180° = \sin 0° = 0$, but $\cos 180° = -1$ and $\cos 0° = +1$. Since $(b_1, b_2) = (\cos\theta, \sin\theta)$, cosine and sine together completely specify each angle.

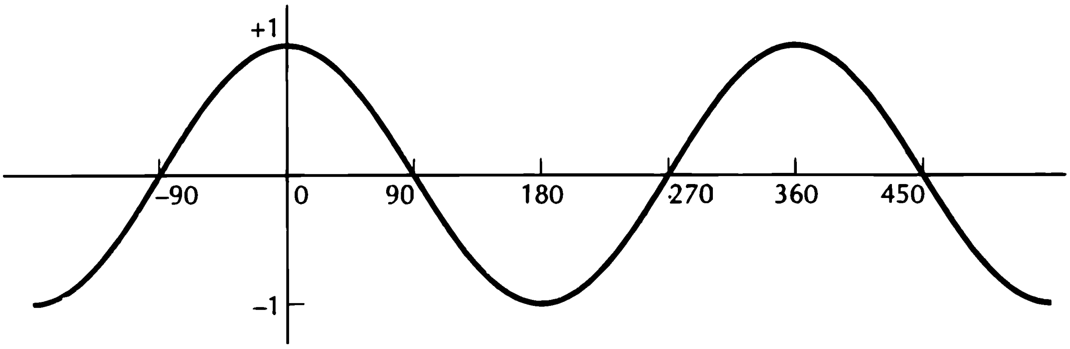



Figure A2.6

*The graph of* $\theta \mapsto \cos \theta$.

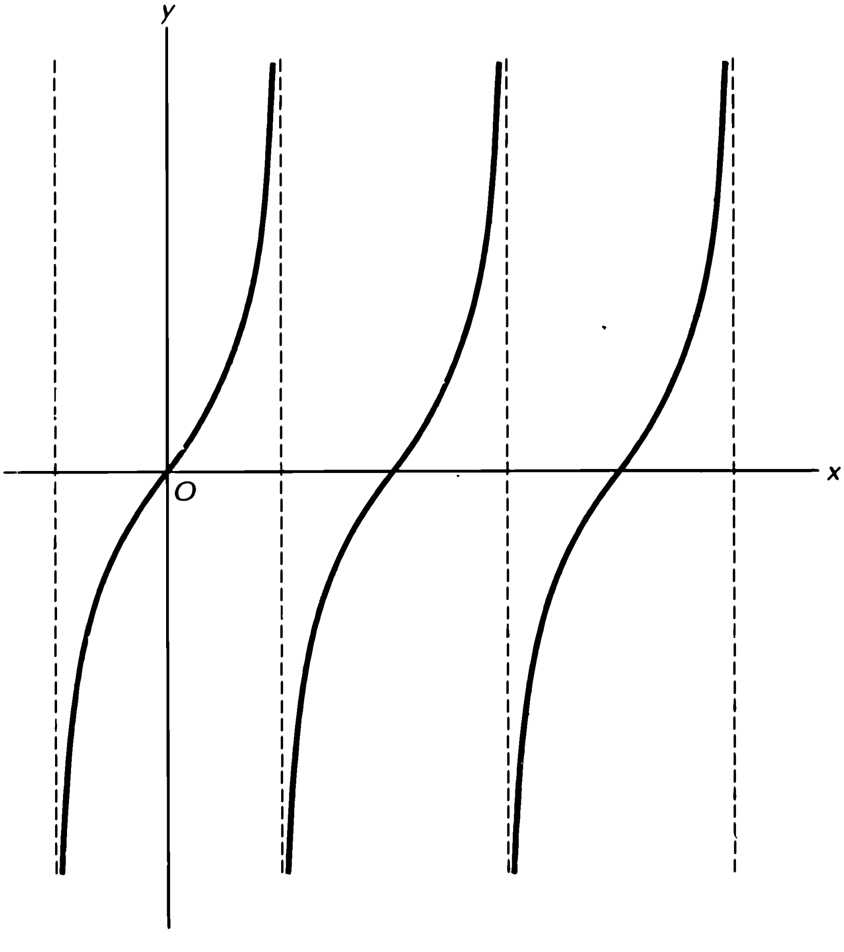



Figure A2.7

*The graph of* $\theta \mapsto \tan \theta$.

In addition, each trig function has its own advantages and its own uses. For example, the sine and tangent functions are natural since as $\theta$ moves from $-90°$ to $0°$ to $+90°$, the sine shadows it while moving from $-1$ to $0$ to $+1$ and the tangent shadows it while moving from $-\infty$ to $0$ to $+\infty$. The tangent function is especially useful in applications because its definition involves the length of the two legs of the defining right triangle and, in fact, exactly tracks the *slope* of the hypotenuse of that triangle. On the other hand, as Figure A2.6 indicates, the cosine function satisfies

$$\cos\theta = \cos(-\theta); \tag{2}$$

that is, the cosine is an *even function*. In particular, the cosine of an angle is independent of whether the angle is measured clockwise or counterclockwise.

## A2.3 THE PYTHAGOREAN THEOREM

For the rest of this section we will use some of the basic Euclidean geometry that one studies in high school to derive useful properties of the trigonometric functions. The key geometric principle is the **Pythagorean Theorem**: the sum of the squares of the legs of a right triangle equals the square of the length of the hypotenuse; $a^2 + b^2 = c^2$ in Figure A2.8.

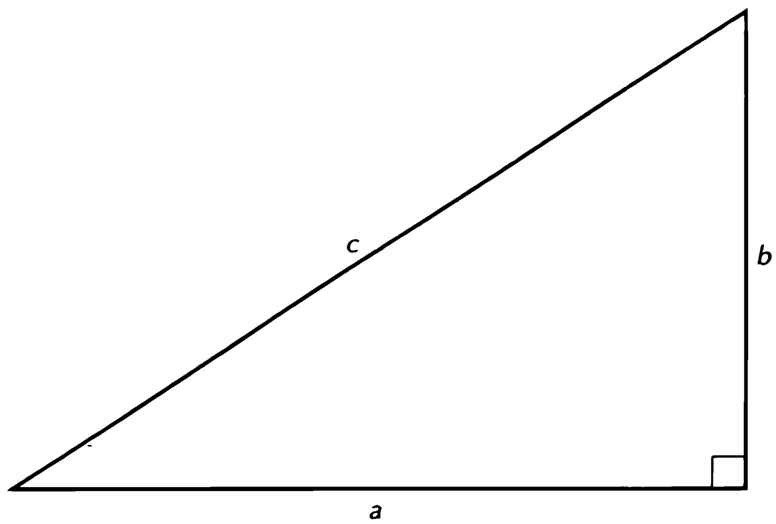



**Figure A2.8** *By the Pythagorean Theorem, $a^2 + b^2 = c^2$.*

The application of the Pythagorean Theorem to Figure A2.4 yields

$$b_1^2 + b_2^2 = 1, \tag{3}$$

since the hypotenuse has length 1. Using the definitions (1) of the trig functions, equation (3) translates to

$$\cos^2\theta + \sin^2\theta = 1 \quad \text{for all angles } \theta. \tag{4}$$

Dividing (3) through by $b_1^2$ yields

$$1 + \left(\frac{b_2}{b_1}\right)^2 = \left(\frac{1}{b_1}\right)^2 \quad \text{or} \quad 1 + \tan^2\theta = \sec^2\theta. \tag{5}$$

Dividing (3) through by $b_2^2$ yields

$$\left(\frac{b_1}{b_2}\right)^2 + 1 = \left(\frac{1}{b_2}\right)^2 \quad \text{or} \quad \cot^2\theta + 1 = \csc^2\theta. \tag{6}$$

## A2.4 EVALUATING TRIGONOMETRIC FUNCTIONS

To understand the trigonometric functions a little better, we evaluate the sine, cosine, and tangent functions at the three important acute angles: 30°, 45°, and 60°. These evaluations rely on the basic result of Euclidean geometry that two sides of a triangle are equal if and only if the angles opposite these sides are equal.

For the 45° angle in Figure A2.9, we have drawn right triangle $OBC$, motivated by Figure A2.2. (We have put aside the unit circle approach for a moment.) Since $\angle O = 45°$ and $\angle OCB = 90°$, $\angle OBC = 45°$. (The angles of a triangle sum to 180°.) Since $\angle COB = \angle OBC$, $\|OC\| = \|CB\|$. If we chose $B$ so that $\|OC\| = \|CB\| = 1$, then by the Pythagorean Theorem,

$$\|OB\|^2 = \|OC\|^2 + \|BC\|^2 = 1 + 1 = 2.$$

So, $\|OB\| = \sqrt{2}$. We conclude that

$$\begin{aligned}
\sin 45° &= \frac{\|BC\|}{\|OB\|} = \frac{1}{\sqrt{2}} \approx 0.7071, \\
\cos 45° &= \frac{\|OC\|}{\|OB\|} = \frac{1}{\sqrt{2}} \approx 0.7071, \\
\tan 45° &= \frac{\|BC\|}{\|OC\|} = \frac{1}{1} = 1.
\end{aligned} \tag{7}$$

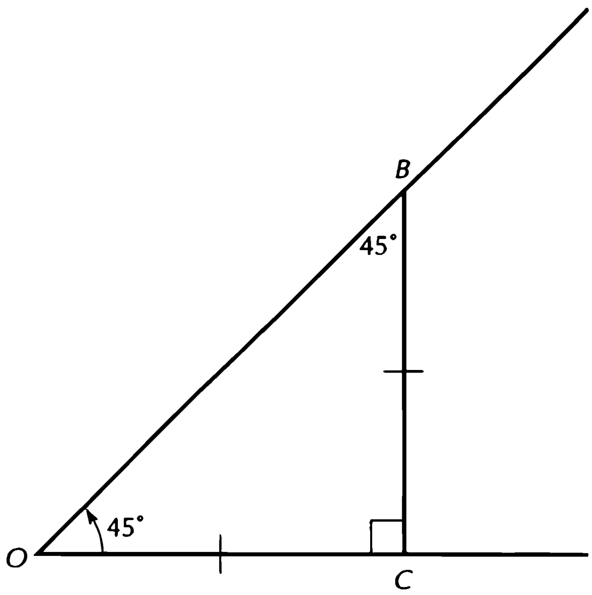
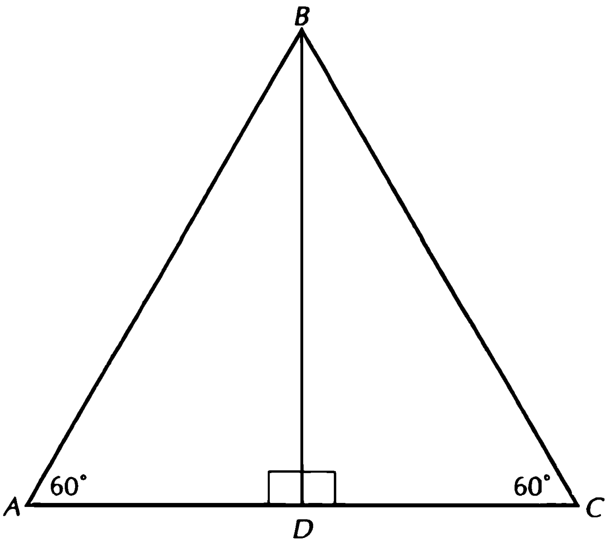



**Figure A2.9** *The 45-45-90 isosceles triangle.*

To compute the values of the trig functions at 30° and 60°, consider the 60-60-60 equilateral triangle $ABC$ in Figure A2.10, and draw the bisector of $\angle B$, $BD$. Since

$$\angle A = \angle C = 60^\circ \quad \text{and} \quad \angle ABD = \angle DBC = 30^\circ, \tag{8}$$

$\angle ADB = \angle CDB = 90^\circ$. We use the right triangle $ABD$ to evaluate the trig functions at 30° and at 60°.




*The 60-60-60 equilateral triangle.*

**Figure A2.10**

Recall from high school geometry that two triangles are congruent if in the two triangles, two corresponding pairs of angles and the sides between them are equal (angle-side-angle rule). Therefore, (8) implies that triangles $ABD$ and $BDC$ are congruent; and so,

$$\|AD\| = \|DC\| = \frac{1}{2}\|AB\|.$$

If we choose units so that $\|AD\| = 1$, then $\|AB\| = 2$, and by the Pythagorean Theorem,

$$\|BD\| = \sqrt{\|AB\|^2 - \|AD\|^2} = \sqrt{3}.$$

It follows that

$$\sin 60^\circ = \frac{\|BD\|}{\|AB\|} = \frac{\sqrt{3}}{2} \approx 0.866,$$

$$\cos 60^\circ = \frac{\|AD\|}{\|AB\|} = \frac{1}{2} = 0.5,$$

$$\tan 60^\circ = \frac{\|BD\|}{\|AD\|} = \sqrt{3} \approx 1.732,$$

$$\sin 30^\circ = \frac{\|AD\|}{\|AB\|} = \frac{1}{2} = 0.5,$$

$$\cos 30^\circ = \frac{\|BD\|}{\|AB\|} = \frac{\sqrt{3}}{2} \approx 0.866,$$

$$\tan 30^\circ = \frac{\|AD\|}{\|BD\|} = \frac{1}{\sqrt{3}} \approx 0.577.$$

These values are consistent with the graphs in Figures A2.5, A2.6, and A2.7.

## A2.5 MULTIANGLE FORMULAS

Finally, we consider the behavior of the trigonometric functions with regard to the sums and differences of angles. The following theorem, presented without proof, summarizes these rules.

**Theorem A2.2** For any two angles $a$ and $b$,

$$\sin(a + b) = \sin a \cos b + \cos a \sin b,$$

$$\sin(a - b) = \sin a \cos b - \cos a \sin b,$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b,$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b.$$

In the special case that $a = b$,

$$\sin 2a = 2 \sin a \cos a,$$

$$\cos 2a = \cos^2 a - \sin^2 a.$$

Theorem A2.2 will be important in deriving the formulas of $(\sin x)'$ and $(\cos x)'$. Of course, it also extends the set of angles whose sines and cosines can be computed exactly. For example, knowing the sin and cos of 45° and 30°, we can compute the sin and cos of 15° and 75°.

## A2.6 FUNCTIONS OF REAL NUMBERS

We have formulated the trigonometric functions as functions of geometric angles. We can easily consider each of them as functions defined on the set of real numbers by associating to any real number $r$ the angle of $r°$. In fact, we implicitly did this when we drew the graphs in Figures A2.5, A2.6, and A2.7.

However, just as there are a number of different measures of distance (inches, feet, yards, miles, centimeters, meters, etc.), there are a number of different measures of an angle. The most intuitive is the *degree*, which is defined by the fact that an angle of 180° yields a straight line and that an angle of 360° describes a complete revolution about the circle.

But why 360? Any number could have served about as well. Since the right angle is a natural cornerstone among angles, some engineers find it convenient to measure angles by what percentage of a right angle they are, using the term "grads" for the underlying unit. In this approach, a right angle is an angle of 100 grads, a 45° angle is an angle of 50 grads (50 percent of a right angle), and a 60° angle is an angle of $66\frac{2}{3}$ grads ($66\frac{2}{3}$ percent of a right angle).

In some sense, a more natural approach is to start with the unit circle — the circle of radius 1 pictured in Figure A2.3 — and to describe an angle by the *length of the arc* of the unit circle which that angle sweeps out. The word "radians" is used to describe this unit of measurement. Since the circumference of a circle of radius 1 is $2\pi$, an angle of 360° is the same as an angle of $2\pi$ radians. A right angle, which corresponds to $1/4$ of the way around the circle, is measured as

$$\frac{1}{4} \cdot 2\pi = \frac{\pi}{2} \text{ radians.}$$

In general, since 360° corresponds to $2\pi$ radians, an angle of $x°$ is an angle of $y$ radians where

$$\frac{1}{360}x = \frac{1}{2\pi}y,$$

so that

$$y = \frac{\pi}{180}x \quad \text{and} \quad x = \frac{180}{\pi}y.$$

The graph of the sine function in radians is presented in Figure A2.11.

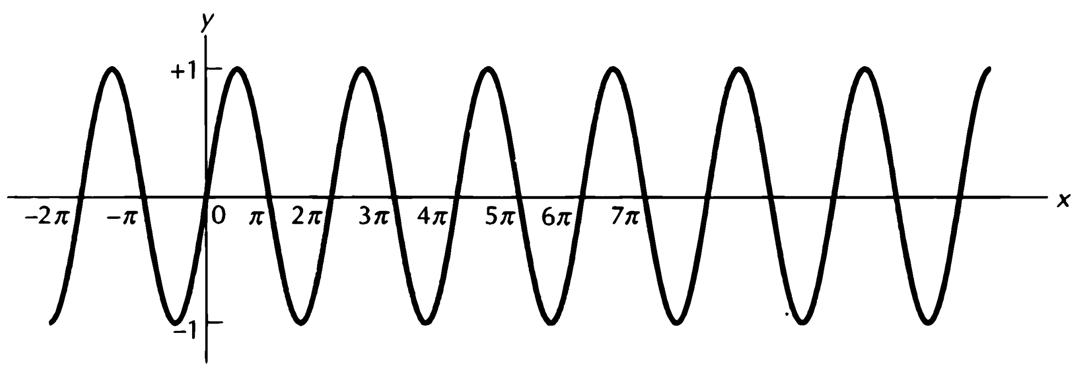



**Figure A2.11** *The graph of $x \longmapsto \sin x$ with $x$ measured in radians.*

If you are evaluating trig functions on a hand-held scientific calculator, you must tell the calculator the units with which you are measuring angles. When they are switched on, most scientific calculators start in degrees mode, and indicate so with DEG showing in a corner of the display window. To change units to radians or grads, push the DRG button on your calculator and watch the corresponding units change to RAD or GRAD in the window.

## A2.7 CALCULUS WITH TRIG FUNCTIONS

When one uses calculus to work with trig functions, it is understood that all angles are measures in radians, because it is in radians that the derivatives have the simplest expressions. The key ingredient in the calculus approach to trig functions are the following two convergence results.

**Theorem A2.3** When angles $x$ are measured in radians,

$$\frac{\sin x}{x} \to 1 \quad \text{as} \quad x \to 0, \tag{9}$$

$$\frac{\cos x - 1}{x} \to 0 \quad \text{as} \quad x \to 0. \tag{10}$$

The proof of these results takes us a little out of our way, and so it is presented at the end of this section. Figure A2.12 presents the intuition behind (9). Since angles are measured in radians, the size $x$ of $\angle BOC$ is the length of arc $BD$ on the

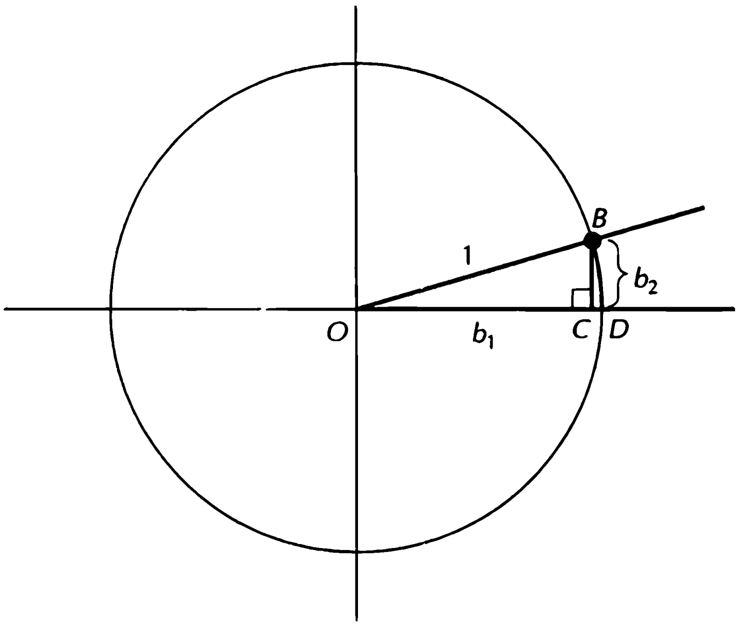



**Figure A2.12** *The geometry behind (9).*

unit circle, and $\sin x = \|BC\| = b_2$. As $x \to 0$, $b_2 = \sin x \to 0$, and the length of the *arc* BD becomes very close to the length of the *segment* $\overline{BC}$. (Use your calculator to see that (9) holds as $x \to 0$ when $x$ is measured in radians.)

Using the results of Theorem A2.3, one can easily compute the expressions for the derivatives of the six trigonometric functions.

**Theorem A2.4** Suppose that angles $x$ are measures in radians. Then,

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x,$$
$$(\tan x)' = \sec x^2, \qquad (\cot x)' = -\csc^2 x,$$
$$(\sec x)' = \tan x \cdot \sec x, \qquad (\csc x)' = -\cot x \cdot \csc x.$$

*Proof* The proof uses the definition of the derivative and the angle-sum formula in Theorem A2.2:

$$\begin{aligned}\frac{\sin(x+h) - \sin x}{h} &= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right).\end{aligned} \tag{11}$$

Now, let $h$ tend to 0. By Theorem A2.3, $(\cos h - 1)/h \to 0$ and $(\sin h)/h \to 1$. Therefore,

$$\begin{aligned}\frac{d}{dx}\sin x &\equiv \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && \text{(def. of derivative)} \\ &= \sin x \cdot 0 + \cos x \cdot 1 && \text{(by (11))} \\ &= \cos x.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\cos(x+h) - \cos x}{h} &= \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \cos x \left(\frac{\cos h - 1}{h}\right) - \sin x \left(\frac{\sin h}{h}\right)\end{aligned}$$

tends to $\cos x \cdot 0 - \sin x \cdot 1$ as $h \to 0$. Therefore, $(\cos x)' = -\sin x$. Since tan, cot, sec, and csc can be expressed as quotients of sin and cos (Theorem A2.1), their derivatives can now be calculated using the quotient rule of differentiation. (Exercise.) ■

## A2.8 TAYLOR SERIES

The Taylor series can be a powerful technique for working with nonlinear functions. As described in Section 30.3, the Taylor series of a function $f: \mathbf{R} \to \mathbf{R}$ about the point $x = 0$ is

$$f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{[3]}(0)x^3 + \frac{1}{4!}f^{[4]}(0)x^4 + \cdots. \qquad (12)$$

Since $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $\sin 0 = 0$, and $\cos 0 = 1$, the even order derivatives of the sine function are 0 at $x = 0$, while the odd order derivatives are $\pm 1$. The Taylor series for the sine function at $x = 0$ is

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots. \qquad (13)$$

Similarly, the Taylor series of the cosine function at $x = 0$ is

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots. \qquad (14)$$

(Exercise.) For each fixed $x$, the consecutive terms of the series in (13) and (14) go to zero (rather rapidly). (Exercise.) It follows — with quite a bit of analysis — that the series in (13) and (14) converge *for any x*.

*Example A2.1* Taylor series yield an effective method for computing values of sin and cos. Let's use this method to compute $\sin(\pi/4)$ to three decimal places. The first four terms in (13) for $x = \pi/4$ are

$$x \approx 0.7853982, \qquad -\frac{x^3}{3!} \approx -0.0807455,$$
$$\frac{x^5}{5!} \approx 0.0024904, \quad \text{and} \quad -\frac{x^7}{7!} \approx -0.0000366 \qquad (15)$$

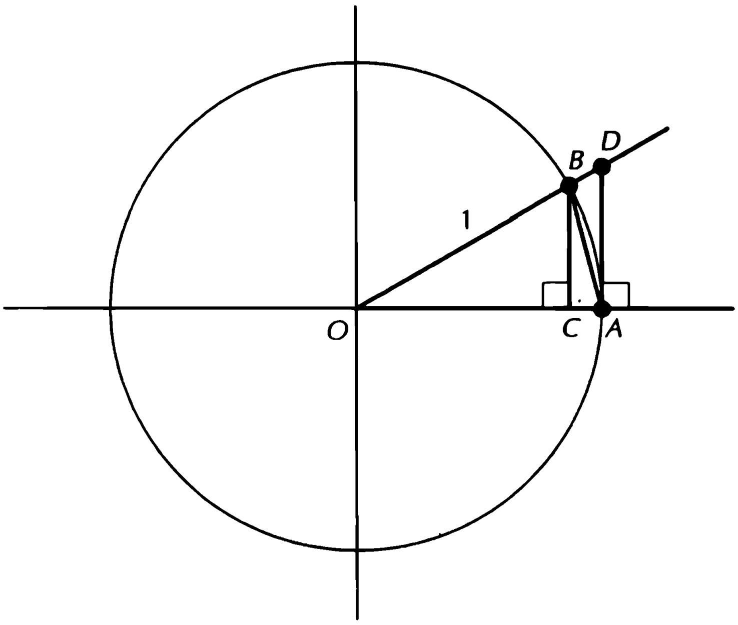
Since further terms have zeros in the first five decimal places, they do not affect the first three digits of the Taylor series approximation of $\sin(\pi/4)$. Adding the four numbers in (15) yields

$$\sin\frac{\pi}{4} \approx 0.7071065,$$

an answer that is, in comparison with the true answer of $1/\sqrt{2} \approx 0.7071068$ in (7), correct to six significant figures.

## A2.9 PROOF OF THEOREM A2.3

We close this section by sketching a proof of the basic convergence results in Theorem A2.3. Draw the unit circle $S$, as in Figure A2.13. Let $\theta$ be an acute angle from the origin whose initial ray is the positive $x$-axis, as in Figures A2.3 and A2.13. Let $B$ be the point where the terminal ray of $\theta$ meets $S$. Let $BC$ be the perpendicular line segment from the point $B$ to the initial ray of $\theta$. Let $A$ be the point $(1, 0)$, and let $AD$ be the vertical line segment from $A$ to the terminal ray of $\theta$.




$\triangle OBA \subset$ *sector* $OBA \subset \triangle ODA$.

**Figure A2.13**

As Figure A2.13 indicates, triangle $OBA \subset$ sector $OBA \subset$ triangle $ODA$. Therefore,

$$\text{area of } \triangle OBA \leq \text{area of sector } OBA \leq \text{area of } \triangle ODA. \tag{16}$$

Now,

$$\text{area of } \triangle OBA = \frac{1}{2} \cdot \|OA\| \cdot \|BC\|$$

$$= \frac{1}{2} \cdot 1 \cdot \sin\theta,$$

$$\text{area of sector } OBA = \frac{\theta}{2\pi} \cdot \text{area inside } S$$

$$= \frac{\theta}{2\pi} \cdot \pi \cdot 1^2 = \frac{1}{2} \cdot \theta,$$

$$\text{area of } \triangle ODA = \frac{1}{2} \cdot \|OA\| \cdot \|AD\|$$
$$= \frac{1}{2} \cdot 1 \cdot \tan\theta.$$

Substituting these calculations into (16) yields

$$\frac{1}{2}\sin\theta \le \frac{1}{2}\theta \le \frac{1}{2}\tan\theta. \tag{17}$$

Divide (17) through by $\frac{1}{2}\sin x$:

$$1 \le \frac{\theta}{\sin\theta} \le \cos\theta,$$

and invert:

$$\frac{1}{\cos\theta} \le \frac{\sin\theta}{\theta} \le 1.$$

As $\theta \to 0$, $\cos\theta \to 1$ and $(\sin\theta)/\theta$ is forced to converge to 1; this proves (9).

To prove (10), note that

$$\begin{aligned}
\frac{\cos x - 1}{x} &= \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} \\
&= \frac{\cos^2 x - 1}{x(\cos x + 1)} \\
&= \frac{-\sin^2 x}{x(\cos x + 1)} && \text{(by (4))} \\
&= \frac{\sin x}{x} \cdot \frac{-\sin x}{\cos x + 1} \\
&\to 1 \cdot \frac{0}{2} = 0, \qquad \text{as } x \to 0 && \text{(by (9))}
\end{aligned}$$

## EXERCISES

**A2.1** Prove Theorem A2.1 from the definitions of the trig functions.

**A2.2** Draw the graphs of the cotangent, secant, and cosecant functions.

**A2.3** Prove that $\cos\theta = \sin(90° - \theta)$ and $\sin\theta = \cos(90° - \theta)$.

**A2.4** Evaluate sine, cosine, and tangent at 120°, 135°, 150°, 210°, 225°, and 240°

**A2.5** Evaluate cotangent, secant, and cosecant at 30°, 45° and 60°.

**A2.6** Derive formulas for $\cos \frac{1}{2}a$ and $\sin \frac{1}{2}a$ in terms of $\cos a$.

**A2.7** Use Theorem A2.2 and the previous exercise to compute the sine and cosine of 15°, 22.5°, and 75°.

**A2.8** Compare the graphs of $\sin x$ in Figures A2.5 and A2.11 by graphing both on the same coordinate grid.

**A2.9** Use a hand calculator to see that (9) holds when $x$ is measured in radians, but not when $x$ is measured in degrees.

**A2.10** Use $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ and Theorem A2.1 to compute the derivatives of the other four trig functions.

**A2.11** Use the Taylor series to compute $\cos(\pi/4)$ and $\sin(\pi/3)$, and compare your answers with the exact answer computed earlier.

**A2.12** Show that (13) and (14) are the Taylor series (12) for the sine and cosine functions.

**A2.13** For any fixed number $a$ — no matter how large — show that $a^n/n! \to 0$ as $n \to \infty$.

# Complex Numbers

## A3.1 BACKGROUND

The solutions of a quadratic equation

$$ax^2 + bx + c = 0 \tag{1}$$

are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

The only caveat is that the expression $b^2 - 4ac$ under the radical must be positive for the answers (2) to be a pair of real numbers. For example, (2) tells us that the solutions of $x^2 - 4x + 3 = 0$ are $x = 1$ and $x = 3$ (check) and that the solutions of

$$x^2 - 4x + 13 = 0 \tag{3}$$

are

$$x = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2}. \quad \text{(Check.)} \tag{4}$$

Since the square of every real number — positive or negative — is a *positive* number, no real number has a square equal to $-36$; and therefore, equation (3) has no real solution. However, as we will see at the end of this section, there are situations where one needs to work with a solution of equation (3) — even a solution that does not have any meaning. To get around this problem, mathematicians have extended the concept of a number and have created a new kind of number — the complex number — which includes square roots of negative numbers.

The key step in this extension is to let the symbol $i$ stand for $\sqrt{-1}$, so that $i^2 = -1$, and to set up this extended number system so that it has all the properties of the usual real number system, in particular, the six properties in Section A1.2. In this case,

$$\sqrt{-36} = \sqrt{36 \cdot -1} = \sqrt{36} \cdot \sqrt{-1} = 6i,$$

and the "solutions" (4) of equation (3) can be formally written as

$$2 + 3i \quad \text{and} \quad 2 - 3i. \tag{5}$$

### Definitions

The set of complex numbers, often denoted **C**, is the set of real numbers augmented by an extra symbol $i$. A typical complex number has the form $a + bi$, where $a$ and $b$ are real numbers. The formal definition follows.

**Definition** A **complex number** is a number of the form $a + bi$, where $a$ and $b$ are real numbers, and the symbol $i$ formally satisfies $i^2 = -1$. The real number $a$ is called the **real part** of $a + bi$; the real number $b$ is called the **imaginary part** of $a + bi$. A complex number $bi$ whose real part is 0 is called an **imaginary number**.

**Definition** As we will see below, for any complex number $a + bi$, the related complex number $a - bi$ plays a number of important roles. The complex number $a - bi$ is called the **complex conjugate** of $a + bi$. When the symbol $z$ is used to denote the complex number $a + bi$, its conjugate $a - bi$ is written as $\bar{z}$:

$$\overline{a + bi} = a - bi.$$

### Arithmetic Operations

Two complex numbers can be added, subtracted, multiplied, or divided; the result is a new complex number. For example, for real numbers $a_1, a_2, b_1, b_2$:

$$\begin{aligned}
(a_1 + b_1 i) + (a_2 + b_2 i) &= a_1 + a_2 + b_1 i + b_2 i \\
&= (a_1 + a_2) + (b_1 + b_2)i, \\
(a_1 + b_1 i) - (a_2 + b_2 i) &= a_1 - a_2 + b_1 i - b_2 i \\
&= (a_1 - a_2) + (b_1 - b_2)i, \\
(a_1 + b_1 i) \cdot (a_2 + b_2 i) &= (a_1 \cdot a_2) + (a_1 \cdot b_2 i) + (b_1 i \cdot a_2) + (b_1 i \cdot b_2 i) \\
&= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i,
\end{aligned}$$

since $b_1 i \cdot b_2 i = b_1 b_2 i^2 = b_1 b_2(-1)$. In particular,

$$(a + bi)^2 = (a + bi) \cdot (a + bi) = (a^2 - b^2) + 2ab\,i, \tag{6}$$

and the product of a complex number $a + bi$ and its complex conjugate $a - bi$ is

$$(a + bi) \cdot (a - bi) = a^2 + b^2, \tag{7}$$

a positive *real number*.

Division is a little trickier. To obtain a complex number of the form $A + Bi$ from a quotient $(a_1 + b_1 i)/(a_2 + b_2 i)$ of complex numbers, multiply numerator and denominator of the quotient by $a_2 - b_2 i$, the complex conjugate of the denominator $a_2 + b_2 i$, to get a real denominator:

$$\begin{aligned}\frac{a_1 + b_1 i}{a_2 + b_2 i} &= \frac{a_1 + b_1 i}{a_2 + b_2 i} \cdot \frac{a_2 - b_2 i}{a_2 - b_2 i} \\ &= \frac{(a_1 a_2 + b_1 b_2) + (a_2 b_1 - a_1 b_2)i}{a_2^2 + b_2^2} \\ &= \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right) i,\end{aligned}$$

a complex number of the form $A + Bi$.

*Example A3.1* For example,

$$(2 + 3i) + (4 + 5i) = 6 + 8i,$$

$$(2 + 3i) \cdot (4 + 5i) = -7 + 22i,$$

$$\frac{2 + 3i}{4 + 5i} = \frac{2 + 3i}{4 + 5i} \cdot \frac{4 - 5i}{4 - 5i} = \frac{23}{41} + \frac{2}{41} i,$$

$$(2 + 3i) \cdot (2 - 3i) = 13.$$

## A3.2 SOLUTIONS OF POLYNOMIAL EQUATIONS

Complex conjugates arise naturally in the solution of polynomial equations, because, as the following theorem asserts, if a complex number $z_0 = a + bi$ is a solution of a polynomial equation with real coefficients, so is its complex conjugate $\bar{z}_0 = a - bi$. Before proving this statement, we collect the basic properties of the complex conjugation operation.

**Theorem A3.1**

(*a*) A complex number $z = a + bi$ equals its complex conjugate $\bar{z} = a - bi$ if and only if $z$ is real; that is, $z = a$ and $b = 0$.

Furthermore, for any complex numbers $z_1$ and $z_2$,

(b) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and
(c) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.

*Proof* Suppose $a + bi = a - bi$. Subtract $a$ from both sides of this equation: $bi = -bi$. Divide both sides by $i$: $b = -b$. But the only real number $b$ that equals its negative is $b = 0$. Follow these steps in reverse order to prove the converse. The proofs of parts $b$ and $c$ follow directly from the above definitions of the addition and multiplication operations and are left for the exercises. ■

We now show that the complex roots of a real polynomial equation occur in conjugate pairs. For example, the roots of the quadratic (3) are the conjugate complex numbers $2 + 3i$ and $2 - 3i$.

**Theorem A3.2** Consider the polynomial equation

$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n = 0, \tag{8}$$

whose coefficients $c_0, c_1, \ldots, c_n$ are *real numbers*. If $z_0 = a + bi$ is a solution of (8), then so is its complex conjugate $\bar{z}_0 = a - bi$.

*Proof* Suppose that $z_0 = a + bi$ is a specific solution of polynomial equation (8) so that

$$c_0 + c_1z_0 + c_2z_0^2 + \cdots + c_nz_0^n = 0. \tag{9}$$

Take the complex conjugate of both sides of (9):

$$\begin{aligned} 0 &= \overline{c_0 + c_1z_0 + c_2z_0^2 + \cdots + c_nz_0^n} \\ &= \overline{c_0} + \overline{c_1z_0} + \overline{c_1z_0^2} + \cdots + \overline{c_nz_0^n} && \text{(by Theorem A3.2.2)} \\ &= \overline{c_0} + \overline{c_1}\,\overline{z_0} + \overline{c_1}\,\overline{z_0}^2 + \cdots + \overline{c_n}\,\overline{z_0}^n && \text{(by Theorem A3.2.3)} \\ &= c_0 + c_1\overline{z_0} + c_2\overline{z_0}^2 + \cdots + c_n\overline{z_0}^n && \text{(by Theorem A3.2.1).} \end{aligned}$$

The last line states that $\bar{z}_0 = a - bi$ satisfies equation (9).

Of course, formula (2) gives the solution of every *second* order polynomial equation. One can use this formula to check directly that if $a + bi$ is a complex solution of $c_0 + c_1x + c_2x^2 = 0$, so is $a - bi$. (Exercise!) ■

## A3.3 GEOMETRIC REPRESENTATION

Just as real numbers can be represented geometrically on the real number line, as in Figure 2.1, a complex number $a + bi$ can be considered as an ordered pair of real numbers $(a, b)$ and represented as a point in two-dimensional $xy$-space with $x$ coordinate $a$ and $y$ coordinate $b$, as in Figure A3.1. In fact, one can think of the complex numbers $\{a + bi : a, b \in \mathbf{R}\}$ as simply another way of writing the set of all ordered pairs $\{(a, b) : a, b \in \mathbf{R}\}$ in the Cartesian plane. (The only real difference lies in the fact that there is a natural multiplication on the set of complex numbers, but not on the set of ordered pairs in the plane.) In this representation, the horizontal or $x$-axis is called the **real axis**, the vertical or $y$-axis is called the **imaginary axis**, and the Cartesian plane is called the **complex plane**. The complex numbers $a + 0i$ whose imaginary parts are zero can be identified with the real numbers on the real number line.

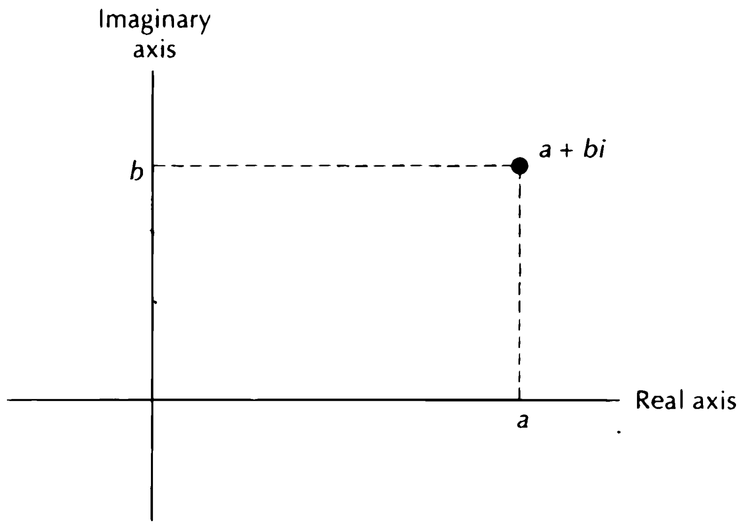



**Figure A3.1** *Representing a + bi in the complex plane.*

In the complex plane, the Euclidean distance of the point $z = a + bi$ from the origin is $\sqrt{a^2 + b^2}$ by the Pythagorean Theorem. However, by (7), this distance can be written as

$$\sqrt{a^2 + b^2} = \sqrt{(a + bi)(a - bi)} = \sqrt{z\bar{z}}. \qquad (10)$$

This real number (10) that represents the "distance" of the complex number $a + bi$ for the complex zero $0 + 0i$ is called the **norm** or **modulus** of $a + bi$. It is written as $|a + bi|$ since it generalizes the absolute value of a real number.

Consider the right triangle drawn in Figure A3.2, whose legs lie along the axes and whose hypotenuse is the vector from the origin to $a + bi$. The length $r$ of this vector is the distance from $(0, 0)$ to $(a, b)$, the modulus $\sqrt{a^2 + b^2}$ of $a + bi$. Let $\theta$ denote the angle that this vector makes with the positive real axis, as pictured in Figure A3.2. The angle $\theta$ is called the **argument** of the complex number $a + bi$. By the definitions of the cosine and sine functions in the last section,

$$\cos\theta = \frac{a}{r} \quad \text{and} \quad \sin\theta = \frac{b}{r},$$

so that

$$a + bi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta). \qquad (11)$$

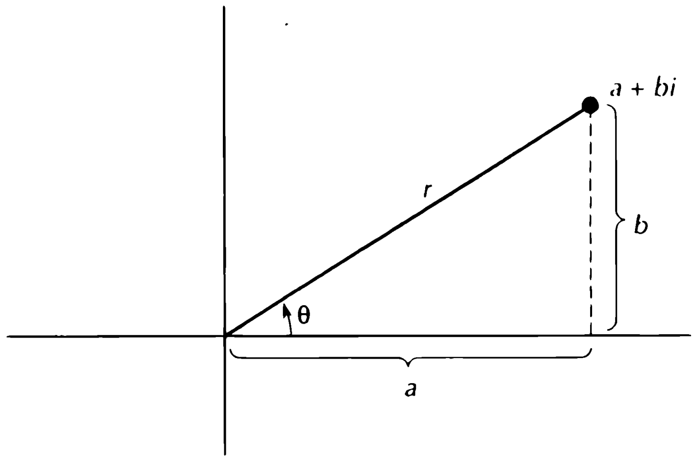
Expression (11) is called the **polar coordinate representation** of $a + bi$. It is especially helpful in evaluating powers of $a + bi$. For example,

$$\begin{aligned}(a+bi)^2 &= [r(\cos\theta + i\sin\theta)]^2 \\ &= r^2(\cos\theta + i\sin\theta)^2 \\ &= r^2(\cos^2\theta - \sin^2\theta + 2\cos\theta\sin\theta\, i) &&\text{(by (6))} \\ &= r^2(\cos 2\theta + i\sin 2\theta) &&\text{(by Theorem A2.2).}\end{aligned}$$

The identity

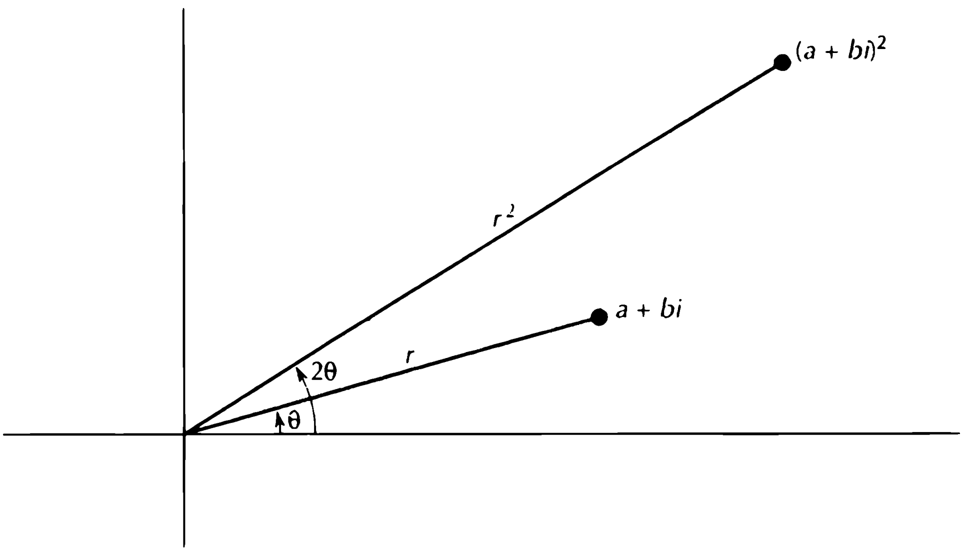
$$(a + bi)^2 = r^2(\cos 2\theta + i\sin 2\theta) \tag{12}$$

makes it easy to locate $(a + bi)^2$ in the complex plane. Its modulus $r^2$ is the square of the modulus $r$ of $a + bi$ and its argument $2\theta$ is twice the argument $\theta$ of $a + bi$, as illustrated in Figure A3.3.




*Polar coordinate representation of a + bi.*

**Figure A3.2**




*Using (12) to locate $(a + bi)^2$ in the plane.*

**Figure A3.3**

The identity (12) generalizes to all powers of $a + bi$. The generalization, called **DeMoivre's theorem**, plays an important role in the solution of linear difference equations in Chapter 23.

**Theorem A3.3 (DeMoivre's theorem)** For any complex number $a + bi$ with polar representation $r(\cos\theta + i\sin\theta)$ and any positive integer $n$,

$$(a + bi)^n = r^n(\cos n\theta + i\sin n\theta). \tag{13}$$

*Proof* The proof is by induction on $n$. We know that identity (13) holds for $n = 1, 2$. Suppose that it holds for $n = k$ so that

$$\left[r(\cos\theta + i\sin\theta)\right]^k = r^k(\cos k\theta + i\sin k\theta). \tag{14}$$

We want to show that it holds for $n = k + 1$. Now,

$$\begin{aligned}\left[r(\cos\theta + i\sin\theta)\right]^{k+1} &= r^k(\cos k\theta + i\sin k\theta)\cdot r(\cos\theta + i\sin\theta)\\ &= r^{k+1}((\cos k\theta\cos\theta - \sin k\theta\sin\theta)\\ &\qquad + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta))\end{aligned} \tag{15}$$

using the rule for the product of complex numbers. But, by Theorem A2.2,

$$\begin{aligned}\cos((k+1)\theta) &= \cos(k\theta + \theta)\\ &= \cos k\theta\cos\theta - \sin k\theta\sin\theta\\ \sin((k+1)\theta) &= \sin(k\theta + \theta)\\ &= \sin k\theta\cos\theta + \sin\theta\cos k\theta.\end{aligned}$$

Therefore, (15) can be rewritten as

$$\left[r(\cos\theta + i\sin\theta)\right]^{k+1} = r^{k+1}\left[\cos(k+1)\theta + i\sin(k+1)\theta\right].$$

This completes the inductive step and verifies that (13) holds for all integers $n$.
■

## A3.4 COMPLEX NUMBERS AS EXPONENTS

In working with linear differential equations, one often encounters the expression $e^{a+bi}$ — the number $e$ with the complex number $a + bi$ as its *exponent*. How does one interpret a complex number as an exponent?

For this discussion, the Taylor series of $x \longmapsto e^x$ will come in handy. The Taylor series representation of $e^x$ about the point $x = 0$ is the infinite series

$$e^x = 1 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots. \qquad (16)$$

The infinite series on the right-hand side of (16) *represents* the function $x \longmapsto e^x$ because it converges to $e^x$ for *every* value of $x$. Therefore, it is natural to use the power series (16) to define *complex powers* of $e$.

**Definition** For any complex number $z = a + bi$, define $e^z$ to be the limit of the infinite series

$$1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + \cdots. \qquad (17)$$

Notice that for real numbers $a$, that is, for complex numbers $a + 0i$ with zero imaginary part, this definition (17) gives the usual value $e^a$. What happens for pure imaginary numbers $0 + bi$ with zero real part? Plug $z = ib$ into (17), and separate terms without $i$'s from terms with $i$'s:

$$\begin{aligned} e^{ib} &= 1 + ib + \frac{1}{2!}(ib)^2 + \frac{1}{3!}(ib)^3 + \frac{1}{4!}(ib)^4 + \frac{1}{5!}(ib)^5 + \cdots \\ &= \left(1 - \frac{1}{2!}b^2 + \frac{1}{4!}b^4 - \cdots\right) + i\left(b - \frac{1}{3!}b^3 + \frac{1}{5!}b^5 - \cdots\right), \end{aligned}$$

since $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on. But, by the discussion at the end of Appendix A2, the series in the first set of brackets is the Taylor series for $\cos b$, and the series in the second set of brackets is the Taylor series for $\sin b$. We conclude that, by the above definition of complex exponentiation, for any real number $b$,

$$e^{ib} = \cos b + i \sin b. \qquad (18)$$

Identity (18) is called **Euler's equation**. If we take $b$ to be $\pi$ in (18), the straight-line angle in radians, then

$$e^{i\pi} = -1 \quad \text{or} \quad -e^{i\pi} = 1, \qquad (19)$$

since $\cos \pi = -1$ and $\sin \pi = 0$. Equation (19) is an aesthetic combination of the three esoteric numbers $e$, $i$, and $\pi$. For example, successful math department intramural baseball teams have been known to boast: "We're number $-e^{i\pi}$"

Complex exponentiation retains the important properties that exponentiation with real numbers enjoys. For example, for any two complex numbers $z_1$ and $z_2$,

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}. \qquad (20)$$

To prove (20), write out the Taylor series of $e^{z_1}$ and of $e^{z_2}$, multiply them out term by term, and then collect terms to recover the Taylor series of $e^{z_1+z_2}$. (Exercise.)

Finally, we use Euler's equation (18) and (20) to derive a simpler expression for $e^{a+bi}$:

$$\begin{aligned} e^{a+bi} &= e^a \cdot e^{bi} \\ &= e^a (\cos b + i \sin b). \end{aligned} \tag{21}$$

Some texts use the simple formula (21) to define complex exponentiation $e^{a+bi}$ instead of the Taylor series definition (17).

## A3.5 DIFFERENCE EQUATIONS

We motivated the existence of complex numbers by asserting that mathematicians want *every* quadratic equation (1) to have a pair of solutions, even if the proposed solutions are not real. We close this section by expanding on this theme, in particular by explaining the need for formal solutions of equation (1).

Consider the simple linear difference equation

$$x_{n+1} = 1.05\, x_n, \tag{22}$$

where $x_n$ denotes the amount of some quantity in time period $n$. For example, $x_n$ might be the money in an (inactive) savings account after $n$ years, in which case the multiplier 1.05 is 1 plus the annual 5 percent interest rate. Since the amount in the account at the end of any year is 1.05 times the amount present at the end of the previous year, an account that opens with $x_0$ dollars will contain $(1.05)^n x_0$ dollars after $n$ years. In other words, a solution of (22) is

$$x_n = (1.05)^n x_0, \tag{23}$$

where $x_0$ is the initial deposit.

Often, one studies a slightly more complex dynamic in which the amount $x_n$ present in any one year depends on the amounts present in each of the past *two* years: $x_{n-1}$ and $x_{n-2}$; for example,

$$x_n = 4x_{n-1} - 3x_{n-2}. \tag{24}$$

Motivated by the solution $(1.05)^n$ of the simpler equation (23), one looks for solutions of (24) of the form $x_n = a^n$ by plugging $x_n = a^n$ into (24) and solving for the unknown parameter $a$:

$$a^n = 4a^{n-1} - 3a^{n-2}. \tag{25}$$

Dividing both sides of (25) by $a^{n-2}$ yields

$$a^2 = 4a - 3 \quad \text{or} \quad a^2 - 4a + 3 = 0. \tag{26}$$

The solutions of this quadratic are $a = 3$ and $a = 1$; and so $x_n = 3^n$ and $x_n = 1^n$ are solutions of (24). (Check).

Now consider the difference equation

$$x_n = 4x_{n-1} - 13x_{n-2}. \tag{27}$$

Looking for a solution of the form $x_n = a^n$, we once again plug $x_n = a^n$ into (27) and are led to the quadratic equation

$$a^2 = 4a - 13, \quad \text{or} \quad a^2 - 4a + 13 = 0. \tag{28}$$

The *only* solutions of (28) are the complex numbers $2 + 3i$ and $2 - 3i$, as we saw at the beginning of this section. As is shown in Chapter 23 and in the exercises below, one can use DeMoivre's theorem to manipulate these two solutions to obtain two *real* solutions:

$$x_n = 13^{n/2} \cos n\theta_0 \quad \text{and} \quad x_n = 13^{n/2} \sin n\theta_0, \tag{29}$$

where $\theta_0 \approx 0.588$ radians (33.69°). The complex solutions of (28) are needed to begin the process of solving the dynamic (27). Furthermore, oscillatory behavior is an important phenomenon in all the sciences; but difference equations that have oscillatory solutions, like (24) or (27), are exactly the ones whose underlying quadratic equations have complex roots.

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## EXERCISES

**A3.1** Use the quadratic formula (2) to show that $2 + 3i$ and $2 - 3i$ are the solutions of (1). Then, verify that they are indeed solutions by plugging each back into (1) and carrying out the multiplications and additions.

**A3.2** Let $z_1 = 2 - 3i$, $z_2 = 3 + 4i$, and $z_3 = 1 + i$. Compute $z_1 + z_2$, $z_1 - z_3$, $z_1 \cdot z_2$, $z_1 \cdot z_3$, $z_1/z_3$, $z_1 \cdot \bar{z}_1$, $z_1^3$, $z_1 \cdot \bar{z}_3$.

**A3.3** Write $1/(a + bi)$ in the form $A + Bi$.

**A3.4** Prove parts *b* and *c* of Theorem A3.1.

**A3.5** Prove that complex conjugation preserves subtraction and division too.

**A3.6** Use the quadratic formula (2) to show directly that if $\alpha + \beta i$ is a solution of (1), so is $\alpha - \beta i$.

**A3.7** Find all three solutions of $x^3 - 1 = 0$ and of $x^3 + 1 = 0$.

**A3.8** Write $e^{1+i}$, $e^{\pi i/2}$, and $e^{2-\pi i}$ without complex numbers as exponents.

**A3.9** Use the definition (17) of complex exponentiation to prove (20).

**A3.10** Verify that $x_n = 3^n$ and $x_n = 1^n$ are solutions of difference equation (24). Show that $x_n = c_1 3^n + c_2 1^n$ is a solution for any constants $c_1$ and $c_2$.

**A3.11** *a*) Show that $x_n = k_1(2+3i)^n + k_2(2-3i)^n$ is a solution of (27) for any constants $k_1$ and $k_2$, real or complex.

*b*) In this expression for $x_n$, replace $k_1$ by any complex number $c_1 + c_2 i$, replace $k_2$ by the conjugate $c_1 - c_2 i$, and replace $(2 \pm 3i)^n$ by their polar representations from DeMoivre's theorem.

*c*) Carry out the multiplications and summations in this new expression for $x_n$ to obtain a solution that is a sum of the expressions in (29).

# Integral Calculus

## A4.1 ANTIDERIVATIVES

After a while, the student of calculus is expected to know how to compute derivatives so well that he or she can reverse the process. In this vein, an **antiderivative** of a function $f(x)$ is a function $F(x)$ whose derivative is the original $F$: $F' = f$. The function $F$ is also called the **indefinite integral** of $f$ and written $F(x) = \int f(x)\,dx$. The usual laws of differentiation yield the following table of indefinite integrals, where $C$ denotes an arbitrary constant:

$$\int af(x)\,dx = a\int f(x)\,dx \qquad \int (f+g)\,dx = \int f\,dx + \int g\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln x + C$$

$$\int e^x\,dx = e^x + C \qquad \int e^{f(x)} f'(x)\,dx = e^{f(x)} + C$$

$$\int (f(x))^n f'(x)\,dx = \frac{1}{n+1}(f(x))^{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{f(x)} f'(x)\,dx = \ln f(x) + C$$

*Example A4.1*

$$\int \left(4x^2 + x^{1/2} - \frac{3}{x}\right) dx = \frac{4x^3}{3} + \frac{x^{3/2}}{3/2} - 3\ln x + C$$

$$= \frac{4}{3}x^3 + \frac{2}{3}x^{3/2} - 3\ln x + C.$$

*Example A4.2* To illustrate the next-to-last rule in the above list, let's try to compute the antiderivative of $(x^3 + 3x^2 + 1)^3(x^2 + 2x)$. By the Power Rule, the only viable candidate is some *constant* multiple of $(x^3 + 3x^2 + 1)^4$. We take the derivative of $k(x^3 + 3x^2 + 1)^4$ and then try to find the appropriate constant $k$.

By the Power Rule,

$$\left[k(x^3 + 3x^2 + 1)^4\right]' = 4k(x^3 + 3x^2 + 1)^3 \cdot (3x^2 + 6x). \tag{1}$$

The latter expression will equal $(x^3 + 3x^2 + 1)^3 \cdot (x^2 + 2x)$ if and only if

$$4k(3x^2 + 6x) = (x^2 + 2x),$$

or

$$k = \frac{x^2 + 2x}{4(3x^2 + 6x)} = \frac{x^2 + 2x}{12(x^2 + 2x)} = \frac{1}{12}.$$

We conclude that

$$\int (x^3 + 3x^2 + 1)^3 (x^2 + 2x)\, dx = \frac{1}{12}(x^3 + 3x^2 + 1)^4 + C.$$

*Example A4.3* If we try to use this method to compute the antiderivative of $(x^3 + 3x^2 + 1)^3 \cdot (x^2 + 3x)$, we would once again look for a candidate of the form $k(x^3 + 3x^2 + 1)^4$. Using (1), the derivative of this candidate function will equal $(x^3 + 3x^2 + 1)^3 \cdot (x^2 + 3x)$ if and only if

$$4k(3x^2 + 6x) = x^2 + 3x,$$

or

$$k = \frac{x^2 + 3x}{4(3x^2 + 6x)} = \frac{x^2 + 3x}{12(x^2 + 2x)}, \tag{2}$$

which cannot be further simplified. Since we assumed $k$ constant in our differentiation (1) but found in (2) that it couldn't be constant in order for our candidate to work, we conclude that we cannot use this method to find the desired antiderivative.

## Integration by Parts

Another convenient rule for computing antiderivatives is the converse of the Product Rule. The Product Rule states that, for two differentiable functions $u(x)$ and $v(x)$,

$$(u \cdot v)' = u' \cdot v + u \cdot v'.$$

Taking antiderivatives of both sides, we find

$$u \cdot v = \int u' \cdot v + \int u \cdot v',$$

which is usually written as

$$\int u(x) \cdot v'(x)\, dx = u(x) \cdot v(x) - \int u'(x) \cdot v(x)\, dx \tag{3}$$

and is called **integration by parts**. It is especially useful when $u(x)$ is a function, like $x^k$, whose derivative $u'$ is simpler than $u$ itself and when the antiderivative $v$ of $v'$ is reasonable to work with.

*Example A4.4* Use integration by parts to integrate $xe^{2x}$ with $u(x) = x$ and $v'(x) = e^{2x}$. Since the corresponding $u'(x) = 1$ and $v(x) = \frac{1}{2}e^{2x}$, we find by (3)

$$\begin{aligned}\int xe^{2x}\, dx &= x \cdot \frac{1}{2}e^{2x} - \int 1 \cdot \frac{1}{2}e^{2x}\, dx + C\\ &= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\, dx + C\\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.\end{aligned}$$

## A4.2 THE FUNDAMENTAL THEOREM OF CALCULUS

For numbers $a$ and $b$, the **definite integral** of $f(x)$ from $a$ to $b$ is $F(b) - F(a)$, where $F(x)$ is an antiderivative of $f$. We write this as

$$\int_a^b f(x)\, dx = F(b) - F(a), \quad \text{where } F' = f.$$

The definite integral plays an important role when one wants to aggregate or sum the values of a continuous function. Consider a function that is continuous for $a \le x \le b$. Divide the interval $[a, b]$ into $N$ equal subintervals, each of length $\Delta = (b - a)/N$. Let $x_0, x_1, \ldots, x_N$ denote the endpoints of these subintervals:

$$x_0 = a,\ x_1 = a + \Delta,\ x_2 = a + 2\Delta, \ldots,\ x_N = a + N\Delta = b.$$

Form the sum

$$\begin{aligned}f(x_1)(x_1 - x_0) &+ f(x_2)(x_2 - x_1) + \cdots + f(x_N)(x_N - x_{N-1})\\ &= \sum_{i=1}^{N} f(x_i)\, \Delta.\end{aligned} \tag{4}$$

which is called a **Riemann sum**. The **Fundamental Theorem of Calculus** states that if we iterate this process, each time dividing $[a, b]$ into smaller and smaller subintervals, in the limit we obtain the definite integral $\int_a^b f(x)\,dx$:

$$\lim_{\Delta \to 0} \sum_{i=1}^{N} f(x_i)\Delta = \int_a^b f(x)\,dx.$$

*Example A4.5* Consider the function $f(x) = x^2$ between $x = 0$ and $x = 2$. If we divide $[0, 2]$ into 10 equal parts, each of length 0.2, then (4) becomes

$$f(0.2) \cdot 0.2 + f(0.4) \cdot 0.2 + f(0.6) \cdot 0.2 + \cdots$$
$$+ f(1.8) \cdot 0.2 + f(2) \cdot 0.2 = 3.08.$$

If we next divide $[0, 2]$ into 20 equal subintervals, (4) becomes

$$f(0.1) \cdot 0.1 + f(0.2) \cdot 0.1 + f(0.3) \cdot 0.1 + \cdots$$
$$+ f(1.9) \cdot 0.1 + f(2) \cdot 0.1 = 2.87.$$

If we keep increasing the number of subintervals, we obtain a sequence that tends to $8/3 \approx 2.667$. On the other hand,

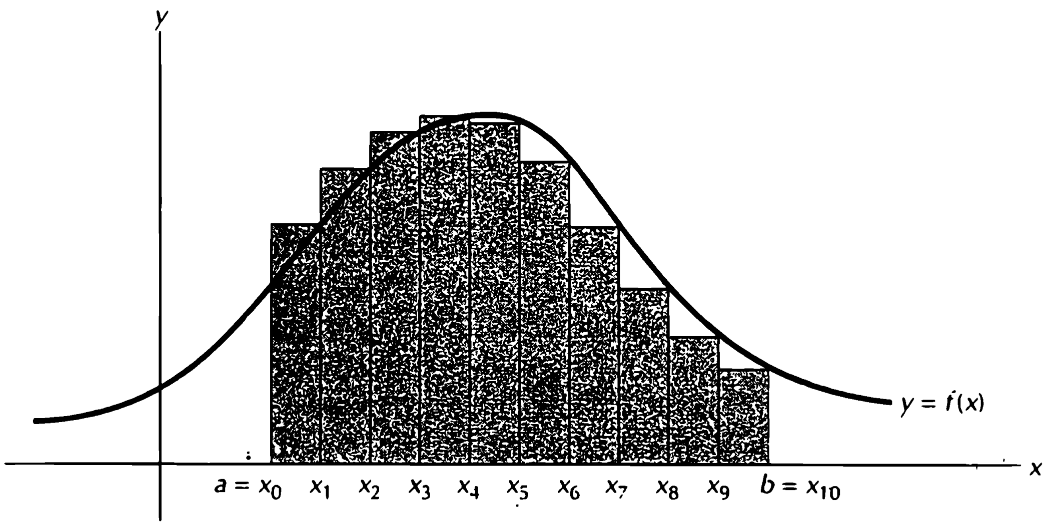
$$\int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_{x=0}^{x=2} = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}.$$

The Fundamental Theorem holds under more general circumstances. First, we need not partition $[a, b]$ into *equal* subintervals, as long as, in the limit process, the length of the largest subinterval goes to zero. Second, we need not always evaluate $f$ at the right endpoint in each subinterval $[x_{i-1}, x_i]$, as we did in (4). We can evaluate $f$ at *any* point in each subinterval.

## A4.3 APPLICATIONS

### Area under a Graph

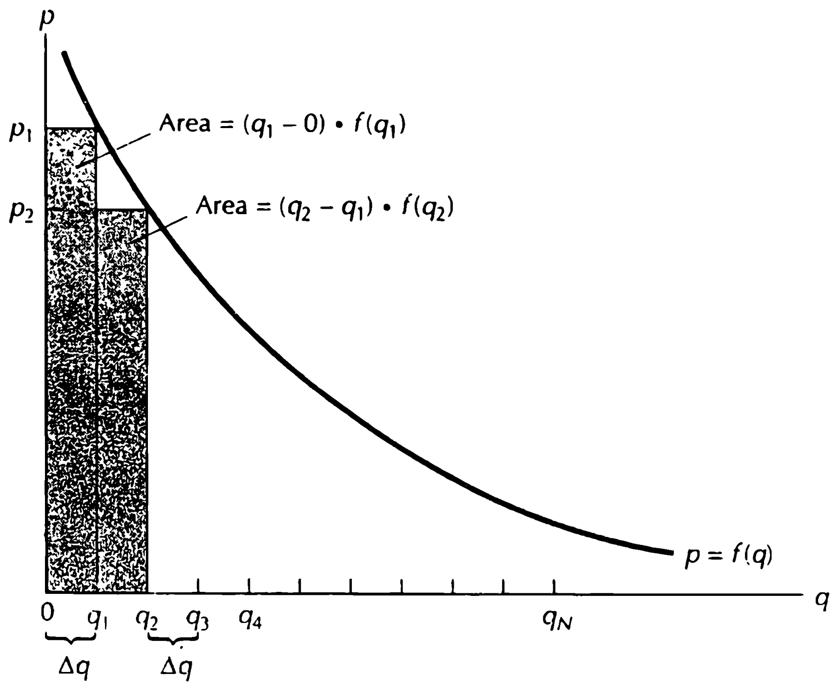
If $f$ is a positive function on $[a, b]$, as pictured in Figure A4.1, each $f(x_i)(x_i - x_{i-1})$ in the Riemann sum is the area of the rectangle that has base $[x_{i-1}, x_i]$ and height $f(x_i)$. The sum of the areas of these rectangles approximates the area under the graph of $f$. As we take finer subdivisions, the corresponding rectangles give better and better approximations to the region under the graph of $f$. By the Fundamental Theorem of Calculus, the area under the graph of $f$ from $a$ to $b$ is $\int_a^b f(x)\,dx$.




*The Riemann sums of a positive function f approximate the area under the graph.* **Figure A4.1**

## Consumer Surplus

Let $p = f(q)$ be the market (inverse) demand function relating selling price to quantity demanded for some commodity $Q$, as pictured in Figure A4.2. Think of $Q$ as a major purchase item, like a house or car, so that most consumers will buy only one of $Q$. In this case, it is convenient to think of the demand function $q = f^{-1}(p)$ as counting how many consumers have reservation price $\geq p$.




*An inverse demand function.* **Figure A4.2**

In order to compute the consumers' total willingness to pay for $Q$, let's suppose that the supplier sells $Q$ in small lots $\Delta q$, with $q_n = n \cdot \Delta q$ for $n = 1, 2, \ldots, N$. According to the inverse demand function, the supplier can charge $p_1 = f(q_1)$ dollars and still sell all of the first lot, with corresponding consumer expenditure of $q_1 \cdot p_1 = (q_1 - 0) \cdot f(q_1)$. Next, the firm offers the second $\Delta q$ lot, with $\Delta q = q_2 - q_1$. In all, $q_2$ units will be sold. According to the inverse demand function, the firm can charge $p_2 = f(q_2)$ dollars and still sell this second lot. Total consumer expenditure is now $f(q_1)(q_1 - 0) + f(q_2)(q_2 - q_1)$, an amount represented by the combined area of the two rectangles in Figure A4.2. Continue this process, offering $\Delta q$ units for sale at each step. Total consumer expenditure will be $\sum_1^n f(q_i)(q_i - q_{i-1})$. This is the Riemann sum of the inverse demand function in Figure A4.2. By the Fundamental Theorem of Calculus, if $\Delta q$ is small enough, this Riemann sum will be well approximated by the area under the graph of the inverse demand function. As a result, the area under the graph of the inverse demand function from $q = 0$ to $q = q^*$ is often called the **total willingness to pay** for $q^*$ units. On the other hand, since $q^*$ units of commodity $Q$ were actually sold at price $p^* = f(q^*)$, the actual expenditure was $q^* \cdot f(q^*)$. The total willingness to pay minus the actual expenditure is called the **consumer surplus**. Both of these concepts play important roles in the evaluation of the benefits of a project in benefit-cost analysis.

### Present Value of a Flow

Suppose that $P(t)$ represents a flow of income over time $t$ in years from $t = a$ to $t = b$. More precisely, $P(t)$ is the annual rate at which income is flowing in at time $t$. Let's compute the present value of this flow if the interest rate is $r$. Partition the time axis from $a$ to $b$ into discrete intervals of equal length $\Delta$. In subinterval $[t_{i-1}, t_i]$, income of approximately $P(t_i)(t_i - t_{i-1})$ will be achieved, since $P(t)$ is in units of income per year and $(t_i - t_{i-1})$ represents the fraction of a year under consideration. The present value of the income in this period will be $e^{-rt_i}P(t_i)(t_i - t_{i-1})$. To get the present value of the entire flow, we add all these present values over subintervals together:

$$\text{PV} = \sum_i e^{-rt_i}P(t_i)(t_i - t_{i-1}).$$

By the Fundamental Theorem of Calculus, we can use

$$\int_{t=a}^{b} e^{-rt}P(t)\,dt$$

to represent the present value of the entire flow.

## EXERCISES

**A4.1** Find the indefinite integral of each of the following functions:

*a)* $4x^6 - x^3$, *b)* $12x^2 - 6x^{1/2} + 3x^{-1/2} - x^{-1}$,

*c)* $6e^{7x}$, *d)* $e^{(3x^2+6x)}(x + 1)$,

*e)* $(x^2 + 2x + 4)^{1/2}(x + 1)$, *f)* $\dfrac{3x^{1/2} + x^{-1/2}}{x^{3/2} + x^{1/2}}$.

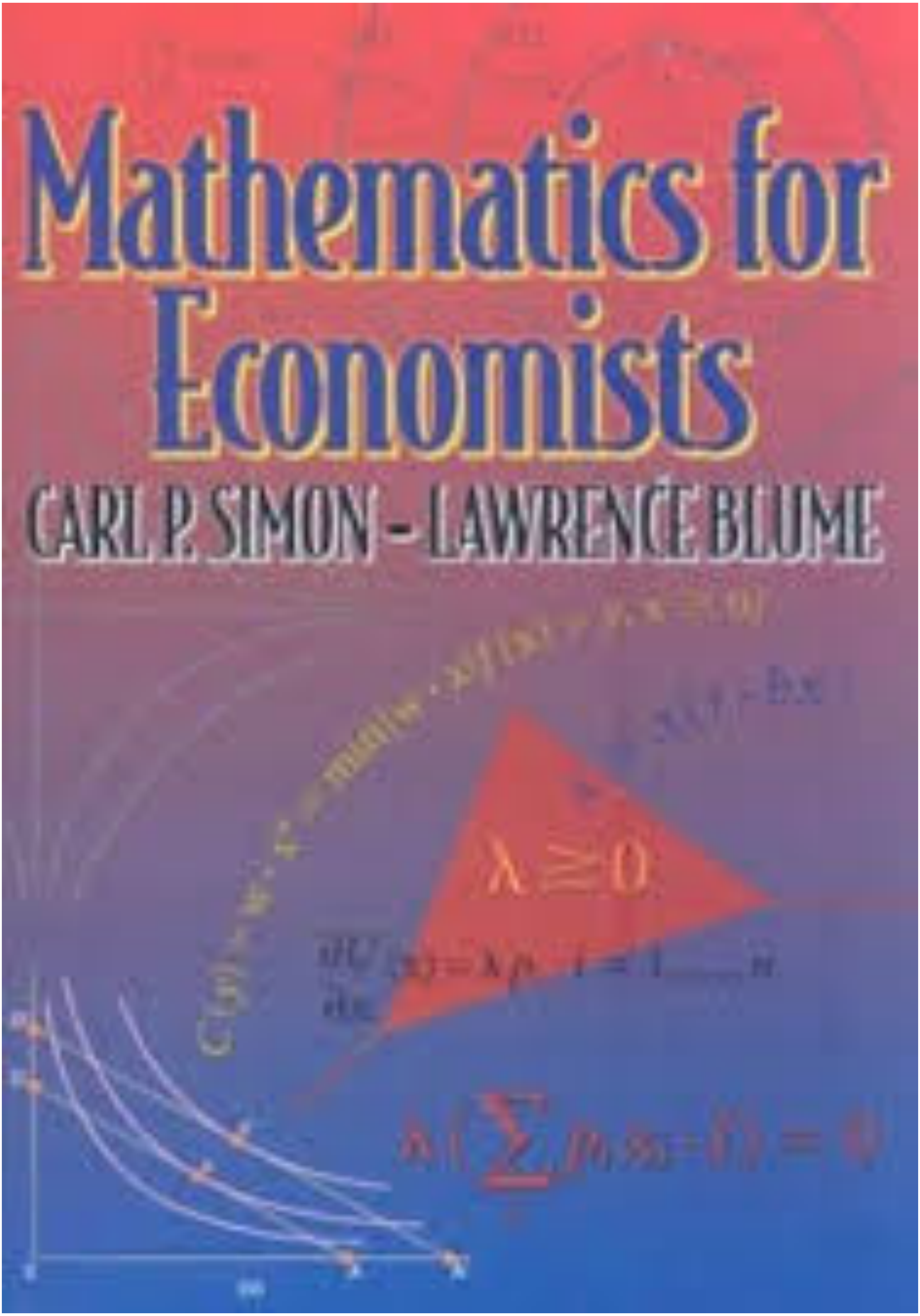
**A4.2** Use integration by parts to integrate:
*a)* $\int x \ln x \, dx$,
*b)* $\int x^2 e^{2x} \, dx$.

**A4.3** Calculate the Riemann sum of $f(x) = x^2$ from $x = 0$ to $x = 2$, dividing $[0, 2]$ into 20 equal subintervals and evaluating $f$ at the *left* endpoint of each subinterval at each step.

**A4.4** Find the area between the graph and the $x$-axis for each of the following functions:
*a)* $\sqrt{x}$, from $x = 1$ to $x = 4$;
*b)* $x \ln x$, from $x = 1$ to $x = e$.

**A4.5** Suppose the commodity $Q$ has inverse demand function $p = 3q^{-1/2}$ and that presently 100 units are being sold. What is the commodity's consumer surplus?

**A4.6** Suppose we know that the interest rate will vary over time according to the expression $r(t)$. What is the present value of a flow of income $P(t)$ from $t = a$ to $t = b$ using this variable interest rate?


Mathematics for Economists
CARL P. SIMON – LAWRENCE BLUME

# Mathematics for Economists

***Mathematics for Economists***, a new text for advanced undergraduate and beginning graduate students in economics, is a thoroughly modern treatment of the mathematics that underlies economic theory.

An abundance of applications to current economic analysis, illustrative diagrams, thought-provoking exercises, careful proofs, and a flexible organization—these are the advantages that Mathematics for Economists brings to today's classroom.

**Carl P. Simon** is professor of mathematics at the University of Michigan. He received his Ph.D. from Northwestern University and has taught at the University of California, Berkeley, and the University of North Carolina. He is the recipient of many awards for teaching, including the University of Michigan Faculty Recognition Award and the Excellence in Education Award.

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# PART V Eigenvalues and Dynamics

# PART VI Advanced Linear Algebra

# PART VIII Appendices

# Preface

For better or worse, mathematics has become the language of modern analytical economics. It quantifies the relationships between economic variables and among economic actors. It formalizes and clarifies properties of these relationships. In the process, it allows economists to identify and analyze those general properties that are critical to the behavior of economic systems.

Elementary economics courses use reasonably simple mathematical techniques to describe and analyze the models they present: high school algebra and geometry, graphs of functions of one variable, and sometimes one-variable calculus. They focus on models with one or two goods in a world of perfect competition, complete information, and no uncertainty. Courses beyond introductory micro- and macroeconomics drop these strong simplifying assumptions. However, the mathematical demands of these more sophisticated models scale up considerably. The goal of this text is to give students of economics and other social sciences a deeper understanding and working knowledge of the mathematics they need to work with these more sophisticated, more realistic, and more interesting models.

## WHY THIS BOOK?

We wrote this book because we felt that the available texts on mathematics for economists left unfilled some of the basic needs of teachers and students in this area. In particular, we tried to make the following improvements over other texts.

1. Many texts in this area focus on mathematical *techniques* at the expense of mathematical *ideas* and *intuition*, often presenting a "cookbook approach." Our book develops the student's intuition for how and why the various mathematical techniques work. It contains many more illustrations and figures than competing texts in order to build the reader's geometric intuition. It emphasizes the primary role of calculus in approximating a nonlinear function by a linear function or polynomial in order to build a simple picture of the behavior of the nonlinear function — a principle rich in geometric content.

2. Students learn how to use and apply mathematics by working with concrete examples and exercises. We illustrate every new concept and technique with worked-out examples. We include exercises at the end of every section to give students the necessary experience working with the mathematics presented.

3. This is a book on using mathematics to understand the structure of economics. We believe that this book contains more economics than any other

math-for-economists text. Each chapter begins with a discussion of the economic motivation for the mathematical concepts presented. On the other hand, this is a book on mathematics for economists, not a text of mathematical economics. We do not feel that it is productive to learn advanced mathematics and advanced economics at the same time. Therefore, we have focused on presenting an introduction to the mathematics that students need in order to work with more advanced economic models.

4. Economics is a dynamic field; economic theorists are regularly introducing or using new mathematical ideas and techniques to shed light on economic theory and econometric analysis. As active researchers in economics, we have tried to make many of these new approaches available to students. In this book we present rather complete discussions of topics at the frontier of economic research, topics like quasiconcave functions, concave programing, indirect utility and expenditure functions, envelope theorems, the duality between cost and production, and nonlinear dynamics.

5. It is important that students of economics understand what constitutes a solid proof — a skill that is learned, not innate. Unlike most other texts in the field, we try to present careful proofs of nearly all the mathematical results presented — so that the reader can understand better both the logic behind the math techniques used and the total structure in which each result builds upon previous results. In many of the exercises, students are asked to work out their own proofs, often by adapting proofs presented in the text.

An important motivation for understanding what constitutes a careful proof is the need for students to develop the ability to read an argument and to decide for themselves whether or not the conclusions really do follow from the stated hypotheses. Furthermore, a good proof tells a story; it can be especially valuable by laying bare the underlying structure of a model in such a way that one clearly sees which of the model's component parts are responsible for producing the behavior asserted in the statement of the economic principle. Some readers of this text will go on to draw conclusions from economic models in their own research. We hope that the experience of working with proofs in this text will be a valuable guide to developing one's own ability to read and write proofs.

## WHAT'S IN THIS BOOK?

At the core of modern microeconomics is the hypothesis that economic agents consciously choose their most preferred behavior according to the alternatives available to them. The area of mathematics most relevant to such a study is the maximization or minimization of a function of several variables in which the variables are constrained by equalities and inequalities. This mathematical problem in all the necessary generality, sometimes called the Lagrange multiplier problem, is a focal point of this book. (See especially Chapters 16 to 19.) The chapters of this book are arranged so that this material can be reached quickly and efficiently.

This text begins with overviews of one-variable calculus (Chapters 2 to 4) and of exponentials and logarithms (Chapter 5). One can either cover this material during the first weeks of the class or, more commonly we believe, can ask students to read it on their own as a review of the calculus they have taken. The examples and exercises in these earliest chapters should make either process relatively simple.

The analysis of solutions to optimization problems usually involves studying the solutions to the systems of equations given by the first-order conditions. The first half of this book focuses on the study of such systems of equations. We first develop a rather complete theory of the solutions of *linear* systems, focusing on such questions as: Does a solution exist? How many are there? What happens to the solution as the equations change a little? (Chapters 6 to 10.) We then turn to the study of the more realistic and more complex *nonlinear* systems (Chapters 11 to 15). We apply the metaprinciple of calculus to this study of nonlinear systems: the best way to study the behavior of the solutions of a nonlinear system is to examine the behavior of a closely related *linear* system of equations. Finally, we pull all this material together in Chapters 16 to 19 in our discussion of optimization problems — unconstrained and constrained — that is the heart of this text.

Chapters 20 through 25 treat two other basic mathematical issues that arise in the study of economic models. Chapters 20 and 21 give an in-depth presentation of *properties* of economic relationships, such as homogeneity, concavity, and quasiconcavity, while Chapter 22 illustrates how these properties arise naturally in economic models. Furthermore, there are often natural *dynamics* in economic processes: prices adjust, economies grow, policies adapt, economic agents maximize over time. Chapters 23, 24, and 25 introduce the mathematics of dynamic systems, focusing on the eigenvalues of a matrix, linear difference equations, and linear and nonlinear differential equations.

This book is laid out so that one can get to the fundamental results and consequences of constrained optimization problems as quickly as possible. In some cases, for example, in the study of determinants, limits of sequences, and compact sets, there are important topics that are slightly off the beaten path to the study of constrained optimization problems. To keep the presentation as flexible as possible, we have placed the description of these topics in the last five chapters of this book. Chapter 26 presents details about the properties of determinants outlined in Chapter 9. Chapter 27 completes the application of matrix algebra in Chapters 7 and 8 to the determination of the size of the set of solutions of a linear system, ending with a discussion of the Fundamental Theorem of Matrix Algebra. Chapter 28 presents economic applications of the Fundamental Theorem. Chapter 29 does some fine-tuning on the study of sets and sequences introduced in Chapter 12. Chapter 30 collects some of the more complex proofs of the multivariable analysis presented in Chapters 13, 14, and 15. In classroom presentations the material in any of these last five chapters can be presented: 1) right after the corresponding material in the earlier chapter, 2) at the end of the course, or 3) not at all, depending on the amount of time available or the needs of the students.

## COORDINATION WITH OTHER COURSES

Often the material in this course is taught concurrently with courses in advanced micro- and macroeconomics. Students are sometimes frustrated with this arrangement because the micro and macro courses usually start working with constrained optimization or dynamics long before these topics can be covered in an orderly mathematical presentation.

We suggest a number of strategies to minimize this frustration. First, we have tried to present the material so that a student can read each introductory chapter in isolation and get a reasonably clear idea of how to work with the material of that chapter, even without a careful reading of earlier chapters. We have done this by including a number of worked exercises with descriptive figures in every introductory chapter.

Often during the first two weeks of our first course on this material, we present a series of short modules that introduces the language and formulation of the more advanced topics so that students can easily read selected parts of later chapters on their own, or at least work out some problems from these chapters.

Finally, we usually ask students who will be taking our course to be familiar with the chapters on one-variable calculus and simple matrix theory before classes begin. We have found that nearly every student has taken a calculus course and nearly two-thirds have had some matrix algebra. So this summer reading requirement — sometimes supplemented by a review session just before classes begin — is helpful in making the mathematical backgrounds of the students in the course more homogeneous.

## ACKNOWLEDGMENTS

It is a pleasure to acknowledge the valuable suggestions and comments of our colleagues, students and reviewers: colleagues such as Philippe Artzner, Ted Bergstrom, Ken Binmore, Dee Dechert, David Easley, Leonard Herk, Phil Howrey, John Jacquez, Jan Kmenta, James Koopman, Tapan Mitra, Peter Morgan, John Nachbar, Scott Pierce, Zvi Safra, Hal Varian, and Henry Wan; students such as Kathleen Anderson, Jackie Coolidge, Don Dunbar, Tom George, Kevin Jackson, David Meyer, Ann Simon, David Simon, and John Wooders, and the countless classes at Cornell and Michigan who struggled through early drafts; reviewers such as Richard Anderson, Texas A & M University; James Bergin, Queen's University; Brian Binger, University of Arizona; Mark Feldman, University of Illinois; Roger Folsom, San Jose State University; Femida Handy, York University; John McDonald, University of Illinois; Norman Obst, Michigan State University; John Riley, University of California at Los Angeles; and Myrna Wooders, University of Toronto. We appreciate the assistance of the people at W.W. Norton, especially Drake McFeely, Catherine Wick and Catherine Von Novak. The order of the authors on the cover of this book merely reflects our decision to use different orders for different books that we write.

We dedicate this book to our wives Susan and Maralyn.

PART I

# Introduction

CHAPTER 1

# Introduction

## 1.1 MATHEMATICS IN ECONOMIC THEORY

Within the last 30 years, mathematics has emerged as the "language of economics." Today economists view mathematics as an invaluable tool at all levels of study, ranging from the statistical expression of real-world trends to the development of fully abstract economic systems. This text will provide a broad introduction to the close relationship between mathematics and economics.

On the most basic level, mathematics provides the foundations for empirical propositions about economic variables — propositions like "a 10 percent increase in the price of gasoline causes a 5 percent drop in the demand for gasoline." The mathematical expression of this relationship is the *demand function*. In particular, the above observation can be summarized by the statement "the elasticity of demand for gasoline is −0.5." We learn this empirical relationship by using techniques of statistics, which is itself a branch of mathematics. Using statistics, the economist transforms raw data from the real world into numerical generalizations such as the one just mentioned.

Furthermore, once such a statistical relationship has been formulated, it can be combined with others of the same type. Piece by piece, the economist constructs an entire network of interlocking relationships. This network enables the economist to draw conclusions about economic variables that are related to each other only indirectly. Starting with the information that the demand for gasoline (within a certain community) falls half as much as its price rises, the economist might explore how the price of gasoline is related to the price of oil, the cost of living, or the demand for electricity.

At the same time, the role of mathematics in economics extends far beyond the domain of statistical technique. For example, economists construct mathematical representations of markets and communities to understand better how they work. The very process of making a model forces the economist to pick out the most important aspects of a situation and then try to express them mathematically. The finished model provides a structured basis for further study. It is never possible to comprehend all the subtle social, cultural, and economic dimensions of a real-world situation at any one time. However, a mathematical model reduces the complexity of the real world to manageable proportions.

In fact, if we think of a model simply as the reduction and organization of subject matter for study, it is clear that models are not unique to mathematical analysis. Even social sciences such as sociology or anthropology, whose techniques are more "literary" than mathematical, rely heavily on models of some sort, in both the exploration and the presentation of their material. At the same time, there are many reasons why mathematical modeling is particularly helpful in economics.

For one thing, a mathematical model forces the economist to define terms precisely. The economist must state the underlying assumptions clearly before embarking on a complex train of thought. Right from the start, the exact nature of the abstraction the economist is working with is clear not just in the economist's mind, but in the mind of every person who reads the work. As a result, discussion about the real-world relevance of the model is likely to be sharply focused. It may even be possible to translate the theoretical model into statistical formulas, so that its validity can be tested with data from the real world.

Mathematics is used not just to organize facts, but to actively generate and explore new theoretical ideas. Often, economists use mathematical techniques such as logical deduction to derive theorems which apply to a wide variety of economic situations, instead of just to a specific local or national community. Consider, for example, the statement "competitive market allocations of resources are Pareto optimal," a theorem of central importance in most intermediate courses on microeconomic theory. In simplified form, this theorem asserts that in a competitive market system, when markets clear so that supply balances demand, any feasible change in consumption or production that improves the lot of some people will make some others worse off. In marked contrast to statements like "demand for gasoline falls half as much as the price of gasoline rises," this theorem does not originate in direct observation of the day-to-day world. Nor is it expressed statistically. Instead, it is a universal principle logically derived from an idealized, mathematical description of various markets. Because the mathematics used in developing the theorem is so far removed from direct observation, it is impossible to empirically test the theorem's ultimate truth or falsity. Only its applicability to the world economy or to the economy of a particular country or region is ever open to question.

Mathematics is not only a powerful tool for gaining insights from models of the economy; it is also needed to broaden the applicability of a model that is too narrowly constructed to be useful. Exercises in undergraduate economics texts, for instance, usually limit themselves, for the sake of simplicity, to the production or sale of two goods. The advanced student or working economist uses mathematics to extend these textbook models so that they address more information at one time — taking into account inflation, additional goods, additional competitors, or any number of other factors. At this point, let's work through a specific example of this latter use of mathematical modeling in economics. We will see how mathematics is used to increase the scope of a simple geometric model familiar from intermediate microeconomic theory.

## 1.2 MODELS OF CONSUMER CHOICE

### Two-Dimensional Model of Consumer Choice

When we study the neoclassical model of consumer choice in an intermediate microeconomic theory course, we usually assume that the consumer has only two goods to choose from — for the purposes of this discussion, gadgets and widgets. Let $x_1$ be a variable which represents the amount of gadgets purchased by our consumer, and let $x_2$ be a variable representing the consumer's purchases of widgets. The pair $(x_1, x_2)$ represents a choice of an amount for both goods and is called a "commodity bundle." If we assume that $x_1$ and $x_2$ could be any nonnegative numbers, then the set of all possible commodity bundles can be represented geometrically as the nonnegative quadrant in the plane. We will call this quadrant "commodity space." In Figure 1.1 the number of gadgets in a commodity bundle is measured on the horizontal axis, while the number of widgets is measured on the vertical axis.

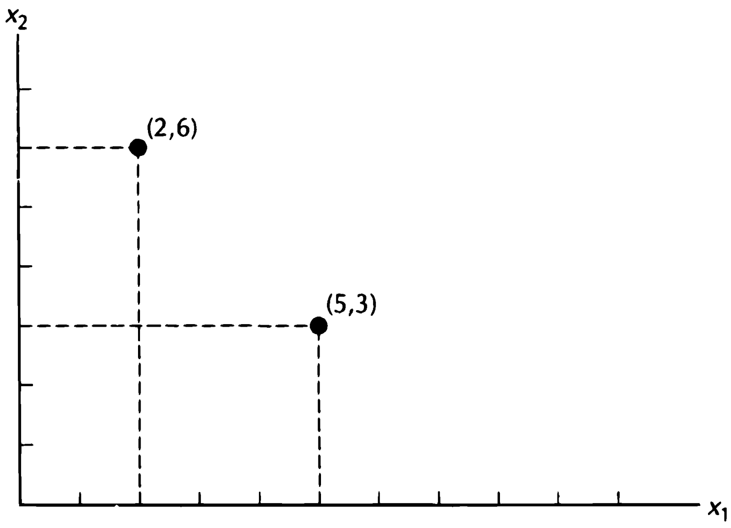



**Figure 1.1** *Two commodity bundles in commodity space.*

Consumers have preferences about commodity bundles in commodity space: Given any two commodity bundles, the consumer either prefers one bundle to the other or is indifferent between the two. If the consumer's preferences satisfy some consistency hypotheses, they can be represented by a utility function. A utility function assigns a real number to each commodity bundle. If the consumer prefers commodity bundle $(x_1, x_2)$ to bundle $(y_1, y_2)$, then the utility function assigns a higher number to $(x_1, x_2)$ than to $(y_1, y_2)$. We write $U(x_1, x_2)$ for the number assigned by the utility function to bundle $(x_1, x_2)$. We usually depict this situation by drawing a sampling of the consumer's indifference curves in commodity space, as shown in Figure 1.2. The utility function assigns the same number to all bundles on any given indifference curve. In other words, the consumer is indifferent

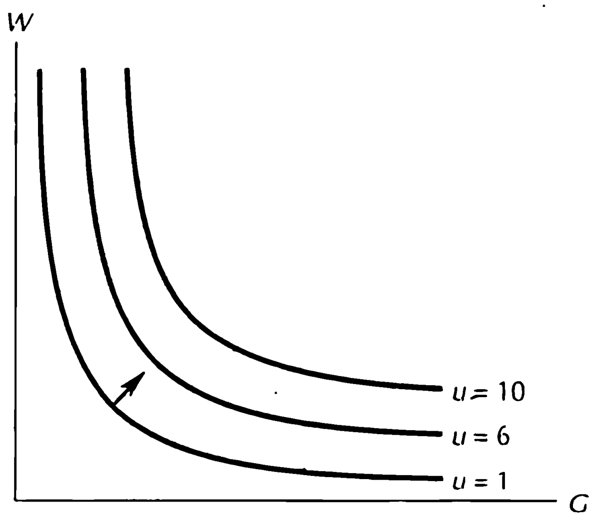


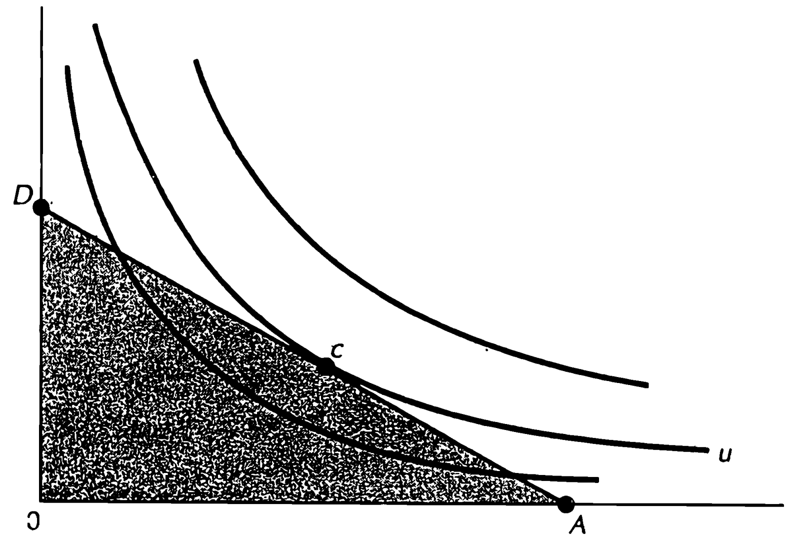

**Figure 1.2** *Indifference curves in commodity space.*

between any two bundles on the same indifference curve. The arrow in Figure 1.2 indicates the direction of preference. Commodity bundles on indifference curves far from the origin are preferred to those on indifference curves near the origin to indicate that this consumer prefers "more" to "less."

We use this representation of consumer preferences to describe the consumer's choice. Suppose a consumer is confronted with a set $B$ of commodity bundles and is asked to choose among them. The consumer will choose so as to maximize his or her utility function on the set $B$. The problem of maximizing a given function on a given set is a mathematical problem.

We have just described a very simple mathematical model of consumer choice. This model has abstracted from — ignored — many aspects of choice that, in some contexts, we would consider very important. For example, how did the consumer "learn" enough about the products to make an informed choice? How does the consumer use this information in making a choice? More generally, where did the consumer's preferences come from, and how are they influenced by the environment in which the decision is being made? Some choice activities are habitual; for example, the decision to light a cigarette. We have said nothing about habit formation in our model. Some choices are regulated by social custom; for example, the decision made by a corporate executive to wear a suit to work. Again, the role of social custom is not explicit in our model. By ignoring these and other aspects of choice, we have constructed a simple, easily understandable model of choice behavior. However, the fact that potentially important factors have been ignored may limit the usefulness of this simple model. For some applications, a more sophisticated model may be required.

Fortunately, we are not interested in using this model to explain all choice behavior. We are interested only in those choices which arise in markets. We describe these choice situations as follows: Associated with each commodity is a price: $p_1$ for the price of gadgets and $p_2$ for the price of widgets. Our consumer has $M$ dollars to divide among the two goods. The consumer cannot spend more




*Budget set OAD and indifference curves.*

**Figure 1.3**

money than he or she has. The cost of commodity bundle $(x_1, x_2)$ is $p_1x_1 + p_2x_2$. This cost cannot exceed $M$. Our theory need only apply to choice sets of the form

$$B = \{(x_1, x_2) : x_1 \geq 0,\ x_2 \geq 0,\ p_1x_1 + p_2x_2 \leq M\}.$$

These are the **budget sets** that the consumer could conceivably face.[1]

Budget sets are easy to visualize. In the commodity space, draw the line segment given by the equation $p_1x_1 + p_2x_2 = M$. Everything on or under this line is affordable. These are the points in the triangle $OAD$ in Figure 1.3.

The maximization problem is also easy to visualize. The consumer will choose from the budget set so as to be on as high an indifference curve as possible. In Figure 1.3 commodity bundle $c$ is the most preferred commodity bundle in $OAD$. Optimal bundle $c$ — sometimes called the consumer's **bundle demanded** at prices $p_1$ and $p_2$ — can be characterized by the fact that the indifference curve $u$, of which $c$ is a member, lies completely outside the budget set except at the point $c$, where it is tangent to the budget line. This is usually stated as: At $c$, the consumer's marginal rate of substitution (the slope of the indifference curve through $c$) equals the price ratio (the slope of the budget line).

In this two-dimensional setting, various thought experiments can be performed: What happens to the demand for gadgets as the price of gadgets increases? As the price of widgets increases? As income increases? These experiments are sometimes referred to as **comparative statics** problems. The experiments of increasing the consumer's income $M$ and the price $p_1$ of gadgets are performed in Figures 1.4 and 1.5.

[1]This set notation will be used throughout the book. In words, $B$ is the set of all pairs of numbers $(x_1, x_2)$ such that both numbers are nonnegative and the inequality $p_1x_1 + p_2x_2 \leq M$ is satisfied.

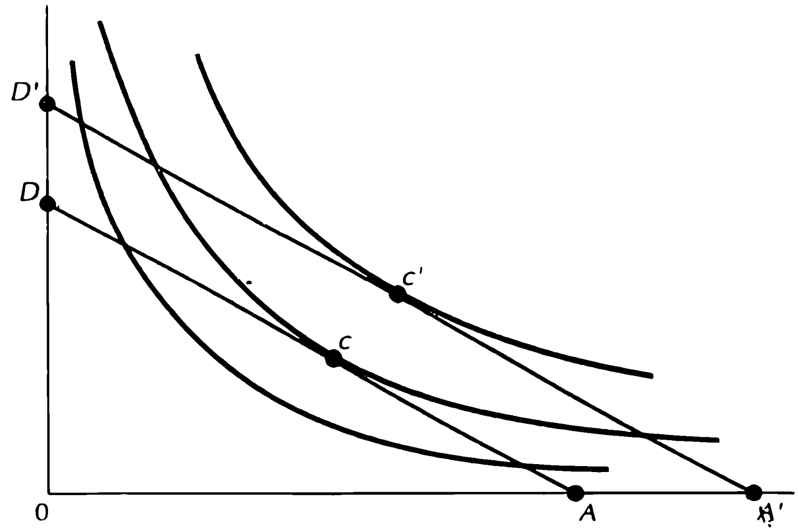



**Figure 1.4** *Effects of increasing M.*

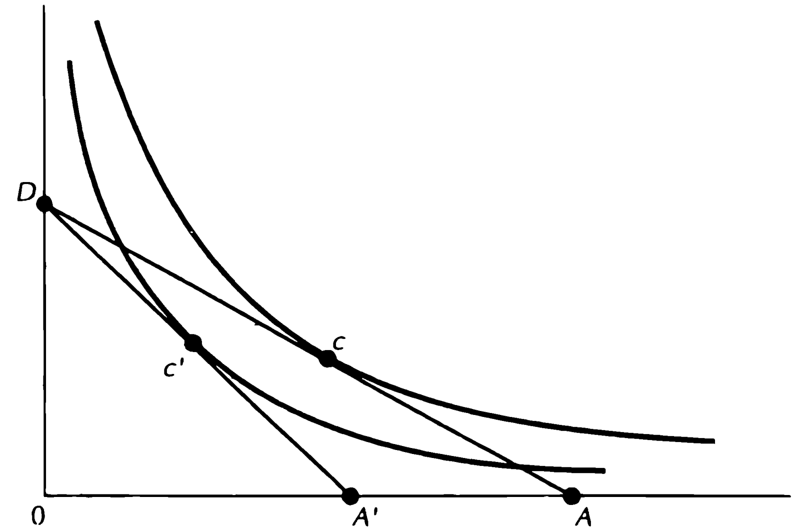



**Figure 1.5** *Effects of increasing $p_1$.*

In intermediate microeconomics classes, we record the results of these experiments in graphs, such as demand curves and Engel curves. At this point we begin to see some of the limits of this geometric approach. Even in this simplest two-good case, demand for any one good depends on three things: the price of the good, the price of the other good, and income. There is no possible way to represent these relationships simultaneously in a two-dimensional picture. Thus we are left with the rather unsatisfactory method of shifting demand curves around when we want to talk about changes in income or changes in the price of the other good. We also have no convenient way to talk rigorously about how demand is affected by the shape of indifference curves. In intermediate microeconomic theory we typically examine two polar cases — straight line (perfect substitute) indifference curves and right angle (perfect complement) indifference curves. But these are rare special cases. Furthermore, we need to know how results we might discover in this setting are affected by relaxing the hypothesis that there are only two goods.

### Multidimensional Model of Consumer Choice

None of these questions can be answered in our geometric framework. We must turn to other mathematical techniques; in particular, multivariate calculus and matrix algebra. To do this, we need to pose the problem analytically. Suppose that our model economy has $n$ goods. Commodity bundles are now lists $(x_1, x_2, \ldots, x_n)$, and a utility function assigns a number $U(x_1, \ldots, x_n)$ to each such list $(x_1, \ldots, x_n)$. The consumer's maximization problem can be stated in the following way

$$\text{maximize } U(x_1, \ldots, x_n)$$

subject to the constraints

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n \le M,$$
$$x_1 \ge 0, \ldots, x_n \ge 0.$$

The system of mathematical equations that one uses to describe the "tangency" conditions when there are $n$ unknowns rather than 2 unknowns is complex. It contains $2n + 1$ different equations and $2n + 1$ unknowns. The study of all the questions of the preceding paragraph reduces to the study of this system of equations. These questions appear in the mathematical analysis as questions about the existence of solutions to the equation system, and questions about how the solutions to the system change with changes in the parameters, such as prices and income. In this text we will discuss ideas and techniques of multivariable calculus and linear algebra that provide sharp answers to these questions.

CHAPTER 2

# One-Variable Calculus: Foundations

A central goal of economic theory is to express and understand relationships between economic variables. These relationships are described mathematically by functions. If we are interested in the effect of one economic variable (like government spending) on one other economic variable (like gross national product), we are led to the study of functions of a single variable — a natural place to begin our mathematical analysis.

The key information about these relationships between economic variables concerns how a change in one variable affects the other. How does a change in the money supply affect interest rates? Will a million dollar increase in government spending increase or decrease total production? By how much? When such relationships are expressed in terms of *linear* functions, the effect of a change in one variable on the other is captured by the "slope" of the function. For more general *nonlinear* functions, the effect of this change is captured by the "derivative" of the function. The derivative is simply the generalization of the slope to nonlinear functions. In this chapter, we will define the derivative of a one-variable function and learn how to compute it, all the while keeping aware of its role in quantifying relationships between variables.

## 2.1 FUNCTIONS ON $R^1$

### Vocabulary of Functions

The basic building blocks of mathematics are numbers and functions. In working with numbers, we will find it convenient to represent them geometrically as points on a number line. The **number line** is a line that extends infinitely far to the right and to the left of a point called the **origin**. The origin is identified with the number 0. Points to the right of the origin represent positive numbers and points to the left represent negative numbers. A basic unit of length is chosen, and successive intervals of this length are marked off from the origin. Those to the right are numbered $+1$, $+2$, $+3$, etc.; those to the left are numbered $-1$, $-2$, $-3$, etc. One can now represent any *positive* real number on the line by finding that point to the *right* of the origin whose distance from the origin in the chosen units is that number. Negative numbers are represented in the same manner, but by moving to the *left*. Consequently, every real number is represented by exactly one point on the line, and each point on the line represents one and only one number. See Figure 2.1. We write $\mathbf{R}^1$ for the set of all real numbers.

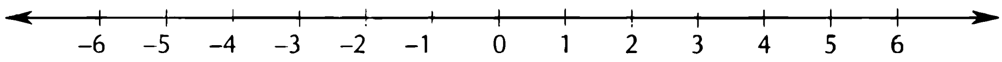



**Figure 2.1** *The number line* $\mathbf{R}^1$.

A **function** is simply a rule which assigns a number in $\mathbf{R}^1$ to each number in $\mathbf{R}^1$. For example, there is the function which assigns to any number the number which is one unit larger. We write this function as $f(x) = x + 1$. To the number 2 it assigns the number 3 and to the number $-3/2$ it assigns the number $-1/2$. We write these assignments as

$$f(2) = 3 \quad \text{and} \quad f(-3/2) = -1/2.$$

The function which assigns to any number its double can be written as $g(x) = 2x$. Write $g(4) = 8$ and $g(-3) = -6$ to indicate that it assigns 8 to 4 and $-6$ to $-3$, respectively.

Often, we use one variable, say $x$, for the input of the function and another variable, say $y$, for the output of the function. In this notation, we would write the above two functions $f$ and $g$ as

$$y = x + 1 \qquad \text{and} \qquad y = 2x,$$

respectively. The input variable $x$ is called the **independent variable**, or in economic applications, the **exogenous variable**. The output variable $y$ is called the **dependent variable**, or in economic applications, the **endogenous variable**.

## Polynomials

Analytically speaking, the simplest functions are the **monomials**, those functions which can be written as $f(x) = ax^k$ for some number $a$ and some positive integer $k$; for example,

$$f_1(x) = 3x^4, \quad f_2(x) = -x^7, \quad \text{and} \quad f_3(x) = -10x^2. \tag{1}$$

The positive integer exponent $k$ is called the **degree** of the monomial; the number $a$ is called a **coefficient.** A function which is formed by adding together monomials is called a **polynomial.** For example, if we add the three monomials in (1), we obtain the polynomial

$$h(x) = -x^7 + 3x^4 - 10x^2,$$

where we write the monomial terms of a polynomial in order of decreasing degree. For any polynomial, the highest degree of any monomial that appears in it is called the **degree** of the polynomial. For example, the degree of the above polynomial $h$ is 7.

There are more complex types of functions: **rational functions,** which are ratios of polynomials, like

$$y = \frac{x^2+1}{x-1}, \quad y = \frac{x^5+7x}{5}, \quad y = \frac{x-1}{x^3+3x+2}, \quad \text{and} \quad y = \frac{x^2-1}{x^2+1}; \qquad (2)$$

**exponential functions,** in which the variable $x$ appears as an exponent, like $y = 10^x$; **trigonometric functions,** like $y = \sin x$ and $y = \cos x$; and so on.

## Graphs

Usually, the essential information about a function is contained in its graph. The **graph** of a function of one variable consists of all points in the Cartesian plane whose coordinates $(x, y)$ satisfy the equation $y = f(x)$. In Figure 2.2 below, the graphs of the five functions mentioned above are drawn.

## Increasing and Decreasing Functions

The basic geometric properties of a function are whether it is increasing or decreasing and the location of its local and global minima and maxima. A function is **increasing** if its graph moves upward from left to right. More precisely, a function $f$ is increasing if

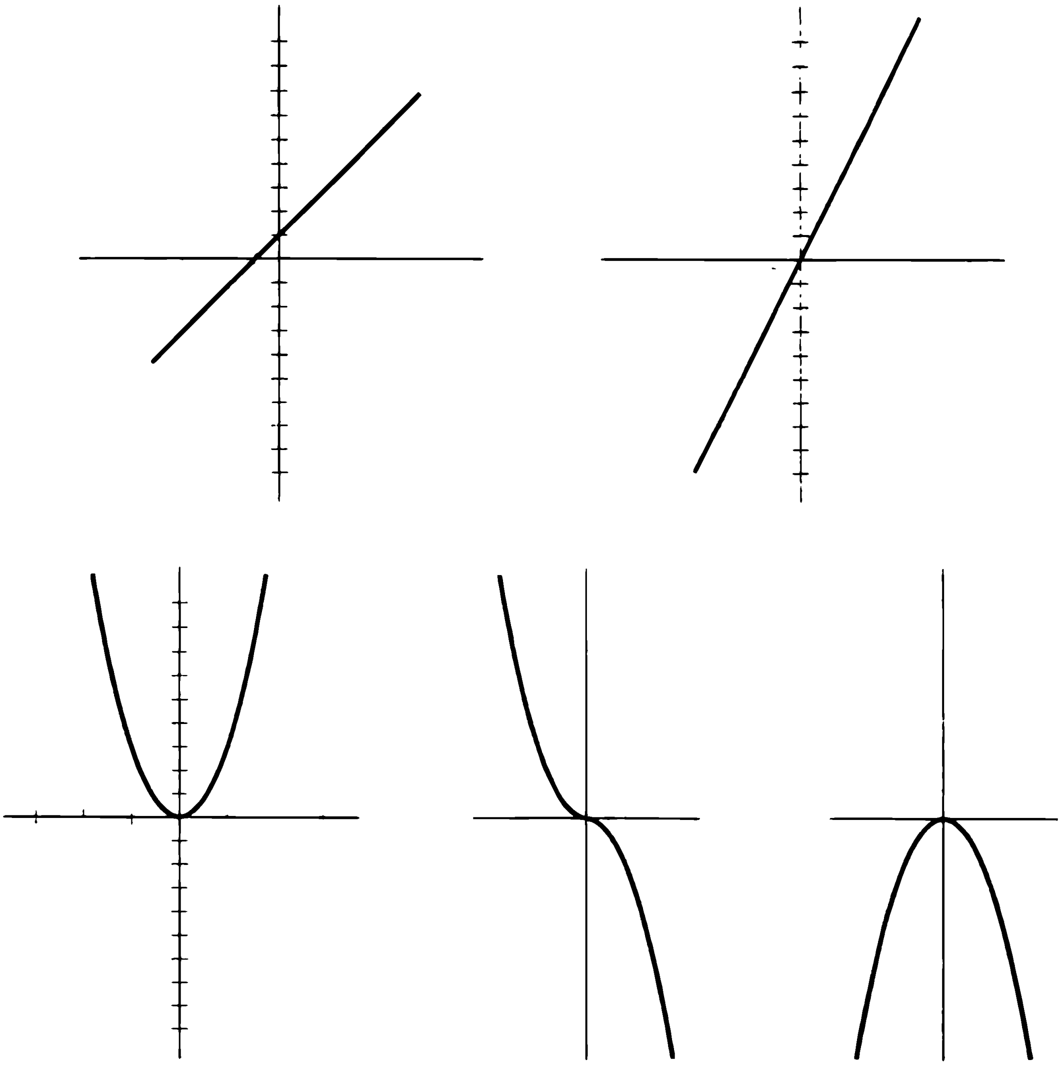
$$x_1 > x_2 \quad \text{implies that} \quad f(x_1) > f(x_2).$$

The functions in the first two graphs of Figure 2.2 are increasing functions. A function is **decreasing** if its graph moves downward from left to right, i.e., if

$$x_1 > x_2 \quad \text{implies that} \quad f(x_1) < f(x_2).$$

The fourth function in Figure 2.2, $f_2(x) = -x^7$, is a decreasing function.

The places where a function changes from increasing to decreasing and vice versa are also important. If a function $f$ changes from decreasing to increasing at $x_0$, the graph of $f$ turns upward around the point $(x_0, f(x_0))$, as in Figure 2.3. This implies that the graph of $f$ lies above the point $(x_0, f(x_0))$ around that point. Such a point $(x_0, f(x_0))$ is called a **local** or **relative minimum** of the function $f$. If the graph of a function $f$ *never* lies below $(x_0, f(x_0))$, i.e., if $f(x) \geq f(x_0)$ for all $x$, then $(x_0, f(x_0))$ is called a **global** or **absolute minimum** of $f$. The point $(0, 0)$ is a global minimum of $f_1(x) = 3x^4$ in Figure 2.2.



*The graphs of* $f(x) = x + 1$, $g(x) = 2x$, $f_1(x) = 3x^4$, $f_2(x) - x^7$, *and* $f_3(x) = -10x^2$.

**Figure 2.2**

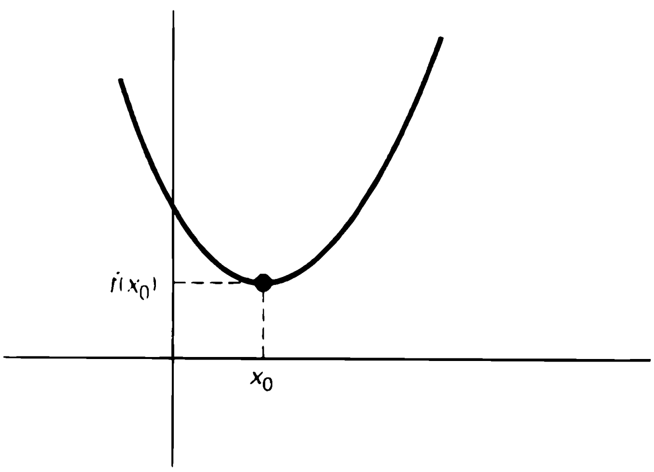



**Figure 2.3**

*Function f has a minimum at* $x_0$.

Similarly, if function $g$ changes from increasing to decreasing at $z_0$, the graph of $g$ cups downward at $(z_0, g(z_0))$ as in Figure 2.4, and $(z_0, g(z_0))$ is called a **local or relative maximum** of $g$; analytically, $g(x) \leq g(z_0)$ for all $x$ near $z_0$. If $g(x) \leq g(z_0)$ for *all* $x$, then $(z_0, g(z_0))$ is a **global or absolute maximum** of $g$. The function $f_3 = -10x^2$ in Figure 2.2 has a local and a global maximum at (0, 0).

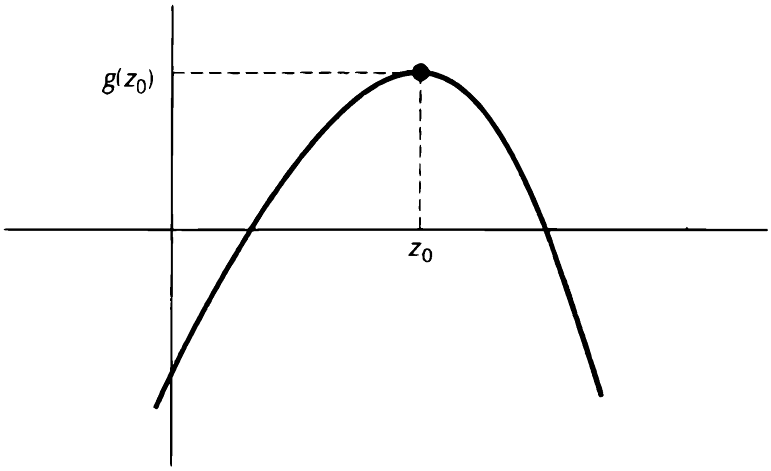



**Figure 2.4** *Function $g$ has a maximum at $z_0$.*

## Domain

Some functions are defined only on proper subsets of $\mathbf{R}^1$. Given a function $f$, the set of numbers $x$ at which $f(x)$ is defined is called the **domain** of $f$. For each of the five functions in Figure 2.2, the domain is all of $\mathbf{R}^1$. However, since division by zero is undefined, the rational function $f(x) = 1/x$ is not defined at $x = 0$. Since it is defined everywhere else, its domain is $\mathbf{R}^1 - \{0\}$. There are two reasons why the domain of a function might be restricted: mathematics-based and application-based. The most common mathematical reasons for restricting the domain are that one cannot divide by zero and one cannot take the square root (or the logarithm) of a negative number. For example, the domain of the function $h_1(x) = 1/(x^2 - 1)$ is all $x$ except $\{-1, +1\}$, and the domain of the function $h_2(x) = \sqrt{x - 7}$ is all $x \geq 7$.

The domain of a function may also be restricted by the application in which the function arises. For example, if $C(x)$ is the cost of producing $x$ cars, $x$ is naturally a positive integer. The domain of $C$ would be the set of positive integers. If we redefine the cost function so that $F(x)$ is the cost of producing $x$ *tons* of cars, the domain of $F$ is naturally the set of nonnegative real numbers:

$$\mathbf{R}_+ \equiv \{x \in \mathbf{R}^1 : x \geq 0\}.$$

The nonnegative half-line $\mathbf{R}_+$ is a common domain for functions which arise in applications.

**Notation** If the domain of the real-valued function $y = f(x)$ is the set $D \subset \mathbf{R}^1$, either for mathematics-based or application-based reasons, we write

$$f: D \to \mathbf{R}^1.$$

### Interval Notation

Speaking of subsets of the line, let's review the standard notation for intervals in $\mathbf{R}^1$. Given two real numbers $a$ and $b$, the set of all numbers between $a$ and $b$ is called an **interval**. If the endpoints $a$ and $b$ are excluded, the interval is called an **open interval** and written as

$$(a, b) \equiv \{x \in \mathbf{R}^1 : a < x < b\}.$$

If both endpoints are included in the interval, the interval is called a **closed interval** and written as

$$[a, b] \equiv \{x \in \mathbf{R}^1 : a \leq x \leq b\}.$$

If only one endpoint is included, the interval is called **half-open** (or **half-closed**) and written as $(a, b]$ or $[a, b)$. There are also five kinds of **infinite intervals**:

$$\begin{aligned}
(a, \infty) &\doteq \{x \in \mathbf{R}^1 : x > a\}, \\
[a, \infty) &= \{x \in \mathbf{R}^1 : x \geq a\}, \\
(-\infty, a) &= \{x \in \mathbf{R}^1 : x < a\}, \\
(-\infty, a] &= \{x \in \mathbf{R}^1 : x \leq a\}, \\
(-\infty, +\infty) &= \mathbf{R}^1.
\end{aligned}$$

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## EXERCISES

**2.1** For each of the following functions, plot enough points to sketch a complete graph. Then answer the following questions:
*a*) Where is the function increasing and where is it decreasing?
*b*) Find the local and global maxima and minima of these functions:

| | | |
|---|---|---|
| *i*) $y = 3x - 2$; | *ii*) $y = -2x$; | *iii*) $y = x^2 + 1$; |
| *iv*) $y = x^3 + x$; | *v*) $y = x^3 - x$; | *vi*) $y = \lvert x \rvert$. |

**2.2** In economic models, it is natural to assume that total cost functions are increasing functions of output, since more output requires more input, which must be paid for. Name two more types of functions which arise in economics models and are naturally increasing functions. Name two types of such functions that are naturally decreasing functions. Name one type that would probably change from increasing to decreasing.

2.3 The degree of a rational function is the degree of its polynomial numerator minus the degree of its polynomial denominator. Any integer — positive, negative, or zero — can be the degree of a rational function. What is the degree of each of the rational functions in (2)?

2.4 What is the domain of each of the following functions:

$$a)\ y = \frac{1}{x-1}; \qquad b)\ y = \frac{1}{\sqrt{x-1}}; \qquad c)\ y = \frac{1}{\sqrt{x^2+1}};$$

$$d)\ y = \frac{x}{x^2-1}; \qquad e)\ y = \sqrt{1-x^2}; \qquad f)\ y = \frac{1}{\sqrt{1-x^2}-1}.$$

2.5 What is the domain of each of the four rational functions in (2)?

2.6 What is the natural domain of the economics functions mentioned in Exercise 2.2?

## 2.2 LINEAR FUNCTIONS

The simplest possible functions are the polynomials of degree 0: the constant functions $f(x) = b$. Since such functions assign the same number $b$ to every real number $x$, they are too simple to be interesting. The simplest *interesting* functions are the polynomials of degree one: functions $f$ of the form

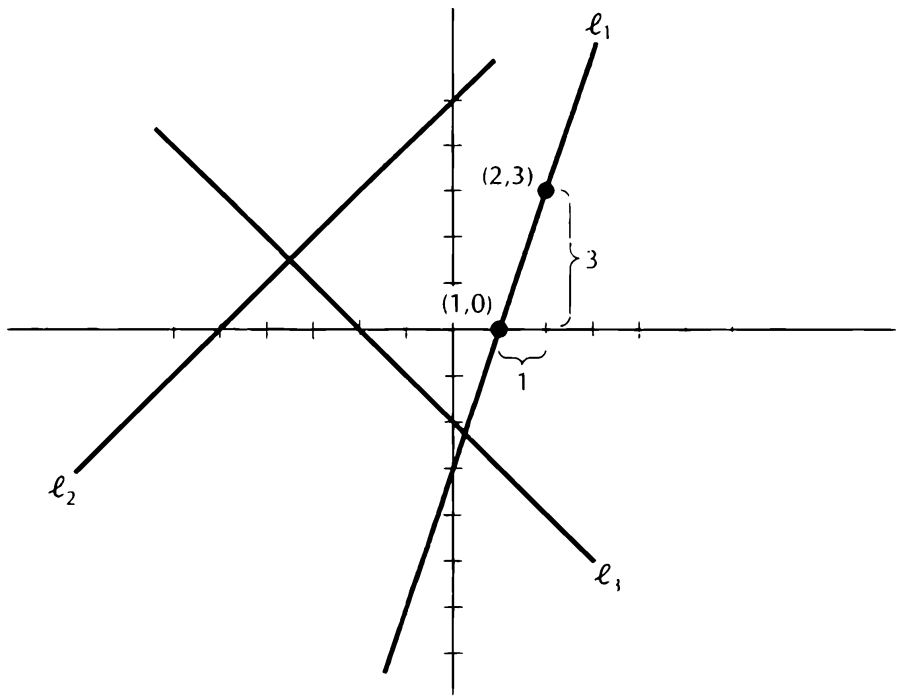
$$f(x) = mx + b.$$

Such functions are called **linear functions** because they are precisely the functions whose graphs are straight lines, as will now be demonstrated.

### The Slope of a Line in the Plane

First, let's look at the geometry of lines in the Cartesian plane. The main characteristic which distinguishes one line from another is its steepness, which we call the **slope** of the line. A natural way to measure the slope of a line is to start at any point $(x_0, y_0)$ on the line and move along the line so that the $x$-coordinate increases by *one unit*. The corresponding change in the $y$-coordinate is called the slope of the line.

*Example 2.1* For example, if we start at the point (1, 0) on the line $\ell_1$ in Figure 2.5 and move along the line until we reach the point whose $x$-coordinate is 2, we will be at the point (2, 3). Since $y$ increases by 3 units in this process, we say that the slope of line $\ell_1$ in Figure 2.5 is 3. The diagonal line $\ell_2$ in Figure 2.5 makes a 45° angle with the horizontal. Its slope is +1, since when $x$ increases by one unit, so does $y$ as one moves up $\ell_2$. The slope of line $\ell_3$, which makes an angle of −45° with the horizontal in Figure 2.5, is −1. Lines steeper than




*Some slopes in the plane.*

**Figure 2.5**

$\ell_2$ have slopes between +1 and $+\infty$. Lines which slope upward but are flatter than $\ell_2$ have slopes between 0 and +1. Horizontal lines have slope zero. Lines which slope downward from left to right, like $\ell_3$, have negative slope.

We need to convince ourselves that the slope of a line is independent of the starting point in the computation of the slope. To compute the slope of the line in Figure 2.6, we can start at the point $(x_1, y_1)$ and move to the point $(x_1 + 1, y_1')$ in triangle #1. In this case we compute the slope as $y_1' - y_1$, a number which is the ratio of the two legs of right triangle #1. If we start instead at the point $(x_2, y_2)$ and move to $(x_2 + 1, y_2')$, we compute a slope of $y_2' - y_2$, the ratio of the two legs in triangle #2. Note that the corresponding sides of triangles #1 and #2 are parallel to each other. By fundamental results of plane geometry, triangles #1 and #2 are similar to each other and therefore the ratios of corresponding sides are equal:

$$\frac{y_2' - y_2}{1} = \frac{y_1' - y_1}{1}.$$

This proves that one computes the same slope for $\ell$ no matter where one starts.

Finally, look at right triangle #3 in Figure 2.6, which is formed by moving from $(x_3, y_3)$ to $(x_4, y_4)$ along $\ell$. Coordinate $x_4$ is not necessarily $x_3 + 1$. By the same geometric analysis, triangle #3 is similar to triangles #1 and #2. Therefore, the corresponding ratios are all equal:

$$\frac{y_4 - y_3}{x_4 - x_3} = \frac{y_2' - y_2}{1} = \frac{y_1' - y_1}{1} = \text{slope of } \ell.$$

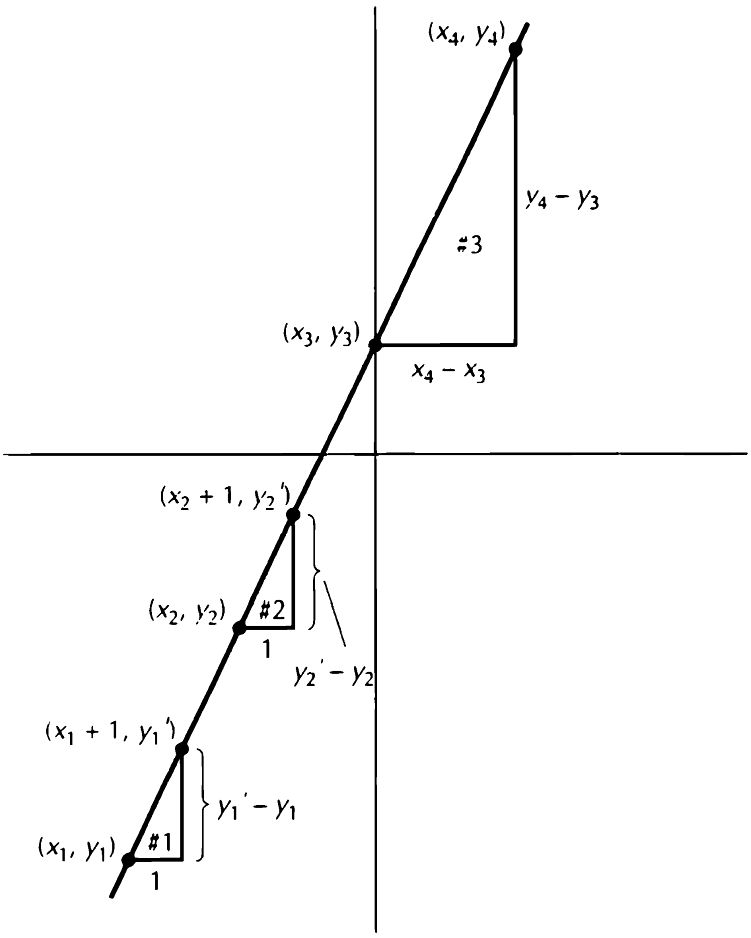



**Figure 2.6** *Computing the slope of line $\ell$ three ways.*

This use of two arbitrary points of a line to compute its slope leads to the following most general definition of the slope of a line.

**Definition** Let $(x_0, y_0)$ and $(x_1, y_1)$ be arbitrary points on a line $\ell$. The ratio

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

is called the **slope** of line $\ell$. The analysis in Figure 2.6 shows that the slope of $\ell$ is independent of the two points chosen on $\ell$. The same analysis shows that two lines are **parallel** if and only if they have the same slope.

*Example 2.2* The slope of the line joining the points (4, 6) and (0, 7) is

$$m = \frac{7 - 6}{0 - 4} = -\frac{1}{4}.$$

This line slopes downward at an angle just less than the horizontal. The slope of the line joining (4, 0) and (0, 1) is also $-1/4$; so these two lines are parallel.

### The Equation of a Line

We next find the equation which the points on a given line must satisfy. First, suppose that the line $\ell$ has slope $m$ and that the line intercepts the $y$-axis at the point $(0, b)$. This point $(0, b)$ is called the **$y$-intercept** of $\ell$. Let $(x, y)$ denote an arbitrary point on the line. Using $(x, y)$ and $(0, b)$ to compute the slope of the line, we conclude that

$$\frac{y-b}{x-0} = m,$$

or $$y - b = mx; \qquad \text{that is,} \qquad y = mx + b.$$

The following theorem summarizes this simple calculation.

> **Theorem 2.1** The line whose slope is $m$ and whose $y$-intercept is the point $(0, b)$ has the equation $y = mx + b$.

### Polynomials of Degree One Have Linear Graphs

Now, consider the general polynomial of degree one $f(x) = mx + b$. Its graph is the locus of all points $(x, y)$ which satisfy the equation $y = mx + b$. Given any two points $(x_1, y_1)$ and $(x_2, y_2)$ on this graph, the slope of the line connecting them is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1}$$

$$= \frac{m(x_2 - x_1)}{x_2 - x_1} = m.$$

Since the slope of this locus is $m$ everywhere, this locus describes a straight line. One checks directly that its $y$-intercept is $b$. So, polynomials of degree one do indeed have straight lines as their graphs, and it is natural to call such functions **linear functions**.

In applications, we sometimes need to construct the formula of the linear function from given analytic data. For example, by Theorem 2.1, the line with slope $m$ and $y$-intercept $(0, b)$ has equation $y = mx + b$. What is the equation of the line with slope $m$ which passes through a more general point, say $(x_0, y_0)$? As in the proof of Theorem 2.1, use the given point $(x_0, y_0)$ and a generic point on the line $(x, y)$ to compute the slope of the line:

$$\frac{y - y_0}{x - x_0} = m.$$

It follows that the equation of the given line is $y = m(x - x_0) + y_0$, or

If, instead, we are given two points on the line, say $(x_0, y_0)$ and $(x_1, y_1)$, we can use these two points to compute the slope $m$ of the line:

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

We can then substitute this value for $m$ in (3).

*Example 2.3* Let $x$ denote the temperature in degrees Centigrade and let $y$ denote the temperature in degrees Fahrenheit. We know that $x$ and $y$ are linearly related, that $0^o$ Centigrade or $32^o$ Fahrenheit is the freezing temperature of water and that $100^o$ Centigrade or $212^o$ Fahrenheit is the boiling temperature of water. To find the equation which relates degrees Fahrenheit to degrees Centigrade, we find the equation of the line through the points (0, 32) and (100, 212). The slope of this line is

$$\frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}.$$

This means that an increase of $1^o$ Centigrade corresponds to an increase of $9/5^o$ Fahrenheit. Use the slope 9/5 and the point (0, 32) to express the linear relationship:

$$\frac{y - 32}{x - 0} = \frac{9}{5} \quad \text{or} \quad y = \frac{9}{5}x + 32.$$

## Interpreting the Slope of a Linear Function

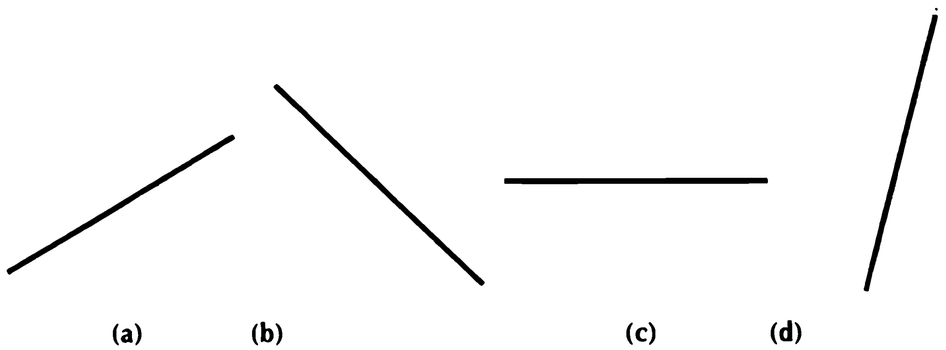
The slope of the graph of a linear function is a key concept. We will simply call it the **slope of the linear function.** Recall that the slope of a line measures how much $y$ changes as one moves along the line increasing $x$ by one unit. Therefore, the slope of a linear function $f$ measures how much $f(x)$ increases for each unit increase in $x$. It measures the rate of increase, or better, the **rate of change** of the function $f$. Linear functions have the same rate of change no matter where one starts.

For example, if $x$ measures time in hours, if $y = f(x)$ is the number of kilometers traveled in $x$ hours, and $f$ is linear, the slope of $f$ measures the number of kilometers traveled *each* hour, that is, the **speed** or **velocity** of the object under study in kilometers per hour.

This view of the slope of a linear function as its rate of change plays a key role in economic analysis. If $C = F(q)$ is a linear cost function which gives the total cost $C$ of manufacturing $q$ units of output, then the slope of $F$ measures the increase in the total manufacturing cost due to the production of one more unit. In effect, it is the cost of making one more unit and is called the **marginal cost.** It plays a central role in the behavior of profit-maximizing firms. If $u = U(x)$ is a linear utility function which measures the utility $u$ or satisfaction of having an income of $x$ dollars, the slope of $U$ measures the added utility from each additional dollar of income. It is called the **marginal utility of income**. If $y = G(z)$ is a linear function which measures the output $y$ achieved by using $z$ units of labor input, then its slope tells how much additional output can be obtained from hiring another unit of labor. It is called the **marginal product of labor**. The rules which characterize the utility-maximizing behavior of consumers and the profit-maximizing behavior of firms all involve these marginal measures, since the decisions about whether or not to consume another unit of some commodity or to produce another unit of output are based not so much on the total amount consumed or produced to date, but rather on how the *next item* consumed will affect total satisfaction or how the *next item* produced will affect revenue, cost, and profit.

## EXERCISES

**2.7** Estimate the slope of the lines in Figure 2.7.




*Four lines in the plane.*

**Figure 2.7**

**2.8** Find the formula for the linear function whose graph:

*a)* has slope 2 and $y$-intercept $(0, 3)$,
*b)* has slope $-3$ and $y$-intercept $(0, 0)$,
*c)* has slope 4 and goes through the point $(1, 1)$,
*d)* has slope $-2$ and goes through the point $(2, -2)$,
*e)* goes through the points $(2, 3)$ and $(4, 5)$,
*f)* goes through the points $(2, -4)$ and $(0, 3)$.

**2.9** Assuming that each of the following functions are linear, give an economic interpretation of the slope of the function:

*a)* $F(q)$ is the revenue from producing $q$ units of output;
*b)* $G(x)$ is the cost of purchasing $x$ units of some commodity;
*c)* $H(p)$ is the amount of the commodity consumed when its price is $p$;
*d)* $C(Y)$ is the total national consumption when national income is $Y$;
*e)* $S(Y)$ is the total national savings when national income is $Y$

## 2.3 THE SLOPE OF NONLINEAR FUNCTIONS

We have just seen that the slope of a linear function as a measure of its marginal effect is a key concept for linear functions in economic theory. However, nearly all functions which arise in applications are nonlinear ones. How do we measure the marginal effects of these nonlinear functions?

Suppose that we are studying the nonlinear function $y = f(x)$ and that currently we are at the point $(x_0, f(x_0))$ on the graph of $f$, as in Figure 2.8. We want to measure the rate of change of $f$ or the steepness of the graph of $f$ when $x = x_0$. A natural solution to this problem is to draw the tangent line to the graph of $f$ at $x_0$, as pictured in Figure 2.8. Since the tangent line very closely approximates the graph of $f$ around $(x_0, f(x_0))$, it is a good proxy for the graph of $f$ itself. Its slope, which we know how to measure, should really be a good measure for the slope of the nonlinear function at $x_0$. We note that for nonlinear functions, unlike linear functions, the slope of the tangent line will vary from point to point.

We use the notion of the tangent line approximation to a graph in our daily lives. For example, contractors who plan to build a large mall or power plant and farmers who want to subdivide large plots of land will generally assume that they are working on a *flat plane*, even though they know that they are working on a rather *round planet*. In effect, they are working with the tangent plane to the earth and the computations that they make on it will be exact to 10 or 20 decimal places — easily close enough for their purposes.

So, we define the slope of a nonlinear function $f$ at a point $(x_0, f(x_0))$ on its graph as the slope of the tangent line to the graph of $f$ at that point. We call the slope of the tangent line to the graph of $f$ at $(x_0, f(x_0))$ the **derivative** of $f$ at $x_0$, and we write it as

$$f'(x_0) \qquad \text{or} \qquad \frac{df}{dx}(x_0).$$

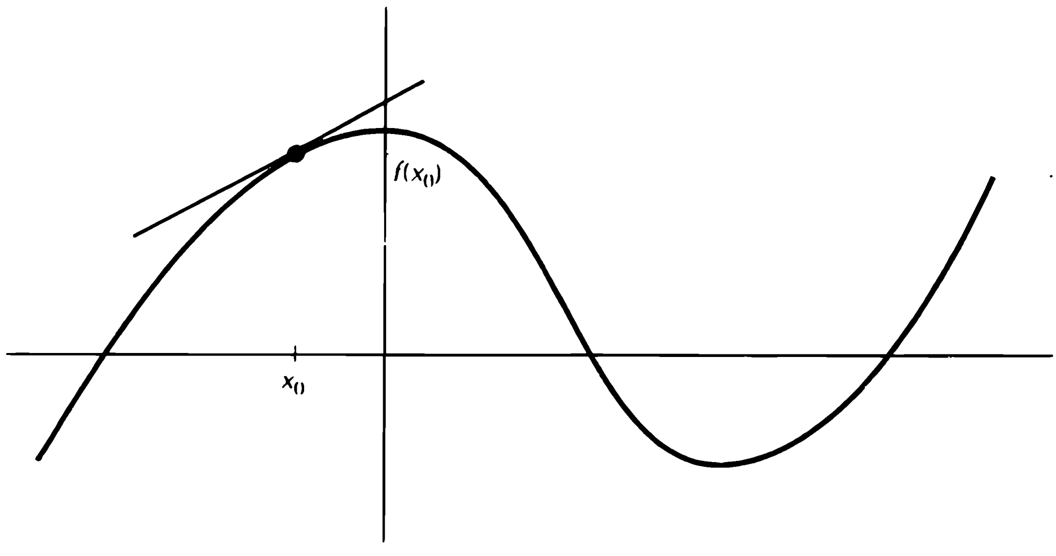
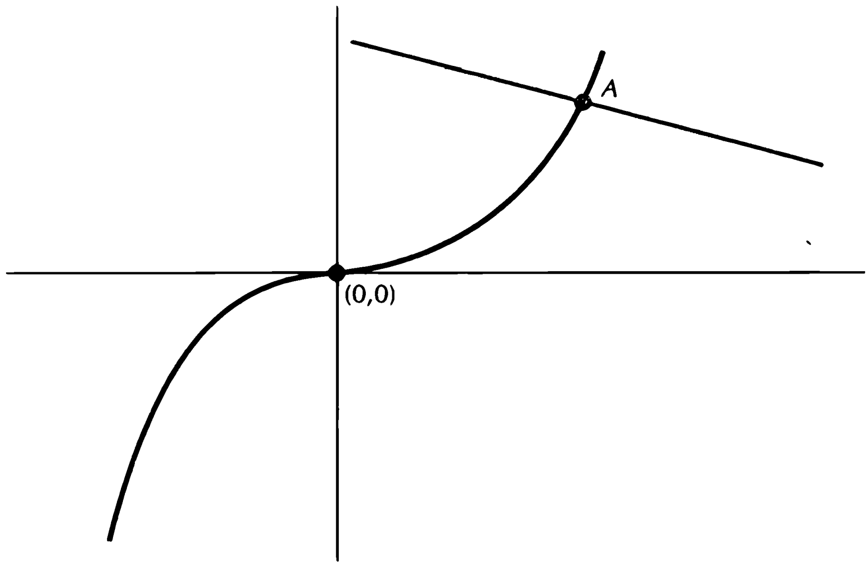



**Figure 2.8** *The graph of a nonlinear function.*

The latter notation comes from the fact that the slope is the change in $f$ divided by the change in $x$, or $\Delta f/\Delta x$, where we follow the convention of writing a capital Greek delta $\Delta$ to denote change.

Since the derivative is such an important concept, we need an analytic definition that we can work with. The first step is to make precise the definition of the tangent line to the graph of $f$ at a point. Try to formulate just such a definition. It is not "the line which meets the graph of $f$ in just one point," because point $A$ in Figure 2.9 shows that we need to add more geometry to this first attempt at a definition. We might expand our first attempt to "the line which meets the graph of $f$ at just one point, but does not cross the graph." However, the $x$-axis in Figure 2.9 is the true tangent line to the graph of $y = x^3$ at (0, 0), and it does indeed cross the graph of $x^3$. So, we need to be yet more subtle.




*A tangent line (x-axis) and a nontangent line to the graph of $x^3$.* **Figure 2.9**

Unfortunately, the only way to handle this problem is to use a limiting process. First, recall that a line segment joining two points on a graph is called a **secant line.** Now, back off a bit from the point $(x_0, f(x_0))$ on the graph of $f$ to the point $(x_0 + h_1, f(x_0 + h_1))$, where $h_1$ is some small number. Draw the secant line $\ell_1$ to the graph joining these two points, as in Figure 2.10. Line $\ell_1$ is an approximation to the tangent line. By choosing the second point closer and closer to $(x_0, f(x_0))$, we will be drawing better and better approximations to the desired tangent line. So, choose $h_2$ closer to zero than $h_1$ and draw the secant line $\ell_2$ of the graph of $f$ joining $(x_0, f(x_0))$ and $(x_0 + h_2, f(x_0 + h_2))$. Continue in this way choosing a sequence $\{h_n\}$ of small numbers which converges monotonically to 0. For each $n$, draw the secant line $\ell_n$ through the two *distinct* points on the graph $(x_0, f(x_0))$ and $(x_0 + h_n, f(x_0 + h_n))$. The secant lines $\{\ell_n\}$ geometrically approach the tangent line to the graph of $f$ at $(x_0, f(x_0))$, and their slopes approach the slope of the tangent line. Since $\ell_n$ passes through the two points $(x_0, f(x_0))$ and $(x_0 + h_n, f(x_0 + h_n))$,

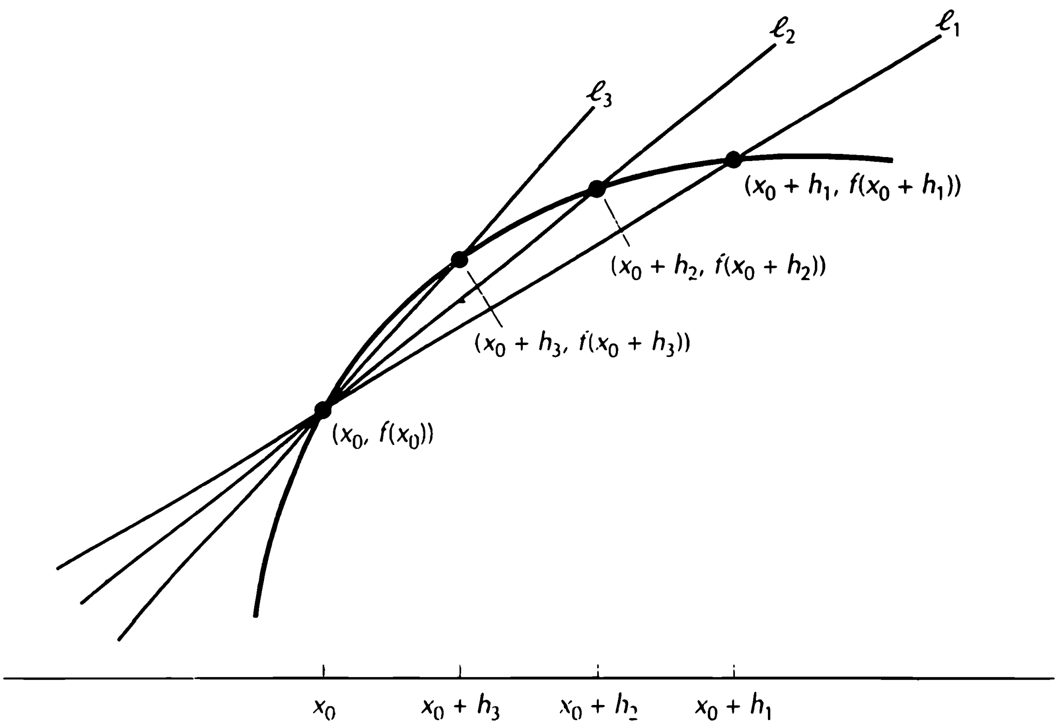



**Figure 2.10** *Approximating the tangent line by a sequence of secant lines.*

its slope is

$$\frac{f(x_0 + h_n) - f(x_0)}{(x_0 + h_n) - x_0} = \frac{f(x_0 + h_n) - f(x_0)}{h_n}.$$

Therefore, the slope of the tangent line is the limit of this process as $h_n$ converges to 0.

**Definition** Let $(x_0, f(x_0))$ be a point on the graph of $y = f(x)$. The **derivative** of $f$ at $x_0$, written

$$f'(x_0) \quad \text{or} \quad \frac{df}{dx}(x_0) \quad \text{or} \quad \frac{dy}{dx}(x_0),$$

is the slope of the tangent line to the graph of $f$ at $(x_0, f(x_0))$. Analytically,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{4}$$

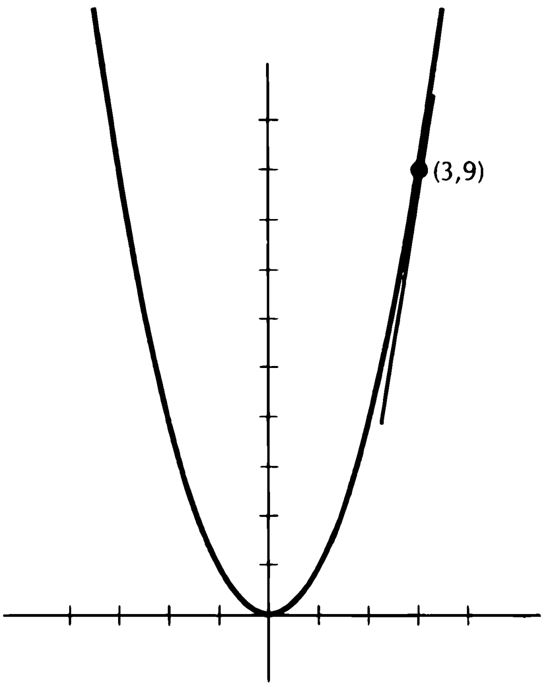
if this limit exists. When this limit does exist, we say that the function $f$ is **differentiable** at $x_0$ with derivative $f'(x_0)$.

## 2.4 COMPUTING DERIVATIVES

*Example 2.4* Let's use formula (4) to compute the derivative of the simplest nonlinear function, $f(x) = x^2$, at the point $x_0 = 3$. Since the graph of $x^2$ is fairly steep at the point (3, 9) as indicated in Figure 2.11, we expect to find $f'(3)$ considerably larger than 1. For a sequence of $h_n$'s converging to zero, choose the sequence

$$\{h_n\} = 0.1, 0.01, 0.001, \ldots, (0.1)^n, \ldots. \tag{5}$$

Table 2.1 summarizes the computations we need to make.




*Tangent line to the graph of $f(x) = x^2$ at $x_0 = 3$.* **Figure 2.11**

As $h_n \to 0$, the quotient in the last column of Table 2.1 approaches 6. Therefore, the slope of the tangent line of the graph of $f(x) = x^2$ at the point (3, 9) is 6; that is, $f'(3) = 6$.

**Table 2.1**

| $h_n$ | $x_0 + h_n$ | $f(x_0 + h_n)$ | $\dfrac{f(x_0 + h_n) - f(x_0)}{h_n}$ |
|---|---|---|---|
| 0.1 | 3.1 | 9.61 | 6.1 |
| 0.01 | 3.01 | 9.0601 | 6.01 |
| 0.001 | 3.001 | 9.006001 | 6.001 |
| 0.0001 | 3.0001 | 9.00060001 | 6.0001 |

*Example 2.5* To prove that $f'(3) = 6$, we need to show that

$$\frac{(3+h_n)^2 - 3^2}{h_n} \longrightarrow 6, \quad \text{as} \quad h_n \longrightarrow 0, \tag{6}$$

for *every* sequence $\{h_n\}$ which approaches zero, not just for the sequence (5). We now prove (6) analytically. For any $h$,

$$\frac{(3+h)^2 - 3^2}{h} = \frac{9 + 6h + h^2 - 9}{h} = \frac{h(6+h)}{h} = 6 + h,$$

which clearly converges to 6 as $h \to 0$. Now, we know for sure that $f'(3) = 6$.

*Example 2.6* Now, add one more degree of generality and compute the derivative of $f(x) = x^2$ at an arbitrary point $x_0$. Let $\{h_n\}$ be an arbitrary sequence which converges to 0 as $n \to \infty$. Then,

$$\frac{f(x_0 + h_n) - f(x_0)}{h_n} = \frac{(x_0 + h_n)^2 - x_0^2}{h_n} = \frac{x_0^2 + 2h_n x_0 + h_n^2 - x_0^2}{h_n}$$

$$= \frac{h_n(2x_0 + h_n)}{h_n} = 2x_0 + h_n,$$

which tends to $2x_0$ as $h_n \to 0$. This calculation proves the following theorem.

**Theorem 2.2** The derivative of $f(x) = x^2$ at $x_0$ is $f'(x_0) = 2x_0$.

Theorem 2.2 and Exercise 2.10 can be summarized by the statement that the derivative of $x^k$ is $kx^{k-1}$ for $k = 0, 1, 2, 3, 4$. We next prove that this statement is true for all positive integers $k$. Later, we'll see that it is true for every real number $k$, including negative numbers and fractions. In the proof of Theorem 2.2 and in the proofs in part *b* of Exercise 2.10, we used the explicit formula for $(x + h)^k$ for small integers $k$. To prove the more general result, we need the general formula for $(x + h)^k$ for any positive integer $k$, a formula we present in the following lemma. Its proof can be found in any precollege algebra text under "binomial expansion."

**Lemma 2.1** For any positive integer $k$,

$$(x+h)^k = x^k + a_1 x^{k-1} h^1 + \cdots + a_{k-1} x^1 h^{k-1} + a_k h^k, \tag{7}$$

where $$a_j = \frac{k!}{j!\,(k-j)!}, \quad \text{for} \quad j = 1, \ldots, k.$$

In particular, $a_1 = k$, $a_2 = k(k-1)/2$, and $a_k = 1$.

**Theorem 2.3** For any positive integer $k$, the derivative of $f(x) = x^k$ at $x_0$ is $f'(x_0) = kx_0^{k-1}$.

*Proof*

$$\frac{(x_0 + h)^k - x_0^k}{h} = \frac{x_0^k + kx_0^{k-1}h^1 + \frac{1}{2}k(k-1)x_0^{k-2}h^2 + \cdots + a_kh^k - x_0^k}{h}$$

$$= \frac{h(kx_0^{k-1} + \frac{1}{2}k(k-1)x_0^{k-2}h + \cdots + a_kh^{k-1})}{h}$$

$$= kx_0^{k-1} + \frac{1}{2}k(k-1)x_0^{k-2}h + \cdots + a_kh^{k-1},$$

which approaches $kx_0^{k-1}$ as $h \to 0$. ■

## Rules for Computing Derivatives

The monomials $x^k$ are the basic building blocks for a large class of functions, including all polynomials and rational functions. To compute the derivatives of functions in these larger classes, we need to know how to take the derivative of a sum, difference, product, or quotient of two functions whose derivatives we know how to compute. First, recall that we add, subtract, divide, and multiply functions in the natural way — just by performing these operations on the values of the functions. For example, if $f(x) = x^3$ and $g(x) = 6x^2$, then the sum, product, and quotient functions constructed from these two are, respectively:

$$(f + g)(x) \equiv f(x) + g(x) = x^3 + 6x^2,$$

$$(f \cdot g)(x) \equiv f(x) \cdot g(x) = x^3 \cdot 6x^2 = 6x^5,$$

$$\left(\frac{f}{g}\right)(x) \equiv \frac{f(x)}{g(x)} = \frac{x^3}{6x^2} = \frac{1}{6}x.$$

The following theorem presents the rules for differentiating the sum, difference, product, quotient, and power of functions. These rules, along with Theorem 2.3, will allow us to compute the derivatives of most elementary functions, including all polynomials and rational functions.

Part *c* of Theorem 2.4 is called the **Product Rule**, part *d* the **Quotient Rule**, and part *e* the **Power Rule**. Note that the derivative behaves very nicely with respect to sums and differences of functions, but the rules for differentiating products and quotients are a bit more complicated. The proof of each statement in Theorem 2.4 requires a rather straightforward manipulation of the definition (4) of the derivative. Parts *a* and *b* should be proved as an illustrative exercise. The

**Theorem 2.4** Suppose that $k$ is an arbitrary constant and that $f$ and $g$ are differentiable functions at $x = x_0$. Then,

$$
\begin{aligned}
a)\quad (f \pm g)'(x_0) &= f'(x_0) \pm g'(x_0),\\
b)\quad (kf)'(x_0) &= k(f'(x_0)),\\
c)\quad (f \cdot g)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0),\\
d)\quad \left(\frac{f}{g}\right)'(x_0) &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2},\\
e)\quad ((f(x))^n)' &= n(f(x))^{n-1} \cdot f'(x),\\
f)\quad (x^k)' &= kx^{k-1}.
\end{aligned}
$$

proofs of parts $c$, $d$, and $e$ are a little more subtle. The proof of part $f$ is listed as an exercise below for negative integers $k$ and will be carried out for fractions $k$ in Section 4.2.

*Example 2.7* We use Theorems 2.3 and 2.4 to calculate the derivatives of some simple functions.

$$
\begin{aligned}
a)\quad (x^7 + 3x^6 - 4x^2 + 5)' &= 7x^6 + 18x^5 - 8x,\\
b)\quad ((x^2 + 3x - 1)(x^4 - 8x))' &= (2x + 3)(x^4 - 8x)\\
&\quad + (x^2 + 3x - 1)(4x^3 - 8)\\
&= 6x^5 + 15x^4 - 4x^3 - 24x^2 - 48x + 8,\\
c)\quad \left(\frac{x^2 - 1}{x^2 + 1}\right) &= \frac{(2x)(x^2 + 1) - (x^2 - 1)(2x)}{(x^2 + 1)^2}\\
&= \frac{4x}{(x^2 + 1)^2},\\
e)\quad ((x^3 - 4x^2 + 1)^5)' &= 5(x^3 - 4x^2 + 1)^4 \cdot (3x^2 - 8x),\\
f)\quad (3x^{2/3} + 3x^{-1})' &= 2x^{-1/3} - 3x^{-2}.
\end{aligned}
$$

### EXERCISES

**2.10** *a*) Use the geometric definition of the derivative to prove that the derivative of a constant function is 0 everywhere and the derivative of $f(x) = mx$ is $f'(x) = m$ for all $x$.

*b*) Use the method of the proof of Theorem 2.2 to prove that the derivative of $x^3$ is $3x^2$ and the derivative of $x^4$ is $4x^3$.

**2.11** Find the derivative of the following functions at an arbitrary point:

*a*) $-7x^3$, *b*) $12x^{-2}$,

*c*) $3x^{-3/2}$, *d*) $\frac{1}{2}\sqrt{x}$,

*e*) $3x^2 - 9x + 7x^{2/5} - 3x^{1/2}$, *f*) $4x^5 - 3x^{1/2}$,

*g*) $(x^2 + 1)(x^2 + 3x + 2)$, *h*) $(x^{1/2} + x^{-1/2})(4x^5 - 3\sqrt{x})$,

*i*) $\dfrac{x-1}{x+1}$, *j*) $\dfrac{x}{x^2+1}$,

*k*) $(x^5 - 3x^2)^7$, *l*) $5(x^5 - 6x^2 + 3x)^{2/3}$,

*m*) $(x^3 + 2x)^3(4x + 5)^2$.

**2.12** Find the equation of the tangent line to the graph of the given function for the specified value of $x$. [Hint: Given a point on a line and the slope of the line, one can construct the equation of the line.]

*a*) $f(x) = x^2$, $x_0 = 3$; *b*) $f(x) = x/(x^2 + 2)$, $x_0 = 1$.

**2.13** Prove parts *a* and *b* of Theorem 2.4.

**2.14** In Theorem 2.3, we proved that the derivative of $y = x^k$ is $y' = kx^{k-1}$ for all positive integers $k$. Use the Quotient Rule, Theorem 2.4*d*, to extend this result to negative integers $k$.

## 2.5 DIFFERENTIABILITY AND CONTINUITY

As we saw in Section 2.3, a function $f$ is differentiable at $x_0$ if, geometrically speaking, its graph has a tangent line at $(x_0, f(x_0))$, or analytically speaking, the limit

$$\lim_{h_n \to 0} \frac{f(x_0 + h_n) - f(x_0)}{h_n} \tag{8}$$

exists and is the same for every sequence $\{h_n\}$ which converges to 0. If a function is differentiable at every point $x_0$ in its domain $D$, we say that the function is **differentiable**. Only functions whose graphs are "smooth curves" have tangent lines everywhere; in fact, mathematicians commonly use the word "smooth" in place of the word "differentiable."

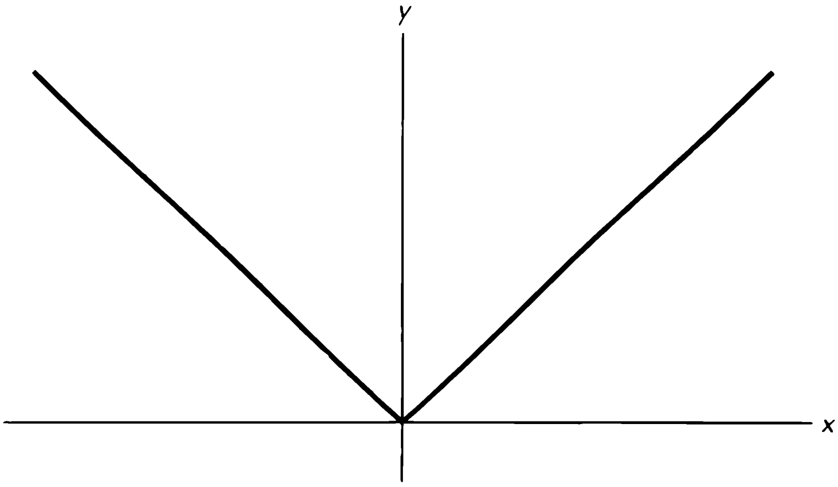



**Figure 2.12** *The graph of $f(x) = |x|$.*

### A Nondifferentiable Function

As an example of a function which is not differentiable everywhere, consider the graph of the absolute value function $f(x) = |x|$ in Figure 2.12. This graph has a sharp corner at the origin. There is no natural tangent line to this graph at (0, 0). Alternatively, as Figure 2.13 indicates, there are infinitely many lines through (0, 0) which lie on one side of the graph and hence would be candidates for the tangent line. Since the graph of $|x|$ has no well-defined tangent line at $x = 0$, the function $|x|$ is not differentiable at $x = 0$.

To see why the *analytic* definition (8) of the derivative does not work for $|x|$, substitute into (8) each of the following two sequences which converge to zero:

$$h_n = \{+.1, +.01, +.001, \ldots, +(.1)^n, \ldots\}$$
$$k_n = \{-.1, -.01, -.001, \ldots, -(.1)^n, \ldots\}.$$

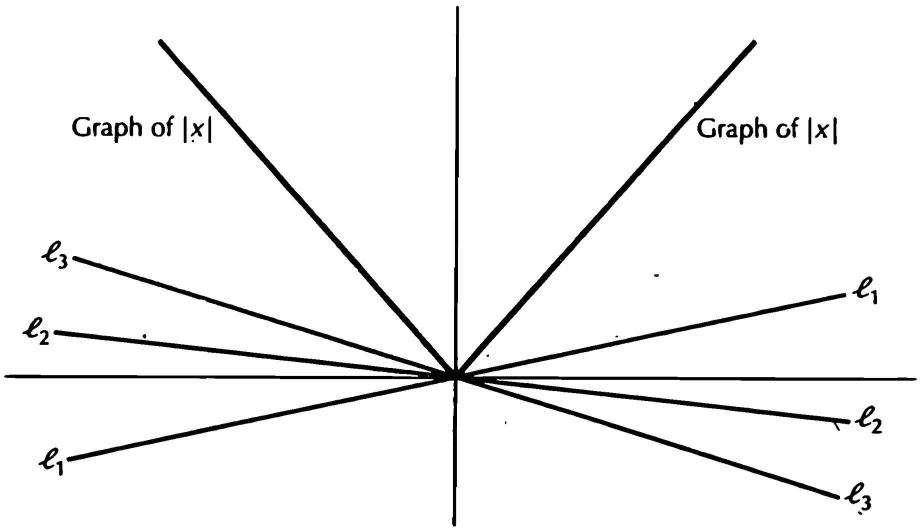



**Figure 2.13** ***Candidates for tangent lines to graph of $|x|$.***

Substituting these sequences into the definition (8) of the derivative, we compute

$$\frac{f(0+h_n)-f(0)}{h_n} = \frac{h_n - 0}{h_n} = +1 \quad \text{for all } n,$$

$$\frac{f(0+k_n)-f(0)}{k_n} = \frac{-k_n - 0}{k_n} = -1 \quad \text{for all } n.$$

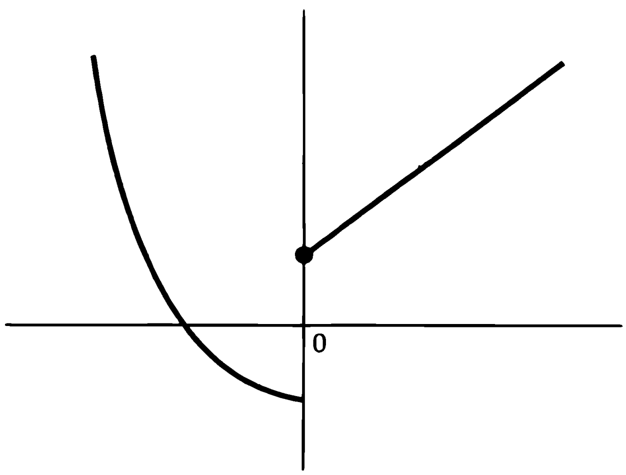
The first sequence is $\{1, 1, \ldots, 1, \ldots\}$, which clearly converges to $+1$; the second sequence is $\{-1, -1, \ldots, -1, \ldots\}$, which clearly converges to $-1$. Since different sequences which converge to 0 yield different limits in (8), the function $|x|$ does *not* have a derivative at $x = 0$.

## Continuous Functions

A property of functions more fundamental than differentiability is that of continuity. From a geometric point of view, a function is **continuous** if its graph has no breaks. Even though it is not differentiable at $x = 0$, the function $f(x) = |x|$ is still continuous. On the other hand, the function

$$g(x) = \begin{cases} x+1, & x \geq 0, \\ x^2 - 1, & x < 0, \end{cases} \tag{9}$$

whose graph is pictured in Figure 2.14, is not continuous at $x = 0$. In this case, we call the point $x = 0$ a **discontinuity** of $g$. It should be clear that the graph of a function cannot have a tangent line at a point of discontinuity. In other words, in order for a function to be differentiable, it must at least be continuous. For functions described by concrete formulas, discontinuities arise when the function is defined by different formulas on different parts of the number line and when the values of these two formulas are different at the point where the formula changes, for example, at the point $x = 0$ in (9).




*The function g given by (9) is discontinuous at x = 0.* **Figure 2.14**

The break in the graph of $g$ at the origin in Figure 2.14 means that there are points on the $x$-axis on either side of zero which are arbitrarily close to each other, but whose values under $g$ are not close to each other. Even though $(-.1)^n$ and $(+.1)^n$ are arbitrarily close to each other, $g((-.1)^n)$ is close to $-1$ while $g((+.1)^n)$ is close to $+1$. As $x$ crosses 0, the value of the function suddenly changes by two units. Small changes in $x$ do not lead to small changes in $g(x)$. This leads to the following more analytic definition of continuity.

**Definition** A function $f: D \rightarrow \mathbf{R}^1$ is **continuous** at $x_0 \in D$ if for *any* sequence $\{x_n\}$ which converges to $x_0$ in $D$, $f(x_n)$ converges to $f(x_0)$. A function is **continuous on a set** $U \subset D$ if it is continuous at every $x \in U$. Finally, we say that a function is continuous if it is continuous at every point in its domain.

The function $g(x)$ defined in (9) does not satisfy this definition at $x = 0$ because

$$\lim_{n \to \infty} f((-.1)^n) = -1; \quad \text{but} \quad f(0) = +1.$$

Most theorems in economic theory require that the function involved be continuous, if not differentiable. Continuity is a reasonable assumption in applications. For example, if $y = f(x)$ is a production function, it is reasonable to assume that a small change in the amount of input $x$ will yield a small change in the corresponding amount $y$ of output produced.

## Continuously Differentiable Functions

If $f$ is a differentiable function, its derivative $f'(x)$ is another function of $x$. It is the function which assigns to each point $x$ the slope of the tangent line to the graph of $f$ at $(x, f(x))$. We can ask whether or not this new function is continuous. Geometrically, the function $f'$ will be continuous if the tangent line to the graph of $f$ at $(x, f(x))$ changes continuously as $x$ changes. If $f'(x)$ is a continuous function of $x$, we say that the original function $f$ is **continuously differentiable**, or $\mathbf{C}^1$ for short.

*Example 2.8* Every polynomial is a continuous function. Since the derivative of a polynomial is a polynomial of one less degree, it is also continuous. Therefore, every polynomial is $C^1$.

---

### EXERCISES

**2.15** Draw a picture of the arguments in the proof that $f(x) = |x|$ does not have a derivative at $x = 0$. Show that $f$ does have a derivative at every point other than 0.

**2.16** For each of the following functions, sketch its graph and describe whether it is continuous and/or differentiable at the point of transition of its two formulas:

$a)\ y = \begin{cases} +x^2, & x \geq 0, \\ -x^2, & x < 0; \end{cases}$ $\quad b)\ y = \begin{cases} +x^2 + 1, & x \geq 0, \\ -x^2 - 1, & x < 0; \end{cases}$

$c)\ y = \begin{cases} x^3, & x \leq 1, \\ x, & x > 1; \end{cases}$ $\quad d)\ y = \begin{cases} x^3, & x < 1, \\ 3x - 2, & x \geq 1. \end{cases}$

**2.17** Which of the functions in the previous exercise are $C^1$ everywhere?

**2.18** Sketch the graph of the function $f(x) = x^{2/3}$ and describe the continuity and differentiability of this function. [The limit in (8) must be finite for the derivative to exist.]

## 2.6 HIGHER-ORDER DERIVATIVES

Let $f$ be a $C^1$ function on $\mathbf{R}^1$. Since its derivative $f'(x)$ is a continuous function on $\mathbf{R}^1$, we can ask whether or not the function $f'$ has a derivative at a point $x_0$. The derivative of $f'(x)$ at $x_0$ is called the **second derivative** of $f$ at $x_0$ and is written

$$f''(x_0) \quad \text{or} \quad \frac{d}{dx}\left(\frac{df}{dx}\right)(x_0) = \frac{d^2f}{dx^2}(x_0).$$

*Example 2.9* The derivative of the function $f(x) = x^3 + 3x^2 + 3x + 1$ is the function $f'(x) = 3x^2 + 6x + 3$. Its derivative, the second derivative of $f$, is $f''(x) = 6x + 6$.

*Example 2.10* Consider the function

$$f(x) = \begin{cases} +\frac{1}{2}x^2, & x \geq 0, \\ -\frac{1}{2}x^2, & x < 0. \end{cases} \tag{10}$$

Since both branches of $f$ equal 0 at the transition point $x = 0$, $f$ is continuous. The same kind of argument shows that $f'$ is continuous, since $f'$ can be written as

$$f'(x) = \begin{cases} +x, & x \geq 0, \\ -x, & x < 0, \end{cases}$$

differentiating both sides of (10). Since $f'$ is continuous, $f$ is $C^1$. However, because $f'(x) = |x|$, $f'$ is not differentiable at $x = 0$, and therefore $f''(x)$ does not exist at $x = 0$. The second derivative of $f$ does exist at all other points, however.

If $f$ has a second derivative everywhere, then $f''$ is a well-defined function of $x$. We will see later that the second derivative has a rich geometric meaning in terms of the shape of the graph of $f$. If $f''$ is itself a continuous function of $x$, then we say that $f$ is **twice continuously differentiable**, or $\mathbf{C^2}$ for short. Every polynomial is a $C^2$ function.

This process continues. If $f$ is $C^2$, so that $x \longmapsto f''(x)$ is a continuous function, we can ask whether $f''$ has a derivative at $x_0$. If it does, we write this derivative as

$$f'''(x_0) \quad \text{or} \quad f^{[3]}(x_0) \quad \text{or} \quad \frac{d^3 f}{dx^3}(x_0).$$

For example, for the cubic polynomial $f(x)$ in Example 2.9, $f'''(x) = 6$. If $f'''(x)$ exists for all $x$ and if $f'''(x)$ is itself a continuous function of $x$, then we say that the original function $f$ is $\mathbf{C^3}$.

This process continues for all positive integers. If $f(x)$ has derivatives of order $1, 2, \ldots, k$ and if the $k$th derivative of $f$,

$$f^{[k]}(x) = \frac{d^k f}{dx^k}(x);$$

is itself a continuous function, we say that $f$ is $\mathbf{C^k}$. If $f$ has a continuous derivative of every order, that is, if $f$ is $C^k$ for every positive integer $k$, then we say that $f$ is $C^\infty$ or "infinitely differentiable." All polynomials are $C^\infty$ functions.

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#### EXERCISES

**2.19** Sketch the graph of the function in (10).

**2.20** Compute the second derivatives of the functions in Exercise 2.11.

**2.21** Discuss the continuity and differentiability of the functions: *a)* $f(x) = x^{5/3}$; *b)* $g(x) = [x]$, the largest integer $\leq x$.

---

## 2.7 APPROXIMATION BY DIFFERENTIALS

This completes our introduction to the fundamental concepts and calculations of calculus. We turn now to the task of using the derivative to shed light on functions. In the next chapter, the derivative will be used to understand functions more completely, to graph functions more efficiently, to solve optimization problems, and to characterize the maximizer or minimizer of a function, especially in economic settings. We begin our discussion of the uses of calculus by showing how the definition of the derivative leads naturally to the construction of the linear approximation of a function. Since this material is the essence of what calculus is about, it is included in this chapter alongside the fundamental concepts of calculus.

Recall that for a linear function $f(x) = mx + b$, the derivative $f'(x) = m$ gives the slope of the graph of $f$ and measures the rate of change or *marginal change* of $f$: the increase in the value of $f$ for every unit increase in the value of $x$.

Let's carry over this marginal analysis to nonlinear functions. After all, this was one of the main reasons for defining the derivative of such an $f$. In formulating the analytic definition of the derivative of $f$, we used the fact that the slope of the tangent line to the graph at $(x_0, f(x_0))$ is well approximated by the slope of the secant line through $(x_0, f(x_0))$ and a nearby point $(x_0 + h, f(x_0 + h))$ on the graph. In symbols,

$$\frac{f(x_0 + h) - f(x_0)}{h} \approx f'(x_0) \tag{11}$$

for $h$ small, where $\approx$ means "is well approximated by" or "is close in value to."

If we set $h = 1$ in (11), then (11) becomes

$$f(x_0 + 1) - f(x_0) \approx f'(x_0); \tag{12}$$

in words, the derivative of $f$ at $x_0$ is a good approximation to the marginal change of $f$ at $x_0$. Of course, the less curved the graph of $f$ at $x_0$, the better is the approximation in (12):

*Example 2.11* Consider the production function $F(x) = \frac{1}{2}\sqrt{x}$. Suppose that the firm is currently using 100 units of labor input $x$, so that its output is 5 units. The derivative of the production function $F$ at $x = 100$,

$$F'(100) = \frac{1}{4}100^{-1/2} = \frac{1}{40} = 0.025,$$

is a good measure of the *additional* output that can be achieved by hiring one more unit of labor, the **marginal product of labor**. The actual increase in output is $F(101) - F(100) = 0.02494\cdots$, pretty close to 0.025.

Even though it is not *exactly* the increase in $y = F(x)$ due to a one unit increase in $x$, economists still use $F'(x)$ as the marginal change in $F$ because it is easier to work with the single term $F'(x)$ than with the difference $F(x + 1) - F(x)$ and because using the simple term $F'(x)$ avoids the question of what unit to use to measure a one unit increase in $x$.

What if the change in the amount of input $x$ is not exactly one unit? Return to (11) and substitute $\Delta x$, the exact change in $x$, for $h$. Multiplying (11) out yields:

$$\Delta y \equiv f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x, \tag{13}$$

or

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x, \tag{14}$$

where we write $\Delta y$ for the exact change in $y = f(x)$ when $x$ changes by $\Delta x$. Once again, the less curved the graph and/or the smaller the change $\Delta x$ in $x$, the better the approximation in (13) and (14).

*Example 2.12* Consider again a firm with production function $y = \frac{1}{2}\sqrt{x}$. Suppose it cuts its labor force $x$ from 900 to 896 units. Let's estimate the change in output $\Delta y$ and the new output $y$ at $x = 896$. We substitute

$$F(x) = \tfrac{1}{2}x^{1/2}, \quad x_0 = 900, \quad \text{and} \quad \Delta x = -4$$

into (13) and (14) and compute that

$$F'(x) = \tfrac{1}{4}x^{-1/2} \quad \text{and} \quad F'(900) = \tfrac{1}{4} \cdot \tfrac{1}{30} = \tfrac{1}{120}.$$

By (13), output will decrease by approximately

$$F'(x_0)\Delta x = \frac{1}{120} \cdot 4 = \frac{1}{30} \text{ units.}$$

By (14), the new output will be approximately

$$F(900) + F'(900)(-4) = 15 - \frac{1}{30} = 14\frac{29}{30} = 14.9666\cdots.$$

The actual new output is $F(896) = 14.9663\cdots$; once again the approximation by derivatives is a good one.

From a mathematical point of view, we can consider (14) as an effective way of approximating $f(x)$ for $x$ close to some $x_0$ where $f(x_0)$ and $f'(x_0)$ are easily computed. For example, in Example 2.12, we computed $\frac{1}{2}\sqrt{896}$, using our familiarity with $\frac{1}{2}\sqrt{900} = 15$.

*Example 2.13* Let's use (14) to estimate the cube root of 1001.5. We know that the cube root of 1000 is 10. Choose $f(x) = x^{1/3}$, $x_0 = 1000$, and $\Delta x = +1.5$. Then,

$$f'(x) = \frac{1}{3}x^{-2/3} \quad \text{and} \quad f'(1000) = \frac{1}{3}(1000)^{-2/3} = \frac{1}{300}.$$

Therefore,

$$f(1001.5) \approx f(1000) + f'(1000) \cdot 1.5 = 10 + \frac{1.5}{300} = 10.005,$$

close to the true value $10.004998\cdots$ of $\sqrt[3]{1001.5}$.

Figure 2.15

Equations (13) and (14) are merely analytic representations of the geometric fact that the tangent line $\ell$ to the graph of $y = f(x)$ at $(x_0, f(x_0))$ is a good approximation to the graph itself for $x$ near $x_0$. As Figure 2.15 indicates, the left-hand sides of (13) and (14) pertain to movement along the graph of $f$, while the right-hand sides pertain to movement along the tangent line $\ell$, because the equation of the tangent line, the line through the point $(x_0, f(x_0))$ with slope $f'(x_0)$, is

$$y = f(x_0) + f'(x_0)(x - x_0) = f(x_0) + f'(x_0)\Delta x.$$

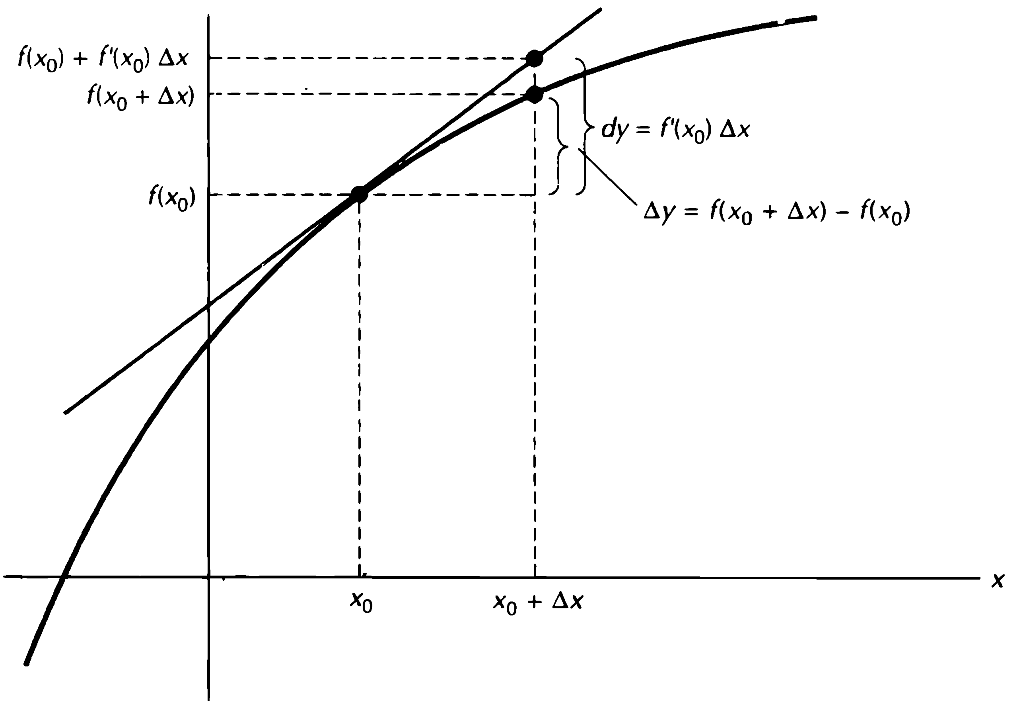
Continue to write $\Delta y$ for the actual change in $f$ as $x$ changes by $\Delta x$, that is, for the change along the graph of $f$, as in Figure 2.15. Write $dy$ for the change in $y$ along the tangent line $\ell$ as $x$ changes by $\Delta x$. Then, (13) can be written as

$$\Delta y \approx dy = f'(x_0)\Delta x.$$

We usually write $dx$ instead of $\Delta x$ when we are working with changes along the tangent line, even though $\Delta x$ is equal to $dx$. The increments $dy$ and $dx$ along the tangent line $\ell$ are called **differentials**. We sometimes write the differential $df$ in place of the differential $dy$. The equation of differentials

$$df = f'(x_0)\,dx \quad \text{or} \quad dy = f'(x_0)\,dx$$

for the variation along the tangent line to the graph of $f$ gives added weight to the notation $\dfrac{df}{dx}$ for the derivative $f'(x)$.




*Comparing $dy$ and $\Delta y$.*

## EXERCISES

**2.22** Suppose that the total cost of manufacturing $x$ units of a certain commodity is $C(x) = 2x^2 + 6x + 12$. Use differentials to approximate the cost of producing the 21st unit. Compare this estimate with the cost of actually producing the 21st unit.

**2.23** A manufacturer's total cost is $C(x) = 0.1x^3 - 0.25x^2 + 300x + 100$ dollars, where $x$ is the level of production. Estimate the effect on the total cost of an increase in the level of production from 6 to 6.1 units.

**2.24** It is estimated that $t$ years from now, the population of a certain town will be $F(t) = 40 - [8/(t + 2)]$. Use differentials to estimate the amount by which the population will increase during the next six months.

**2.25** Use differentials to approximate: *a*) $\sqrt{50}$, *b*) $\sqrt[4]{9997}$, *c*) $(10.003)^5$.

CHAPTER 3

# One-Variable Calculus: Applications

Now that we have defined the derivative and learned how to compute it in Chapter 2, let's put the derivative to work to shed light on some economic relationships. The first step in studying the relationship between two variables is to draw its graph. For nonlinear functions, this can be a difficult task. Sections 3.1 to 3.4 show how the derivative can help us draw graphs more efficiently and more accurately. Furthermore, many economic problems involve the maximization or minimization of some economic entity, for example, maximization of profits or utility and minimization of costs or risk. Section 3.5 demonstrates how to use the derivative of a function both to solve such optimization problems and to derive the economic principles behind these solutions. This chapter closes with a description in Section 3.6 of the main applications of calculus to microeconomics in the study of production, cost, profit, and demand functions.

## 3.1 USING THE FIRST DERIVATIVE FOR GRAPHING

The derivative of a function carries much information about the important properties of the function. In this section, we will see that knowing just the signs of a function's first and second derivatives and the location of only a few points on its graph usually enables us to draw an accurate graph of the function.

### Positive Derivative Implies Increasing Function

As we discussed at the beginning of the last chapter, the most basic information about a function is whether it is increasing or decreasing and where it changes from one to the other. This is exactly the information we get from the sign of the first derivative of the function.

**Theorem 3.1** Suppose that the function $f$ is continuously differentiable at $x_0$. Then,

(*a*) if $f'(x_0) > 0$, there is an open interval containing $x_0$ on which $f$ is increasing, and
(*b*) if $f'(x_0) < 0$, there is an open interval containing $x_0$ on which $f$ is decreasing.

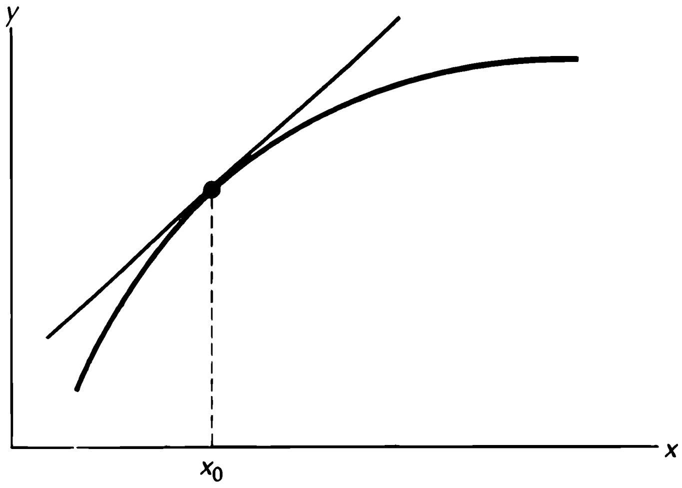



**Figure 3.1** *If $f'(x_0) > 0$, the graph of f slopes upward.*

*Proof* We will sketch a geometric and an analytic proof of part *a*. The proof of part *b* is analogous to that of part *a*.

Figure 3.1 illustrates the simple geometric picture behind the statement of Theorem 3.1. Since $f'(x_0)$ is the slope of the tangent line to the graph of $f$ at $x_0$, $f'(x_0) > 0$ means that the tangent line slopes upward and therefore the graph to which it is tangent slopes upward too.

From an analytic point of view, since $f$ is differentiable at $x_0$,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) > 0.$$

this inequality implies that if $h$ is small and positive, $f(x_0 + h) - f(x_0)$ is positive too. If we write $x_1$ for $x_0 + h$, this statement becomes: for $x_1$ near $x_0$

$$x_1 > x_0 \Longrightarrow f(x_1) > f(x_0).$$

This means that $f$ is an increasing function near $x_0$. ■

The following theorem is the *global* version of Theorem 3.1. The proofs of the first two statements follow from the simple observation that if a function is increasing at each point on an interval, it is increasing on the whole interval. The last two statements follow directly from the first two.

**Theorem 3.2** Let $f$ be a continuously differentiable function on domain $D \subset \mathbf{R}^1$

If $f' > 0$ on interval $(a, b) \subset D$, then $f$ is increasing on $(a, b)$.
If $f' < 0$ on interval $(a, b) \subset D$, then $f$ is decreasing on $(a, b)$.
If $f$ is increasing on $(a, b)$, then $f' \geq 0$ on $(a, b)$.
If $f$ is decreasing on $(a, b)$, then $f' \leq 0$ on $(a, b)$.

Theorems 3.1 and 3.2 are useful in applications in which one has some information about the derivatives of $f$ and needs to know whether or not $f$ is increasing. We will present an example of this phenomenon in Section 3.6 when we prove that if marginal cost is greater than average cost, then average cost is increasing.

## Using First Derivatives to Sketch Graphs

To use Theorem 3.2 to sketch the graph of a given function $f$, we need to find the intervals where $f' > 0$ and the intervals where $f' < 0$. To accomplish this:

(1) First find the points at which $f'(x) = 0$ or $f'$ is not defined. Such points are called **critical points** of $f$. Hopefully, the function under consideration has only finitely many critical points $x_1, x_2, \ldots, x_k$.
(2) Evaluate the function at each of these critical points $x_1, x_2, \ldots, x_k$, and plot the corresponding points on the graph.
(3) Then, check the sign of $f'$ on each of the intervals

$$(-\infty, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k), (x_k, \infty).$$

On any one of these intervals, $f'$ is defined and nonzero. Since $f'(x) = 0$ only when $x = x_1, \ldots, x_k$ and since $f'$ is continuous, $f'$ cannot change sign on any of these intervals; it must be either *always* negative or *always* positive on each. To see whether $f'$ is positive or negative on any one of these intervals, one need only check the sign of $f'$ at one convenient point in that interval.
(4) If $f' > 0$ on interval $I$, draw the graph of $f$ increasing over $I$. If $f' < 0$ on $I$, draw a decreasing graph over $I$.

*Example 3.1* Consider the cubic function $f(x) = x^3 - 3x$. One easily computes that

$$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1),$$

which equals zero only at $x = -1, +1$. These are the critical points of $f$. The corresponding points on the graph of $f$ are $(-1, 2)$ and $(1, -2)$. Next, we check the sign of $f'$ on the three intervals obtained by deleting the critical points from $\mathbf{R}^1$:

$$J_1 = (-\infty, -1), \quad J_2 = (-1, +1), \quad \text{and} \quad J_3 = (+1, +\infty).$$

Choosing a point from each of these three intervals, we note that:

(*a*) $f'(-2) = 9 > 0$, so $f' > 0$ on $J_1$ and $f$ is increasing on $J_1$;
(*b*) $f'(0) = -3 < 0$, so $f' < 0$ on $J_2$ and $f$ is decreasing on $J_2$; and
(*c*) $f'(+2) = 9 > 0$, so $f' > 0$ on $J_3$ and $f$ is increasing on $J_3$.

We have summarized this information on the number line in Figure 3.2, and we have sketched the graph of $f$ in Figure 3.3.

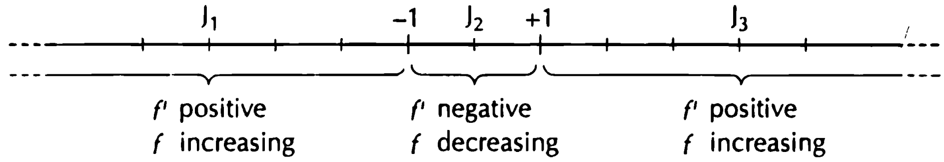



**Figure 3.2** *A summary of first derivative information for $f(x) = x^3 - 3x$.*

Since it is easy to compute, you should include the $y$-intercept $(0, f(0))$ on the graph of $f$ as you sketch it. The $y$-intercept for the function in Example 3.1 is the origin $(0, 0)$. Occasionally, it is straightforward to calculate the $x$-intercepts of $f$, the places where $f(x) = 0$. When this calculation is simple, plot these points on the graph too. For the cubic function in Example 3.1, the $x$-intercepts are the solutions of $f(x) = x(x^2 - 3) = 0$, namely $x = -\sqrt{3},\ 0,\ +\sqrt{3}$.

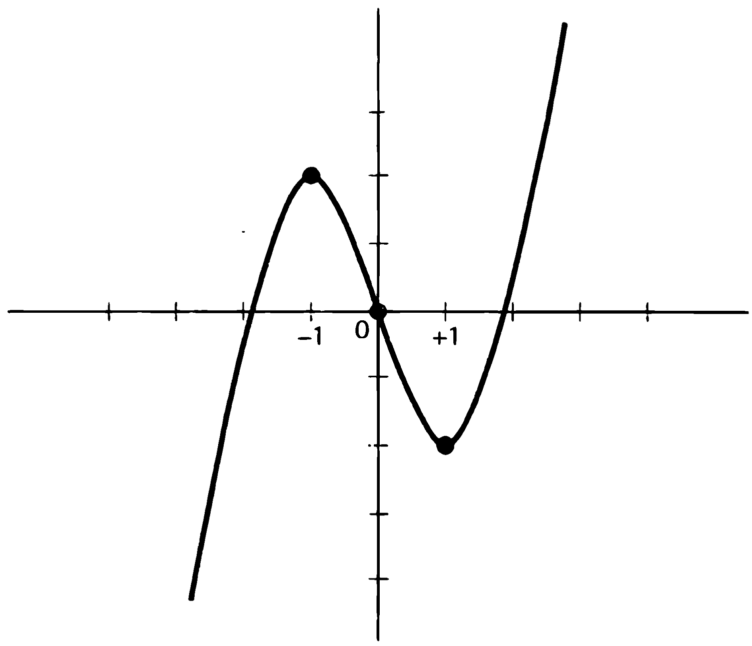



**Figure 3.3** *The graph of $f(x) = x^3 - 3x$.*

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## EXERCISES

**3.1** Use the techniques of this section to sketch the graphs of the following functions:

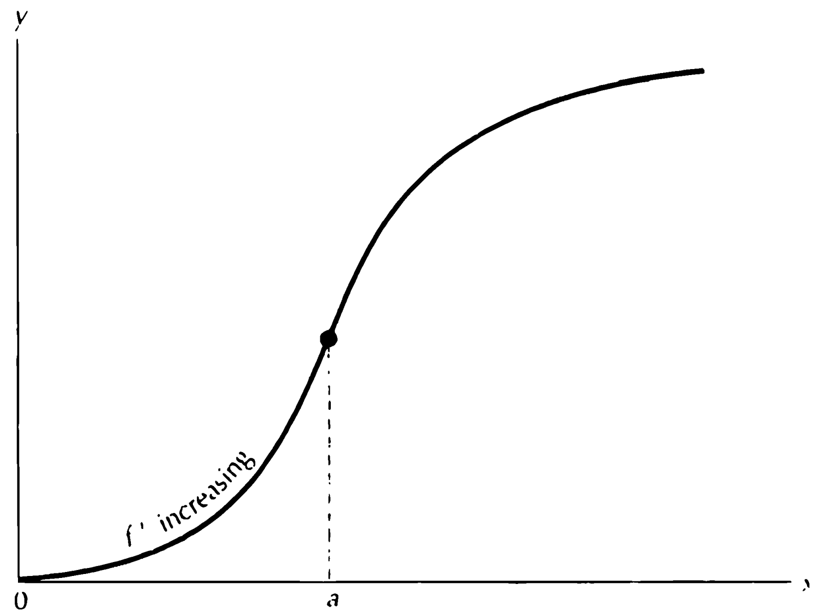
*a)* $x^3 + 3x$, *b)* $x^4 - 8x^3 + 18x^2 - 11$, *c)* $\frac{1}{3}x^3 + 9x + 3$,
*d)* $x^7 - 7x$, *e)* $x^{2/3}$, *f)* $2x^6 - 3x^4 + 2$.

**3.2** Write out the corresponding argument for part *b* of Theorem 3.1.

---

## 3.2 SECOND DERIVATIVES AND CONVEXITY

Frequently, we need to know more about the shape of the graph than where it is increasing and where it is decreasing. Consider, for example, a production function $y = f(x)$, a good example of a function which is naturally increasing. The rate of increase for a production function varies with the number $x$ of workers. At first, the additional output that each new worker adds to the production process increases as specialization and cooperation take place. However, after the gains from specialization are achieved, the additional output per new worker slows down and eventually declines as workers compete for limited space and resources. Figure 3.4 shows the graph of such a production function. Note that it is increasing for all $x$. However, for $x$ between 0 and $a$, its slope (the marginal product of labor) is increasing too; for $x$ bigger than $a$, the slope decreases as $x$ increases.




*A typical production function.*

**Figure 3.4**

Learning curves, which relate amount learned to time elapsed, often have graphs shaped like that of Figure 3.4. Amount learned per unit time — the slope of the curve — is high at first and increasing. However, as the task becomes learned or as the learner's mind reaches its capacity to hold more data, the rate of learning begins to drop.

For $x \in (0, a)$ in Figure 3.4, the slope of $f'(x)$ is an increasing function. By Theorem 3.2, the derivative of $f'$, $f''(x)$, is nonnegative there: $f''(x) \geq 0$ on $(0, a)$. For $x > a$ in Figure 3.4, $f'$ is a decreasing function of $x$; so $f''(x) \leq 0$ on $(a, \infty)$. A differentiable function $f$ for which $f''(x) \geq 0$ on an interval $I$ (so that $f'$ is increasing on $I$) is said to be **concave up** on $I$. A differentiable function $f$ for which $f''(x) \leq 0$ on an interval $I$ (so that $f'$ is decreasing on $I$) is said to be **concave down** on $I$.

An increasing function can be concave up or concave down on its interval of increase. These two cases are illustrated in Figure 3.5. Figure 3.6 shows how a

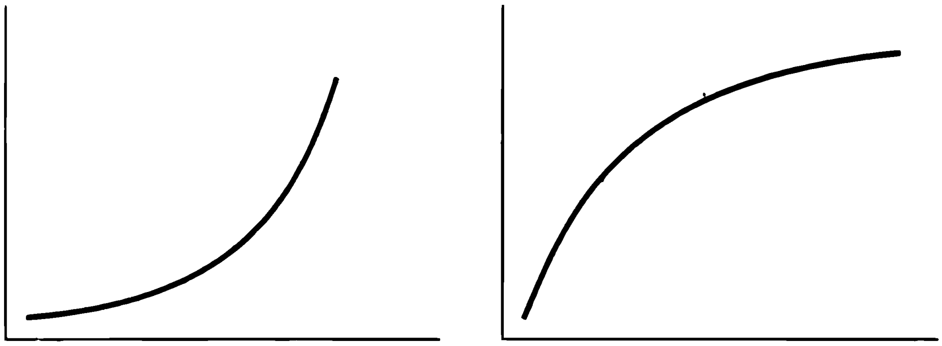


Figure 3.5 *An increasing function can be concave up or concave down.*

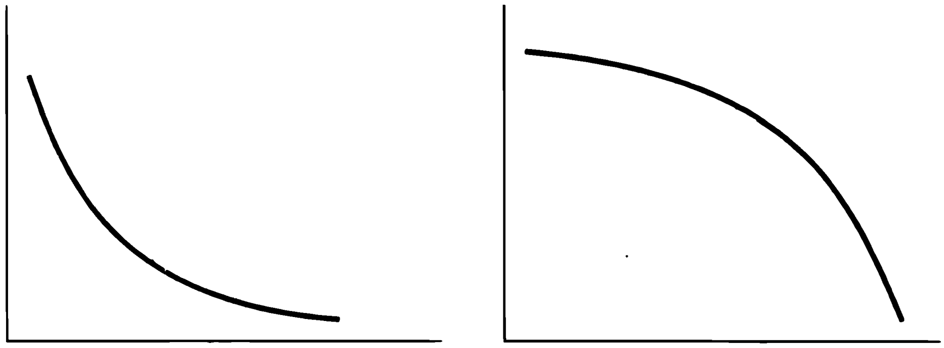


Figure 3.6 *A decreasing function can be concave up or concave down.*

*decreasing* function can be concave up or concave down on its domain. Note that the slope of $f$ is an increasing function of $x$ for a function which is concave up and is a decreasing function of $x$ for a function which is concave down.

There is also a noncalculus definition of concave up and concave down. This characterization follows from the observation that for a function which is concave up, the secant line joining any two points on the graph lies *above* the graph, as illustrated in Figure 3.7. For any two points $a$ and $b$, the set of points between $a$ and $b$ is given by the set $I_{ab} = [a, b]$ of all convex combinations of $a$ and $b$:

$$I_{ab} \equiv \{(1 - t)a + tb : 0 \leq t \leq 1\}.$$

The graph of $f$ above $I_{ab}$ is the set of points

$$\{((1 - t)a + tb, f((1 - t)a + tb)) : 0 \leq t \leq 1\}.$$

On the other hand, the secant line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of $f$ is given by

$$(1 - t)(a, f(a)) + t(b, f(b)) = ((1 - t)a + tb,\ (1 - t)f(a) + tf(b))$$

for $t$ in $[0, 1]$. Therefore, the statement that the secant line lies above the graph of

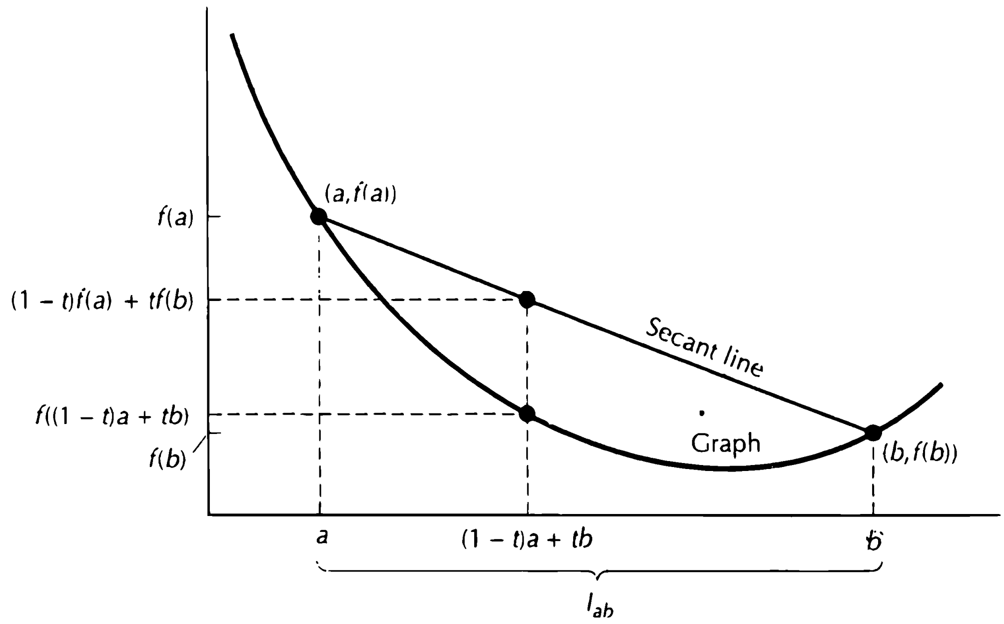



**Figure 3.7** *For a function which is concave up, the secant always lies above the graph.*

$f$ for $x \in I_{ab}$ can be written as

$$(1 - t)f(a) + tf(b) \geq f((1 - t)a + tb) \tag{1}$$

for $0 \leq t \leq 1$. This characterization of concave up is more general than the $f'' \geq 0$ criterion since it also applies to functions which are not differentiable. Therefore, we'll use it as our definition of concave up. In fact, condition (1) *is equivalent to* the condition $f''(x) \geq 0$ on $I_{ab}$ for a $C^2$ function.

**Definition** A function $f$ is called **concave up** or simply **convex** on an interval $I$ if and only if

$$f((1 - t)a + tb) \leq (1 - t)f(a) + tf(b) \tag{2}$$

for all $a, b \in I$ and all $t \in [0, 1]$. A function $f$ is called **concave down** or simply **concave** on interval $I$ if and only if

$$f((1 - t)a + tb) \geq (1 - t)f(a) + tf(b) \tag{3}$$

for all $a, b \in I$ and all $t \in [0, 1]$.

Calculus texts prefer the terms "concave up" and "concave down" to the terms "convex" and "concave." However, functions of more than one variable which satisfy condition (2) or (3) play a central role in economic theory, where the terms "convex" and "concave" are standard.

Of course, knowing where a function is convex or concave is valuable information for sketching its graph. For this purpose, one only needs to know where $f'' > 0$ and where $f'' < 0$. The test to see whether a function is convex or concave mimics the test used in determining whether a function is increasing or decreasing, but uses the second derivative instead of the first. First, one finds those points where $f''(x) = 0$ by solving this equation for $x$. These points are called the **second order critical points** of $f$ or, if the second derivative actually changes sign there, **inflection points** of $f$. These points divide the domain of $f$ into intervals on each of which $f''$ is always positive or always negative. On any one such interval, one need only evaluate $f''$ at a single point in the interval to determine its sign throughout the interval.

*Example 3.2* Let's return to Example 3.1, $f(x) = x^3 - 3x$. Using the first derivative, we determined that $f$ is increasing from $-\infty$ to $x = -1$, decreasing from $x = -1$ to $x = +1$, and increasing again from $x = +1$ to $+\infty$. Using only this first derivative test, we find that the graph of $f$ could conceivably be composed of the three straight line segments shown in Figure 3.8.

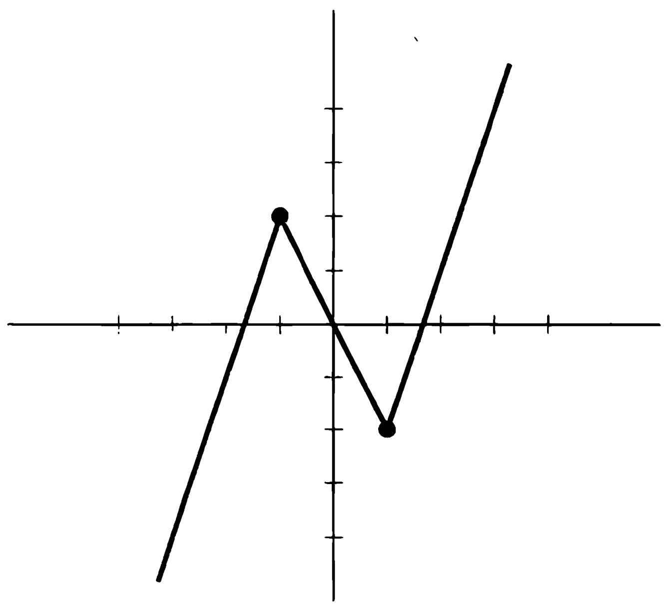


**Figure 3.8** *A candidate for the graph of $f(x) = x^3 - 3x$.*

To make this sketch more accurate, we need to compute the regions of concavity and convexity. These regions can be found simply by computing the second derivative of the original function: $f''(x) = 6x$. We note easily that $f''$ is zero only at 0, that it is negative where $x$ is negative, and that it is positive where $x$ is positive. Therefore, $f$ is concave for $x$ negative and convex for $x$ positive, as shown in Figure 3.3.

## EXERCISES

3.3 For the functions in Exercise 3.1, compute the regions of convexity and concavity and include this information on your graphs.

**3.4** Sketch the graph of a function which has the following properties:

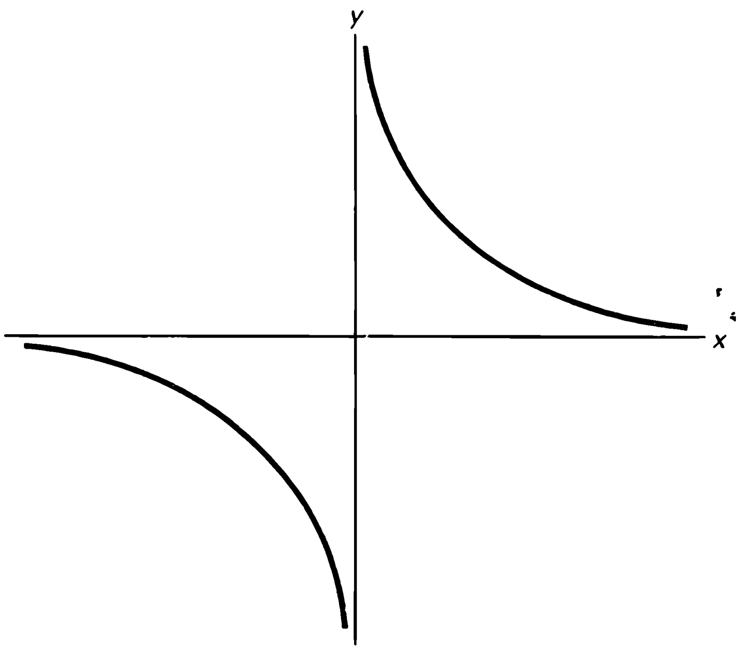
*a*) $f'(x) > 0$ for $x < 1$; *b*) $f'(x) < 0$ for $x > 1$;
*c*) $f''(x) < 0$ for $x < 2$; *d*) $f''(x) > 0$ for $x > 2$.

**3.5** Sketch the graph of a function which has the following properties:
*a*) $f'(x) > 0$ for $-4 < x < -2$ and $2 < x < 4$;
*b*) $f'(x) < 0$ for $-\infty < x < -4$, $-2 < x < +2$, and $4 < x < \infty$;
*c*) $f''(x) > 0$ for $-\infty < x < -3$ and $0 < x < 3$;
*d*) $f''(x) < 0$ for $-3 < x < 0$ and for $3 < x < \infty$.

## 3.3 GRAPHING RATIONAL FUNCTIONS

We complete our discussion of the use of derivatives to sketch the graphs of functions by working with rational functions. Because rational functions have denominators, they are more challenging to graph than polynomials. Furthermore, it is usually easier to visualize the graph of a polynomial than it is the graph of a rational function.

The simplest rational function is $f(x) = 1/x$, whose graph is pictured in Figure 3.9. Since the denominator of a fraction cannot be zero, this function is not defined at $x = 0$. Furthermore, as $x$ approaches 0 from the negative side, the value of $f(x)$ goes to $-\infty$, and as $x$ approaches 0 from the positive side, the value of $f(x)$ goes to $+\infty$. In both cases, the graph of $f$ "cuddles up" to the vertical line through the point $x = 0$, where the function is not defined. Such a vertical line is called a **vertical asymptote** of $f$.




*The graph of* $f(x) = 1/x$.

**Figure 3.9**

In general, if $f$ is a rational function whose denominator is zero at the point $x_0$ (and whose numerator is not zero at $x_0$), then the vertical line $\{x = x_0\}$ is a vertical asymptote of $f$. On either side of this vertical asymptote, the graph of $f$ goes to $+\infty$ or to $-\infty$; one uses calculus techniques to find out which.

In sketching the graph of a rational function, treat the zeros of the denominator of a rational function like the first and second order critical points that arise in the process of finding the signs of $f'$ and of $f''$, because $f'$ and $f''$ can change sign as one crosses a vertical asymptote. In other words, use them to divide the line into intervals on which $f'$ or $f''$ have constant sign. If $f'$ is negative on the interval just to the left of the vertical asymptote, then $f$ must go to $-\infty$ to the left of the asymptote, since $f$ is decreasing there, just as it is for $1/x$ in Figure 3.9. By similar logic, if $f'$ is positive on that interval, then $f$ goes to $+\infty$ just to the left of the vertical asymptote. A similar analysis works for points on the right-hand side of the vertical asymptote.

### Hints for Graphing

(1) Remember that to find the $x$-intercept of a rational function, you need only set the *numerator* equal to zero. If there is no $x$-intercept in a given interval between critical points and/or asymptotes, the graph does not cross the $x$-axis in that interval — an observation which may prove very helpful in sketching an accurate graph.

(2) Since the function, its first derivative, and its second derivative all provide information about each other in the graph of a function, avoid using the word "it" in referring to any of these functions, both in your own mind and in discussing the process of graphing the function with anyone else. If you carefully keep track of which derivative of the function you are working with at any one time, you will save yourself some confusion.

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#### EXERCISES

**3.6** Use calculus to sketch the graph of $16(x + 1)/(x - 2)^2$.

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## 3.4 TAILS AND HORIZONTAL ASYMPTOTES

To complete our guide to drawing graphs of polynomials and rational functions, we turn our attention to the "tails" of the graph — the shape of the graph for large positive and large negative values of $x$.

### Tails of Polynomials

For polynomials, the **leading term** — the monomial of highest degree — determines the shape of the tail of the graph. To see why this is true, consider a concrete example: the cubic $x^3 - 4x^2 + 5x - 6$. If $x$ is very big, say $x = 10^{10}$, then $x^3$ will be $10^{30}$ — a number with 31 digits. On the other hand, $-4x^2$ will be $-4 \cdot 10^{20}$ — a number with only 21 digits. For $x = 10^{10}$, adding $-4x^2$ to $x^3$ will not affect the 10 left-most digits of $x^3$. A calculator which displays only 10 significant digits will not display the effect of this addition at all. The effect of the $5x + 6$ terms is minuscule by comparison. As $x$ gets larger still, the effect of the nonleading terms on the leading term becomes even more insignificant.

In summary, for $|x|$ very large, the graph of a polynomial

$$f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k$$

is determined completely by its leading term $a_0x^k$. To graph the tail of a general polynomial, we need only know how to graph a general monomial. The graph of the monomial $a_0x^k$ is determined by the sign of $a_0$ and the parity of $k$. If $k$ is even, then both tails go to $+\infty$ as $|x| \to \infty$ if $a_0 > 0$, and both tails go to $-\infty$ as $|x| \to \infty$ if $a_0 < 0$. Think of the graphs of $x^2$ and of $-x^2$ as examples. If $k$ is odd, one tail of the graph goes to $+\infty$ and the other to $-\infty$ as $|x| \to \infty$, depending once again on the sign of $a_0$. Think of the graphs of $x^3$ and $-x^3$ as examples of this phenomenon.

### Horizontal Asymptotes of Rational Functions

Next, consider a general rational function:

$$g(x) = \frac{a_0x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k}{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m}, \quad a_0, b_0 \neq 0.$$

For $|x|$ very large, the behavior of the numerator polynomial is determined by its leading term $a_0x^k$ and the behavior of the denominator polynomial is determined by its leading term $b_0x^m$. In other words, for $|x|$ large, the rational function $g$ mirrors the behavior of the monomial

$$\ell(x) = \frac{a_0x^k}{b_0x^m} = \frac{a_0}{b_0}x^{k-m}.$$

In particular, if $k > m$, then $\ell(x)$ is a monomial with a positive degree, and the tails of the rational function $g$ go to $\pm\infty$, just like those of a polynomial. On the other hand, if $k < m$, then $\ell(x) \to 0$ as $|x| \to \infty$, just as $1/x$ does in Figure 3.9. In this case, both tails of $g$ are asymptotic to the $x$-axis as $|x| \to \infty$. We say that the $x$-axis is a **horizontal asymptote** for the graph of $g$. This is the situation which arises in Exercise 3.6. Finally, if $k = m$, then the quotient $\ell(x)$ of the leading terms of $g$ is a nonzero constant $a_0/b_0$. As $|x| \to \infty$, $g(x) \to a_0/b_0$; the graph is asymptotic to the horizontal line $y = a_0/b_0$. This horizontal line is also called a **horizontal asymptote** of the graph of $g$. Look for both vertical and horizontal asymptotes in the graphs of the rational functions in the exercise below.

## EXERCISES

3.7 Sketch the graph of each of the following rational functions:

$a)\ \dfrac{x}{x^2-1};\qquad b)\ \dfrac{x}{x^2+1};\qquad c)\ \dfrac{x^2}{x+1};$

$d)\ \dfrac{x^2+3x}{x^2-1};\qquad e)\ \dfrac{x^2+1}{x};\qquad f)\ \dfrac{1}{x^2+1}.$

3.8 In each of the four graphs below, the graph of the first derivative $f'$ of a function $f$ is sketched. In each case, determine where the function itself is increasing, decreasing, concave up, and concave down. Put this information on a number line and sketch the graph of the function $f$, assuming that $f(0) = 0$.

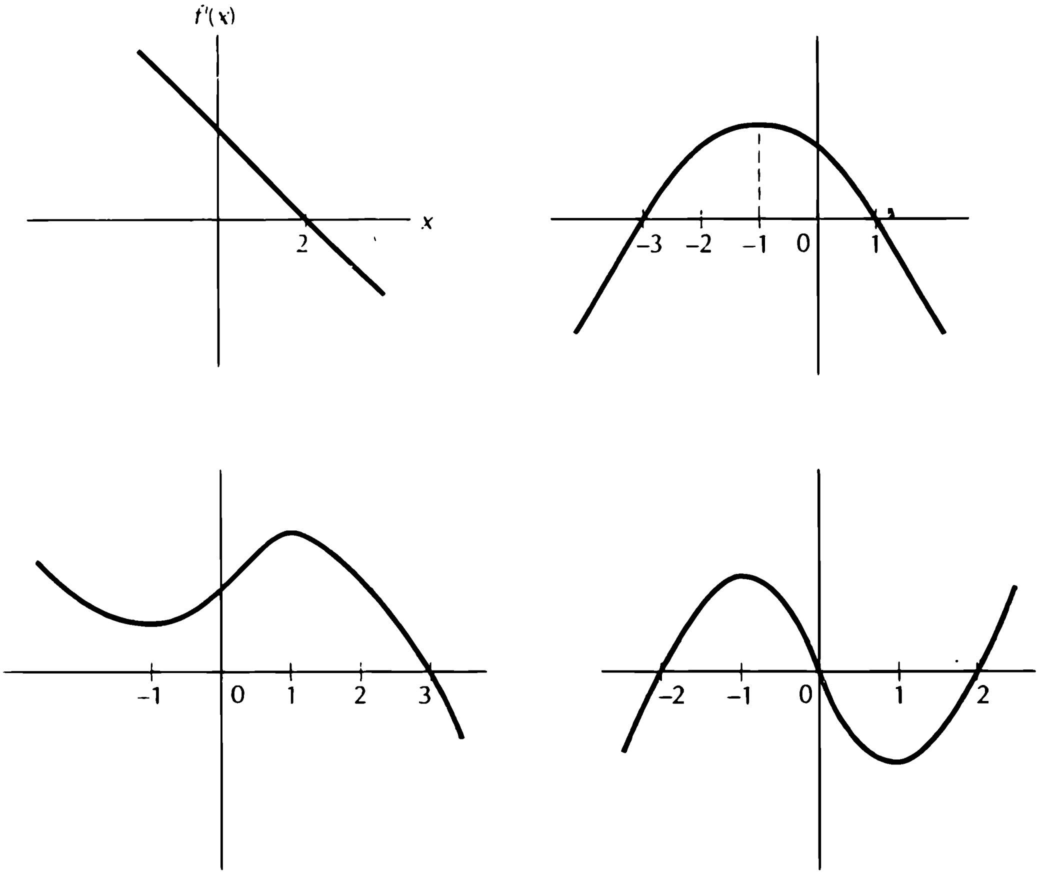



**Figure 3.10** *Graphs of four $f'$'s.*

## 3.5 MAXIMA AND MINIMA

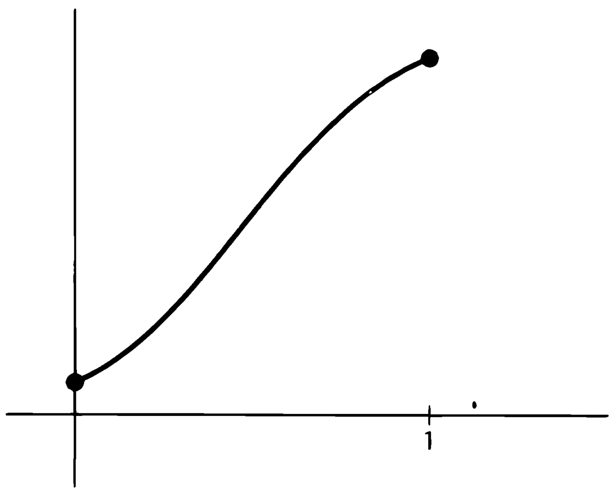
One of the major uses of calculus in mathematical models is to find and characterize maxima and minima of functions. For example, economists are interested in maximizing utility and profit and in minimizing cost. Recall that a function $f$ has a local or relative maximum at $x_0$ if $f(x) \leq f(x_0)$ for all $x$ in some open interval containing $x_0$; $f$ has a global or absolute maximum at $x_0$ if $f(x) \leq f(x_0)$ for all $x$ in the domain of $f$. The function $f$ has a local or relative minimum at $x_0$ if $f(x) \geq f(x_0)$ for all $x$ in some open interval containing $x_0$; $f$ has a global or absolute minimum at $x_0$ if $f(x) \geq f(x_0)$ for all $x$ in the domain of $f$. See the discussion and graphs at the beginning of Chapter 2.

If $f$ has a local maximum (minimum) at $x_0$, we will simply say that $x_0$ is a max (min) of $f$. If we want to emphasize that $f$ has a *global* maximum (minimum) at $x_0$, we will say that $x_0$ is a global max (global min) of $f$.

### Local Maxima and Minima on the Boundary and in the Interior

A max or min of a function can occur at an endpoint of the domain of $f$ or at a point which is not an endpoint — in the "interior" of the domain. These two cases are illustrated in Figures 3.11 and 3.12 for functions whose domains are the closed interval $[0, 1]$. In Figure 3.11, $f$ is increasing on $[0, 1]$ and so its max occurs at the endpoint $x = 1$ of $[0, 1]$. In Figure 3.12, the max of $f$ occurs at $x = 1/3$ in the interior of the domain $[0, 1]$. We will call a max (or min) that occurs at a boundary point of the domain of $f$ a **boundary max** or (**boundary min**). We will call a max (or min) which is not an endpoint of the domain of $f$ an **interior max** (or **interior min**). Of course, if the domain of $f$ is all of $\mathbf{R}^1$ or is an open interval, then any max of $f$ will be an interior max.

The calculus criterion for an interior max or min of $f$ is easy to state and to understand.




*Function with a boundary max at $x = 1$.* **Figure 3.11**

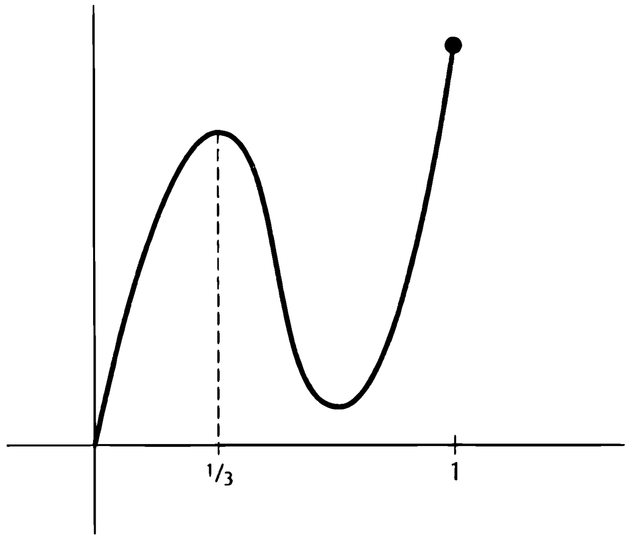



**Figure 3.12** *Function with an interior max at $x = 1/3$.*

**Theorem 3.3** If $x_0$ is an interior max or min of $f$, then $x_0$ is a critical point of $f$.

*Proof* From an analytic point of view, a function is neither increasing nor decreasing on an interval about an interior max or min. By Theorem 3.1, its first derivative cannot be positive or negative there; that is, $f'(x_0)$ must be zero or undefined — $x_0$ is a critical point of $f$. From a geometric point of view, if the graph of $f$ has a tangent line at a max or a min, that tangent line must be horizontal since the graph turns around there, as in Figure 3.13. In other words, $f'(x_0)$ must be zero. ■

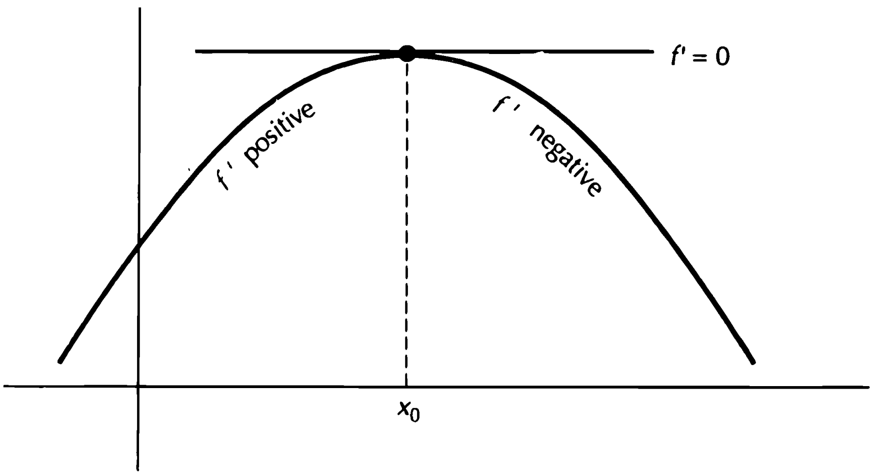



**Figure 3.13** *The graph of $f$ at a max $x_0$.*

*Example 3.3* For the function $f(x) = x^3 - 3x$, pictured in Figure 3.3, the local max and local min occur at the critical points $x = -1$ and $x = +1$, respectively.

*Example 3.4* As Figure 3.11 illustrates, the derivative of $f$ at a *boundary* max or min need not be zero. The production function pictured in Figure 3.4 has domain $[0, \infty)$. Since it is an increasing function, its (global) min occurs at the boundary point $x = 0$, where the derivative of $f$ is not necessarily zero.

## Second Order Conditions

If $x_0$ is a critical point of a function $f$, how can we use calculus to decide whether critical point $x_0$ is a max, a min, or neither? The answer to this question lies in the *second* derivative of $f$ at $x_0$.

**Theorem 3.4**

(*a*) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $x_0$ is a max of $f$;
(*b*) if $f'(x_0) = 0$ and $f''(x_0) > 0$, then $x_0$ is a min of $f$; and
(*c*) if $f'(x_0) = 0$ and $f''(x_0) = 0$, then $x_0$ can be a max, a min, or neither.

*Proof* We will present a proof of part *a* and leave the proof of part *b* as an exercise. From a geometric point of view, $f'(x_0) = 0$ means that the tangent line to the graph of $f$ is horizontal at $x_0$, and $f''(x_0) < 0$ means that the graph curves downward, as in Figure 3.13. These two conditions together imply that $f$ has a local max at $x_0$. From a more analytic point of view, $f''(x_0) < 0$ means that the first derivative $f'$ of $f$ is a decreasing function in an interval about $x_0$. The facts that $f'$ is decreasing and that $f'(x_0) = 0$ mean that $f'$ is positive to the left of $x_0$ and negative to the right of $x_0$. By Theorem 3.1, these two derivative conditions imply that $f$ is increasing to the left of $x_0$ and decreasing to the right of $x_0$. In other words, $f$ has a local max at $x_0$.

To verify statement *c* — that anything can happen when $f'(x_0) = 0$ and $f''(x_0) = 0$ — consider the four graphs in Figure 3.14. Each of these four functions satisfies $f'(0) = 0$ and $f''(0) = 0$. However, 0 is a local min for $f_1$ and a local max for $f_2$, while $f_3$ is strictly increasing at 0 and $f_4$ is strictly decreasing at 0. ■

**Remark** A critical point of $f$ at which the second derivative $f''$ is zero too is called a **degenerate critical point** of $f$. As part *c* of Theorem 3.4 indicates, to determine whether or not any given degenerate critical point is a max, a min, or neither, one needs more information about the function than the sign of its second derivative — information like the sign of $f'$ on a whole interval about the critical point.

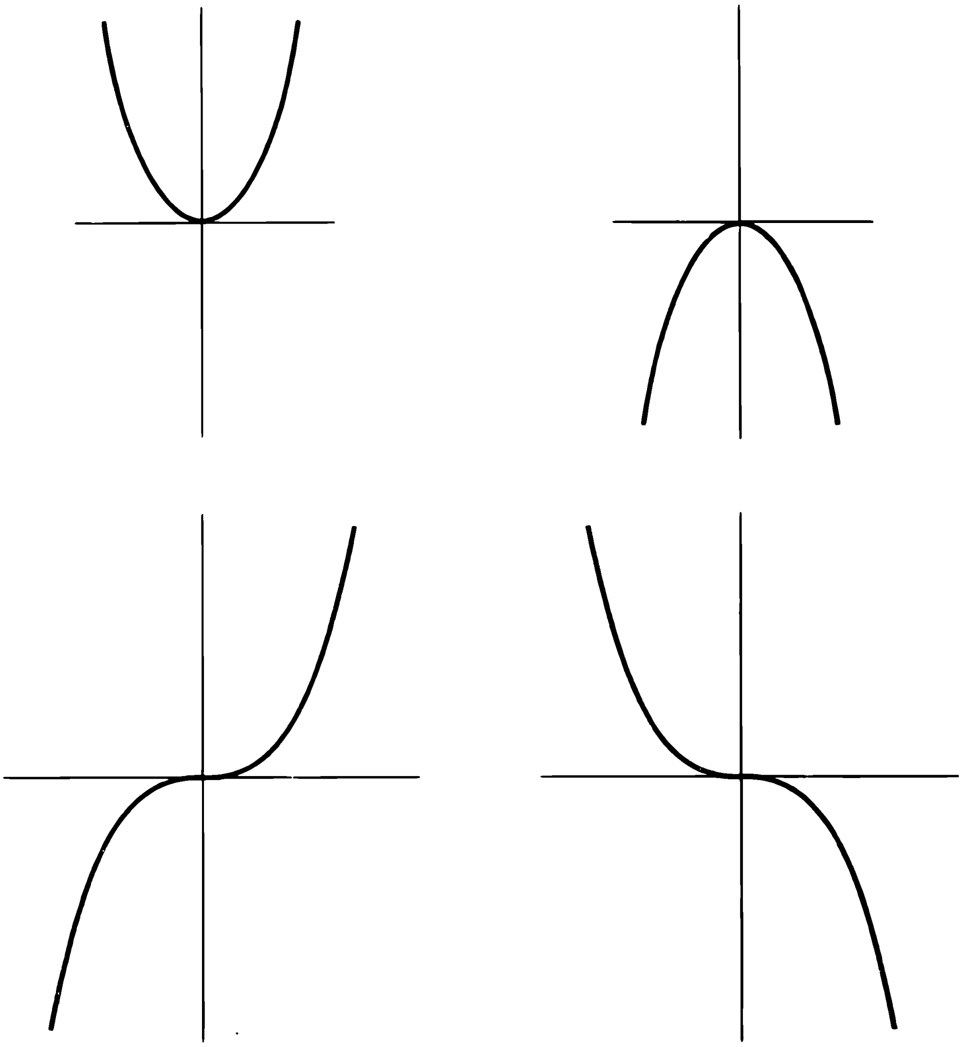


**Figure 3.14** $f_1(x) = x^4$, $f_2(x) = -x^4$, $f_3(x) = x^3$, *and* $f_4(x) = -x^3$.

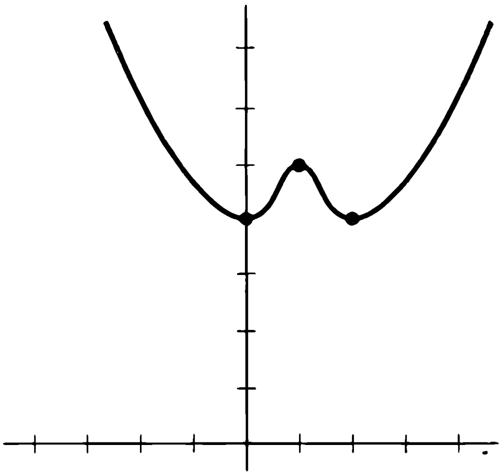
*Example 3.5* Let's use Theorem 3.4 to find the local max and mins of $f(x) = x^4 - 4x^3 + 4x^2 + 4$. The critical points of $f$ are the solutions of

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x-1)(x-2) = 0,$$

that is, $x = 0, 1, 2$. These three points are the only candidates for a max or min of $f$. Let's check the second derivative, $f''(x) = 12x^2 - 24x + 8$, at these three points:

$$f''(0) = 8 > 0, \quad f''(1) = -4 < 0, \quad \text{and} \quad f''(2) = 8 > 0.$$

By Theorem 3.4, $x = 0$ and $x = 2$ are local mins of $f$ and $x = 1$ is a local max. The graph of $f$ is presented in Figure 3.15.



Graph of $f(x) = x^4 - 4x^3 + 4x^2 + 4$.

**Figure 3.15**

## Global Maxima and Minima

Note that $x = 0$ and $x = 2$ are global minima of $f$ in Figure 3.15. However, $x = 1$ is definitely not a global max, since $f$ eventually takes on arbitrarily large values as $x \to \infty$. In Figure 3.12, neither critical point is a global max or min of $f$.

In some problems, we want conditions which will guarantee that a critical point is a *global* max or a *global* min of the function under consideration. For example, if Figure 3.15 represents the profit function of a firm, the firm would be foolish to settle for the local max at $x = 1$, since it can generate arbitrarily large profits by choosing a large value of $x$. In fact, the function $f(x) = x^4 - 4x^3 + 4x^2 + 4$ has no global max.

In general, it is difficult to find a global max of a function or even to prove that a given local max is a global max. There are, however, three situations in which this problem is somewhat easier:

(1) when $f$ has only one critical point in its domain,
(2) when $f'' > 0$ or $f'' < 0$ throughout the domain of $f$, and
(3) when the domain of $f$ is a closed finite interval.

We will examine each of these situations.

## Functions with Only One Critical Point

**Theorem 3.5** Suppose that:

(*a*) the domain of $f$ is an interval $I$ (finite or infinite) in $\mathbf{R}^1$,
(*b*) $x_0$ is a local maximum of $f$, and
(*c*) $x_0$ is the only critical point of $f$ on $I$.

Then, $x_0$ is the global maximum of $f$ on $I$.

*Proof* We will show that if $x_0$ is not the global maximum of $f$, then $f$ must have another critical point on $I$. Suppose there is a point $y_0$ in $I$ with $f(y_0) > f(x_0)$. Suppose further that $y_0 > x_0$. (The case where $y_0 < x_0$ is left as an exercise.) Since $f$ is decreasing just to the right of $x_0$ and eventually increasing again somewhere to the left of $y_0$, it must change from decreasing to increasing somewhere between $x_0$ and $y_0$, say at $z_0$. But then $z_0$ is an interior local minimum of $f$, and therefore is a critical point other than $x_0$ — contradicting the hypothesis that $x_0$ is the only critical point of $f$. Therefore, $x_0$ is the global maximum of $f$ on its domain $I$. ■

### Functions with Nowhere-Zero Second Derivatives

**Theorem 3.6** If $f$ is a $C^2$ function whose domain is an interval $I$ and if $f''$ is never zero on $I$, then $f$ has at most one critical point in $I$. This critical point is a global minimum if $f'' > 0$ and a global maximum if $f'' < 0$.

*Proof* Suppose $f''$ is always positive on domain $I$. By Theorem 3.2, $f'$ is an increasing function on $I$. This means that $f'$ can be zero at at most one point. If there *is* a point $x_0$ where $f'(x_0) = 0$, then $x_0$ is a local minimum of $f$ since $f''(x_0) > 0$. By Theorem 3.5, $x_0$ is the *global* minimum of $f$. ■

It follows from Theorem 3.6, and the fact that a function $f$ is convex if and only if $-f$ is concave, that if $f$ is a $C^2$ convex function, then a critical point of $f$ defined on an interval in $\mathbf{R}^1$ is necessarily a global minimum of $f$. Any local maximum of $f$ must occur on an endpoint of its domain. Similarly, a critical point of a $C^2$ concave function $f$ is necessarily a global maximum of $x$; any local minimum must occur on an endpoint of its domain.

### Functions with No Global Max or Min

A function whose domain is an open interval need not have a global maximum or minimum in its domain. For example, the function $f(x) = x^3 - 3x$, whose graph is pictured in Figure 3.3, has neither a global maximum nor a global minimum since its value goes to $+\infty$ as $x \to \infty$ and to $-\infty$ as $x \to -\infty$. Any strictly increasing or strictly decreasing function whose domain is an *open* interval will not have a maximum or a minimum in its domain. At the same time, there exist functions that have a global minimum but no global maximum in their domain. Examples are the function $x^4$, which is pictured in Figure 3.14, and the function $x^4 - 4x^3 + 4x^2 + 4$, which is pictured in Figure 3.15.

### Functions Whose Domains Are Closed Finite Intervals

However, a famous theorem by Weierstrass states that a continuous function whose domain is a *closed and bounded* interval $[a, b]$ must have both a global maximum and a global minimum in this domain (see Theorem 30.1). Furthermore, as we will now see, there is a natural method for calculating these global extrema.

By Theorem 3.3, an interior maximum or minimum of any function must be a critical point of $f$. The only other candidates for a maximum or minimum are the two endpoints of the domain: $x = a$ and $x = b$. So, if we're looking for the global maximum of a $C^1$ function $f$ with domain $[a, b]$, we need only:

(1) compute the critical points of $f$ by solving $f'(x) = 0$ for $x$ in $(a, b)$,
(2) evaluate $f$ at these critical points and at the endpoints $a$ and $b$ of its domain, and
(3) choose the point from among these that gives the largest value of $f$ in step 2.

*Example 3.6* Suppose that $x$ years after its founding in 1960, the Association of Smart Statisticians had a membership given by the function $f(x) = 2x^3 - 45x^2 + 300x + 500$. In the period between 1960 and 1980, what was its largest and its smallest membership and when were these two extremes realized? Mathematically, this is the problem of maximizing $f(x) = 2x^3 - 45x^2 + 300x + 500$ for $x$ in the closed interval $[0, 20]$. The critical points of $f$, the solutions of

$$\begin{aligned} 0 = f'(x) &= 6x^2 - 90x + 300 \\ &= 6(x^2 - 15x + 50) \\ &= 6(x - 5)(x - 10), \end{aligned}$$

are $x = 5, 10$. To solve the problem, we need only evaluate $f$ at the critical points $x = 5, 10$ and at the boundary points $x = 0, 20$:

$$f(0) = 500, \quad f(5) = 1125, \quad f(10) = 1000, \quad f(20) = 10375.$$

Therefore, the global max occurs at $x = 20$ and the global min occurs at $x = 0$.

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**EXERCISES**

3.9 For each of the following functions $f$ with specified domains $D_1$ and $D_2$, find the global maximum and the global minimum of $f$ on each $D_i$ if they exist. Justify your answers.

| | | | |
|---|---|---|---|
| *a*) $1/x$ | on $D_1 = (1, 2)$ | and | on $D_2 = [1, 2]$, |
| *b*) $x^3 + 3x$ | on $D_1 = (-\infty, +\infty)$ | and | on $D_2 = [0, 1]$, |
| *c*) $x^3 - 3x$ | on $D_1 = [-4, -2]$ | and | on $D_2 = [0, \infty)$, |
| *d*) $x^2/(x + 1)$ | on $D_1 = [0, 10]$ | and | on $D_2 = [0, \infty)$, |
| *e*) $3x^5 - 5x^3$ | on $D_1 = [-2, +2]$ | and | on $D_2 = [-\sqrt{2}, +\sqrt{2}]$, |
| *f*) $x + (1/x)$ | on $D_1 = (0, \infty)$ | and | on $D_2 = (-\infty, 0)$, |

*g*) $1/(1 + x^2)$ on $D_1 = (-\infty, +\infty)$ and on $D_2 = [1, 2]$,

*h*) $3x + 5 + (75/x)$ on $D_1 = [-2, +2]$ and on $D_2 = [1, 10]$.

**3.10** A manufacturer produces gizmos at a cost of $5 each. The manufacturer computes that if each gizmo sells for $x$ dollars, $(15 - x)$ gizmos will be sold. What is the manufacturer's profit function? What price should the manufacturer charge to maximize profit?

**3.11** A manufacturer can produce economics texts at a cost of $5 apiece. The text currently sells for $10, and at this price 10 texts are sold each day. The manufacturer figures that each dollar decrease in price will sell one additional copy each day. Write out the demand and profit functions. What price $x$ maximizes profit?

**3.12** Prove part *b* of Theorem 3.4.

**3.13** Draw the appropriate figure to illustrate the proof of Theorem 3.5, and carry out the proof of Theorem 3.5 for the case in which $y_0 < x_0$.

**3.14** Adapt the argument in the proof of Theorem 3.6 to the case where $f''$ is greater than *or equal to* zero on the domain $I$.

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## 3.6 APPLICATIONS TO ECONOMICS

In this section we discuss some of the ways in which the concepts and techniques of calculus lead to a better understanding of the principles of economics. So far in this chapter, we have used calculus to study the properties of *specific* functions, like

$$x^3 - 3x \quad \text{and} \quad x^4 - 4x^3 + 4x^2 + 4$$

in Examples 3.1 and 3.5, respectively. We now need to move on to a consideration of *general* types of functions which are distinguished, not by their formulas, but by their properties.

### Production Functions

Consider, as an example, a production function $y = f(q)$, which relates the amount of input $q$, say labor input, to the amount of output $y$ that can be produced with $q$ units of input. Because we want a theory that is broadly applicable when modeling the production process, we assume only that the production function we use has the properties pictured in Figure 3.4 and not that it has a specific functional form, like $f(q) = \sqrt{q}$. When we need to make assumptions about a production function, $y = f(q)$ in an economy, we will only assume that:

(1) it is continuous or maybe $C^2$,
(2) it is increasing, and
(3) there is a level of input $a$ such that the production function is concave up, for $0 \le q < a$, and concave down for $q > a$.

If $f$ is $C^2$, these assumptions translate to the following assumptions about the derivative of $f$:

$$\begin{aligned} &(2')\ f'(q) > 0 \text{ for all } q, \text{ and} \\ &(3')\ \text{for some } a \geq 0,\ f''(q) > 0 \text{ for } q \in [0, a) \text{ and } f''(q) < 0 \text{ for } q > a. \end{aligned} \tag{4}$$

Occasionally, to build our intuition or to construct concrete models of economies, we will work with a general *class* of functional forms. For production functions, we often work with the two-parameter family of functions $y = kq^b$, where $k$ and $b$ are positive constants or **parameters**. Depending on the size of $b$, these functions are either always concave up or always concave down for $q > 0$. In particular, if $0 < b < 1$, $a = 0$ in (4) and $f(q)$ is always concave down; if $b > 1$, $a = \infty$ in (4) and $f(q)$ is always concave up. The fact that this class of production functions has two parameters which may be adjusted according to the production process under consideration adds some richness and flexibility to its use. Nevertheless, an economist should be uncomfortable with an economic principle that holds only for an economy governed by production functions in this class.

## Cost Functions

A **cost function** $C(x)$ assigns to each level of *output* $x$, the total cost of producing that much output. Like production functions, cost functions are naturally increasing functions of their argument $x$. However, the independent variable for a cost function is the level of output, while the independent variable for a production function is the level of input.

The derivative $C'(x)$ of a cost function is called the **marginal cost** and is written $MC(x)$. As we discussed in Section 2.7, $MC(x)$ measures the additional cost incurred from the production of one more unit of output when the current output is $x$.

The **average cost function** also plays an important role in economic theory. It is the function

$$AC(x) = \frac{C(x)}{x},$$

which measures the cost per unit produced. Using calculus, one can derive some useful relationships between the marginal cost and average cost functions.

**Theorem 3.7** Suppose the cost function $C(x)$ is a $C^1$ function. Then,

(*a*) if $MC > AC$, $AC$ is increasing,
(*b*) if $MC < AC$, $AC$ is decreasing, and
(*c*) at an interior minimum of $AC$, $AC = MC$.

*Proof* To show whether a function is increasing or decreasing, we need only compute the sign of its first derivative. Using the Quotient Rule, we compute that the first derivative of $AC(x)$ is

$$AC'(x) = \frac{d}{dx}\left(\frac{C(x)}{x}\right) = \frac{C'(x) \cdot x - 1 \cdot C(x)}{x^2}$$
$$= \frac{C'(x) - (C(x)/x)}{x} = \frac{MC - AC}{x}.$$

(*a*) If $MC > AC, AC'(x) > 0$ and $AC(x)$ is increasing.
(*b*) If $MC < AC, AC'(x) < 0$ and $AC(x)$ is decreasing.
(*c*) If $x_0$ is an interior minimum of $AC(x)$, then by Theorem 3.3, $AC'(x_0) = 0$ and $MC(x_0) = AC(x_0)$. ■

Theorem 3.7 has a rich intuitive and geometric content. From an intuitive point of view, the theorem says that if you do better than your average some day, your average goes up that day. On days that you do worse than your average, your average goes down. For baseball fans, a batter who goes hitless in a game will see his batting average drop; a batter who has a "perfect day at the plate" will raise his batting average.

Taking a geometric point of view, consider the graph of cost function $y = C(x)$, as pictured in Figures 3.16 and 3.17. This graph is sometimes called a **cost curve**.

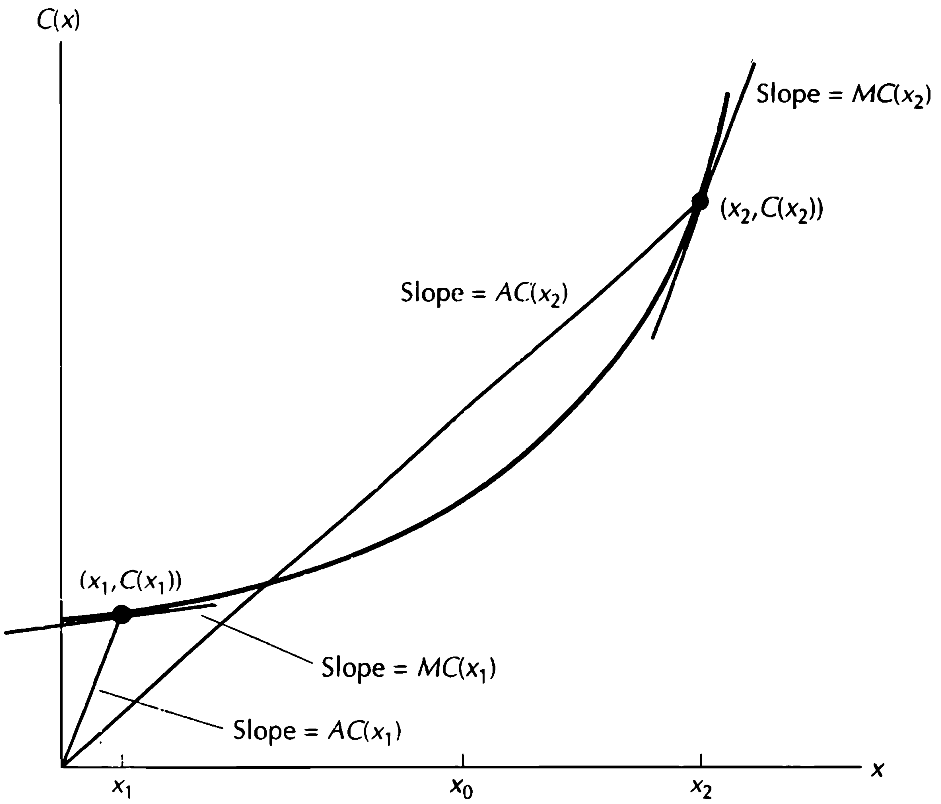
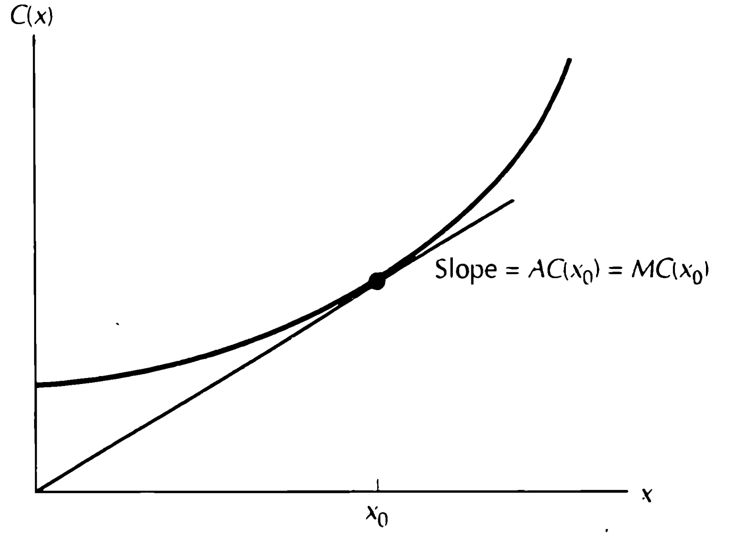



**Figure 3.16** *At $x_1$, $AC > MC$ and $AC$ is decreasing. At $x_2$, $MC > AC$ and $AC$ is increasing.*




*At $x_0$, AC is a minimum and AC = MC.* **Figure 3.17**

The marginal cost at point $x$, $MC(x)$, can be considered as the slope of the tangent line to this curve at the point $(x, C(x))$. The average cost at $x$,

$$AC(x) = \frac{C(x) - 0}{x - 0},$$

can be considered as the slope of the line segment from $(x, C(x))$ to the origin $(0, 0)$. The cost curve $C$ in Figures 3.16 and 3.17 is an increasing function, and $C(0) > 0$ implies that there are some **fixed costs**—costs independent of the amount produced. At the points $(x_1, C(x_1))$ and $(x_2, C(x_2))$ on the cost curve in Figure 3.16, we have drawn the tangent line to the graph, whose slope represents $MC(x_i)$, and the line to the origin, whose slope represents $AC(x_i)$. Note that $AC(x_1) > MC(x_1)$ and that $AC(x)$ decreases as $x$ increases from $x_1$. On the other hand, at the point $(x_2, C(x_2))$, $MC(x_2) > AC(x_2)$ and $AC(x)$ increases as $x$ increases from $x_2$; this is consistent with Theorem 3.7. In Figure 3.17, we have drawn attention to the point $(x_0, C(x_0))$ on the graph where the slope of the line to the origin is at a minimum. At this minimizing point, $(x_0, C(x_0))$, the line to the origin is actually tangent to the graph: $AC(x_0) = MC(x_0)$, as Theorem 3.7 states.

As $x$ increases from 0 to $\infty$ in Figures 3.16 and 3.17, the slope of the line from $(x, C(x))$ to the origin starts very large, decreases past $x_1$, reaches its minimum value at $x_0$, and then increases again as $x$ passes $x_2$ and becomes arbitrarily large. If we graph this slope, that is, if we graph the average cost curve $AC(x)$, versus $x$, we find a U-shaped curve, as pictured in Figure 3.18. We have also drawn the marginal cost curve $MC$ in Figure 3.18. The critical property in this figure is that for $x < x_0$, the $MC$-curve lies below the $AC$-curve while $AC$ is decreasing; for $x > x_0$, the $MC$-curve lies above the $AC$-curve as $AC$ increases. Figure 3.18 plays a major role in the study of the **firm in intermediate micro-economics courses.**

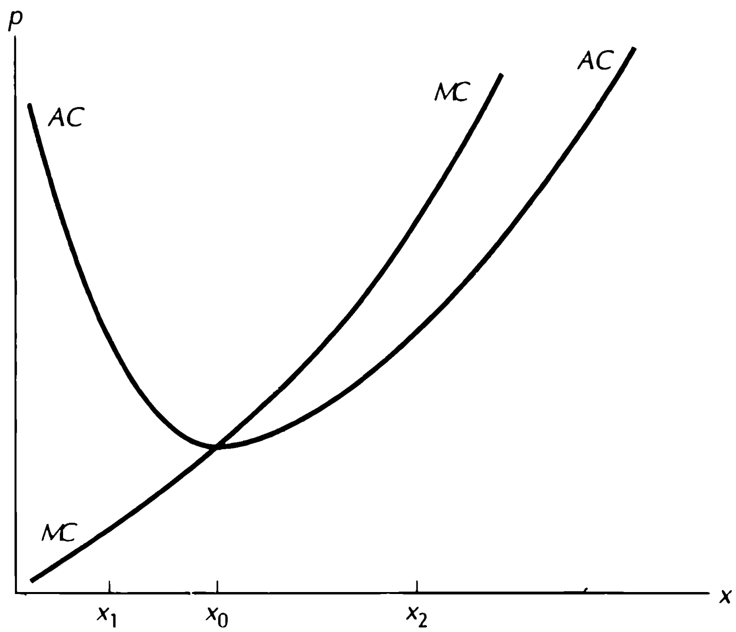



**Figure 3.18** *The AC- and MC-curves.*

### Revenue and Profit Functions

Let $C(x)$ continue to denote a firm's cost function relative to its output $x$. Let $R(x)$ be the firm's **revenue function**, the function which indicates how much money a firm receives for selling $x$ units of its output. Like $C(x)$, $R(x)$ should be an increasing function of $x$. We write $MR(x)$ for the firm's marginal revenue function $R'(x)$. If $p(x)$ is the unit price when the firm's output is $x$ units, then $R(x)$ is simply $p(x) \cdot x$. In a model of **perfect competition**, that is, a model characterized by the assumptions that there are many firms and that no individual firm can control the output price by its productive activity, the price any firm receives for its output is a constant $p(x) = p$, independent of the amount $x$ it produces. In this case, the firm's revenue function is simply the linear function $R(x) = p \cdot x$, and

$$MR = AR = p; \tag{5}$$

marginal revenue and average revenue are equal.

A firm's **profit function** is simply the difference

$$\Pi(x) = R(x) - C(x)$$

between its revenue and its cost at any level of output $x$. Economists often use the uppercase Greek letter pi, $\Pi$, to denote profit. The domain of $\Pi$, $R$, and $C$ is generally the nonnegative half-line $[0, \infty)$. If we assume that the goal of the firm is to choose the output level $x^*$ that maximizes its profit, then by Theorem 3.3, the optimal output level $x^*$ — if not zero — satisfies

$$\frac{d\Pi}{dx}(x^*) = \frac{dR}{dx}(x^*) - \frac{dC}{dx}(x^*) = MR(x^*) - MC(x^*) = 0,$$

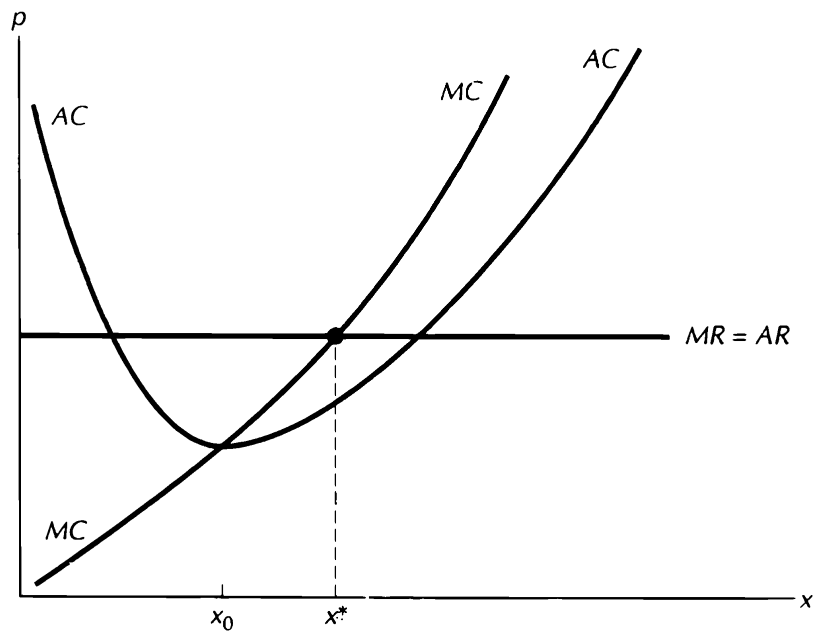
or

$$MR(x^*) = MC(x^*).$$

This principle, that marginal revenue equals marginal cost at the optimal output, is one of the cornerstones of economic theory. It is a rather intuitive guideline. A firm should continue producing output until the cost of producing one more unit ($MC$) is just offset by the revenue that the additional unit will bring in ($MR$). If the firm will receive more for the next unit than that unit will add to its cost ($MR > MC$), then producing that next unit will increase the firm's profit and it should carry out the production. If the cost of making one more unit is more than the revenue that the unit will bring in the market ($MC > MR$), then producing that additional unit will cut into the firm's profit; the firm should have stopped production earlier.

Let's look more carefully at the perfectly competitive case where the revenue function is $R(x) = p \cdot x$. In Figure 3.19, we have drawn a typical average cost ($AC$) and marginal cost ($MC$) curve, as in Figure 3.18. We have added a horizontal line at $y = p$ to represent a firm's marginal revenue ($MR$) and average revenue curve ($AR$), according to (5). The optimal output point $x^*$ — where $MR = MC$ — is darkened in Figure 3.19 at the intersection of the $MR$- and $MC$-curves.

If the market price $p$ of the output were to increase, the $MR$-line $y = p$ in Figure 3.19 would move up and the corresponding optimal output would increase too. At each stage, price $p$ and optimal output $x$ are related by the equation $p = MC(x)$ and the optimal output is represented by a point on the marginal cost




*The AC-, MC-, AR-, and MR-curves for a competitive firm.*

curve. Another way of stating the fact that, for every price, the optimal amount of output a firm will supply lies at the point where the horizontal price line crosses the $MC$-curve is to say that the $MC$-curve gives the locus of the price-optimal output combinations. In the language of economics, the $MC$-curve is the firm's **supply curve**, the curve which relates the market price to the amount produced.

Finally, we bring into the picture the second derivative condition that an interior optimal output $x^*$ must satisfy. Since $\Pi'(x) = p - C'(x)$,

$$\Pi''(x) = 0 - C''(x).$$

At the interior maximizer, $\Pi''(x^*) \leq 0$ by Theorem 3.4. This implies that $C''(x) \geq 0$ and leads to the principle that at its optimal output the firm should be experiencing *increasing marginal cost.*

## Demand Functions and Elasticity

A firm's revenue function $R(x)$ can be written as the product of the amount sold times the unit selling price. In simple models, we assume that the amount sold equals the amount $x$ produced. In the model of perfect competition analyzed in Figure 3.19, we assumed that the selling price is a scalar $p$ that is independent of the amount produced. However, in models of monopoly (an industry with a single firm) and oligopoly (an industry dominated by a small number of firms), there is usually a relationship between the amount $x$ of the product in the market and the price at which the product sells. If this relationship is represented by a function $x = F(p)$, which expresses the amount $x$ consumed in terms of price level $p$, then $F$ is called a **demand function.** If the relationship is expressed by a function $p = G(x)$ which expresses the price $p$ in terms of the amount $x$ being consumed, then $G$ is called an **inverse demand function.** In a single-firm industry, it is the inverse demand function that is the natural factor of the revenue function, since the latter can be written as

$$R(x) = p \cdot x = G(x) \cdot x.$$

Since $G(x) = R(x)/x$, the inverse demand function is also the firm's average revenue function.

Economists, of course, are deeply interested in how changes in price affect changes in demand. The natural measure of this sensitivity is the slope of the demand function, $F'(p)$ or $\Delta x/\Delta p$. As we well know, this *marginal demand* describes the effect of a unit increase in price on the purchasing behavior of consumers. However, this sensitivity indicator has one major disadvantage: it is highly dependent on the units used to measure quantity and price. Suppose, for example, that a 10-cent increase in price will lead to a million-gallon decrease in the consumption of gasoline. The marginal demand is

$$\frac{\Delta x}{\Delta p} = \frac{-10^6}{10} = -10^5$$

if we measure $x$ in gallons and $p$ in cents. However, if we measure $x$ in gallons and $p$ in *dollars*, then the marginal demand changes by a factor of 100 to

$$\frac{\Delta x}{\Delta p} = \frac{-10^6}{10} = -10^7 \tag{6}$$

Finally, if we use a million gallons as our unit of gasoline consumption and the cent as our unit of price, then the marginal demand becomes

$$\frac{\Delta x}{\Delta p} = \frac{-1}{10} = -0.1,$$

100 million times smaller than the measure in (6). Economists would like a measure of the sensitivity of demand to price changes which cannot be manipulated by choice of units and which can be used to compare consumption habits in different countries with different currencies and different measures of weight or volume.

The solution to this problem is to use the percent rate of change instead of the actual change. For any quantity, the **percent rate of change** is the actual change divided by the initial amount:

$$\frac{q_1 - q_0}{q_0} = \frac{\Delta q}{q_0}.$$

Since the numerator and denominator are measured in the same units, the units cancel out in this division process. For example, if the price changes from \$1.25 to \$1.50, the percent rate of change of price is

$$\frac{1.50 - 1.25}{1.25} = \frac{0.25}{1.25} = \frac{1}{5} = 20\,\text{percent}.$$

It will be 20 percent whether we choose dollars, cents, or French franc equivalents as our unit of currency.

To keep the sensitivity measure completely free of units, we will measure both the change in quantity and the change in price in percentage terms. Our measure of sensitivity now becomes the *percent* change in quantity demanded divided by the *percent* change in price,

$$\frac{\Delta x}{x} \Big/ \frac{\Delta p}{p}, \tag{7}$$

in other words, the percent change in demand for each 1 percent rise in price. This sensitivity measure is called the **price elasticity of demand** and usually represented by the Greek letter epsilon, $\varepsilon$. Rewrite the double quotient (7) as a single quotient:

$$\varepsilon = \frac{\Delta x}{x} \Big/ \frac{\Delta p}{p} = \frac{\Delta x}{x} \cdot \frac{p}{\Delta p} = \frac{\Delta x}{\Delta p} \cdot \frac{p}{x} = \frac{\Delta x}{\Delta p} \Big/ \frac{x}{p}. \tag{8}$$

The factor $\Delta x/\Delta p$ in the last two terms is just the marginal demand, while the quotient $x/p$ in the last term is simply the average demand. So, the elasticity can be thought of as the marginal demand divided by the average demand.

The marginal demand can, of course, be well approximated by the slope $F'(p)$ of the demand function $x = F(p)$. Substituting $F'(p)$ for $\Delta x/\Delta p$ and $F(p)$ for $x$ in (8) yields the calculus form of the **price elasticity**:

$$\varepsilon = \frac{F'(p) \cdot p}{F(p)}. \tag{9}$$

**Notation** The discrete version (8) of the price elasticity is called the **arc elasticity** and is usually used to compute $\varepsilon$ when we know only a finite number of price-quantity combinations. The differentiable version (9) of the price elasticity is called the **point elasticity** and is used when a continuous demand curve has been estimated or in proving theorems about price elasticity.

We will soon use (9) to prove an illuminating relationship between elasticity and total revenue or expenditure. First, we take a closer look at the elasticity and, in the process, introduce some more vocabulary. A basic assumption about demand functions is that raising the price of a commodity usually lowers the amount consumed. Mathematically speaking, demand is a decreasing function of price. (We are ignoring the empirically rare phenomenon of a Giffen good — a good for which lower prices lead to lower consumption.) This assumption means that $\Delta x/\Delta p$ in (8) and $F'(p)$ in (9) are negative numbers, as we saw in (6), and therefore that the price elasticity of a good is a negative number. (Some intermediate economics texts define the price elasticity as the absolute value of the expression in (8) or (9) to avoid dealing with negative numbers. We won't.)

A good which is rather insensitive to price changes will have a price elasticity close to zero. Necessities, like fuel oil and medical care, are good examples of this phenomenon. On the other hand, a good for which small price increases lead to large drops in consumption — speaking in terms of percentages — will have a price elasticity that is a large negative number. Luxury items, like Lamborghinis and ermine coats, and items with many close substitutes, like Froot Loops or Cap'n Crunch cereals, are examples of this phenomenon. The following definitions add some precision to these concepts.

**Definition** A good whose price elasticity lies between 0 and $-1$ is called **inelastic**. A good whose price elasticity lies between $-1$ and $-\infty$ is called **elastic**. A good whose price elasticity equals $-1$ is said to be **unit elastic**.

If the price of a good goes up, the change in total expenditure on that good is, at first glance, indeterminate, since expenditure is price times quantity, and the two move in opposite directions. As the next theorem shows, the elasticity of the good in question resolves this ambiguity.

**Theorem 3.8** For an inelastic good, an increase in price leads to an *increase* in total expenditure. For an elastic good, an increase in price leads to a *decrease* in total expenditure.

*Proof* Let $x = F(p)$ be the demand function for the good under study. The total expenditure at price $p$ is

$$E(p) = p \cdot x = p \cdot F(p).$$

To see whether $E(p)$ is increasing or decreasing, we need only check the sign of its first derivative. By the Product Rule,

$$E'(p) = p \cdot F'(p) + 1 \cdot F(p).$$

Divide both sides by the *positive* quantity $F(p)$:

$$\frac{E'(p)}{F(p)} = \frac{p \cdot F'(p)}{F(p)} + 1 = \varepsilon + 1, \tag{10}$$

by (9). If the good is inelastic, $-1 < \varepsilon < 0$ and $\varepsilon + 1 > 0$. In this case, (10) is positive, $E'(p)$ is positive, and therefore $E(p)$ is an increasing function of $p$. Similarly, if the good is elastic, $\varepsilon < -1$ and $\varepsilon + 1 < 0$. Now, each expression in (10) is negative, $E'(p)$ is negative, and therefore $E(p)$ is a decreasing function of $p$. ■

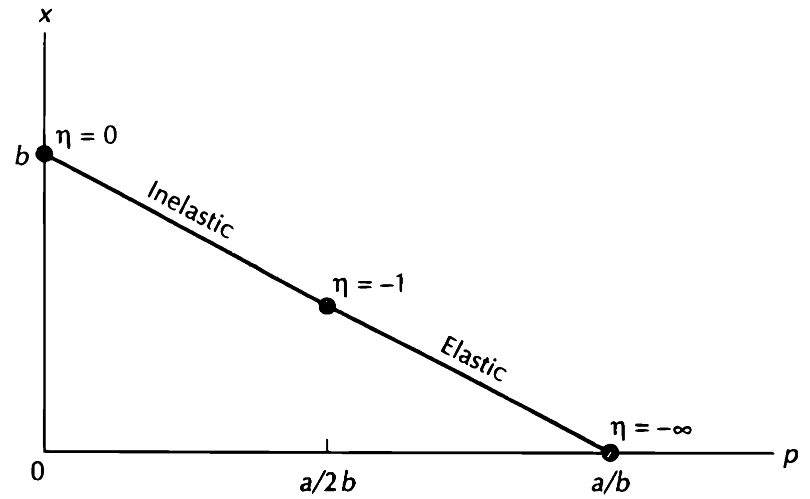
In working with concrete economic models, economists sometimes use specific functional forms for the economy's demand functions, especially **linear demand**

$$x = F(p) \equiv a - bp, \qquad a, b > 0, \tag{11}$$

and **constant elasticity demand**

$$x = F(p) \equiv kp^{-r}, \qquad k, r > 0. \tag{12}$$

For (11), the demand function is a straight line segment with negative slope $-b$ and $x$-intercept $a$, as pictured in Figure 3.20. Since the slope of $F$ differs from




*The graph of the linear demand function $x = a - bp$.* **Figure 3.20** the elasticity of $F$, it should not be surprising that the elasticity varies along the demand curve:

$$\varepsilon \equiv \frac{F'(p) \cdot p}{F(p)} = \frac{-bp}{a - bp} = \frac{1}{1 - (a/bp)}$$

from $\varepsilon = 0$ when $p = 0$ and $x = a$ to $\varepsilon = -\infty$ when $p = a/b$ and $x = 0$.

In Figure 3.20, the graph of the demand function $x = F(p)$ is drawn in what appears to be the most natural way — with the independent variable $p$ measured along the horizontal axis and the dependent variable $x$ measured along the vertical axis. However, for all the other quantity-price relationships we have studied so far — cost, revenue, profit, and their marginal and average manifestations — the output $x$ was naturally the independent variable and was represented along the horizontal $x$-axis while the price variable was naturally the dependent variable and was represented along the vertical $y$-axis. Because this use of the horizontal and vertical axes is the more common situation for economic functions and because we will want to incorporate the demand relationship into our graphical studies of revenue, cost, and profit curves, as in Figure 3.19, as average revenue curves, we will graph the demand relationship with quantity on the horizontal axis and price on the vertical axis.

This section concludes with the incorporation of the demand, average revenue, and marginal revenue curves into our analyses of the cost curves in Figure 3.18. This was done in Figure 3.19 for the case of perfect competition where the average revenue curve was a horizontal line. Now treat the antipodal case of a pure monopolist facing a linear demand curve $x = a - bp$ for its product. The inverse demand curve is

$$p = \frac{a}{b} - \frac{1}{b}x. \tag{13}$$

If the monopolist wants to sell $x$ units, it will have to charge the price $p$ given by (13). Therefore, the monopolist's revenue function is

$$R(x) = \left(\frac{a}{b} - \frac{1}{b}x\right) \cdot x.$$

The marginal revenue is

$$R'(x) = \frac{a}{b} - \frac{2}{b}x,$$

a curve with the same $p$-intercept but with twice the slope of the average revenue (= inverse demand) curve. These curves are sketched in Figure 3.21. The optimal output occurs at the point $x^*$ above which the $MR$- and $MC$-curves cross. The corresponding selling price $p^*$ can be read off the demand curve above $x^*$ (*not* from

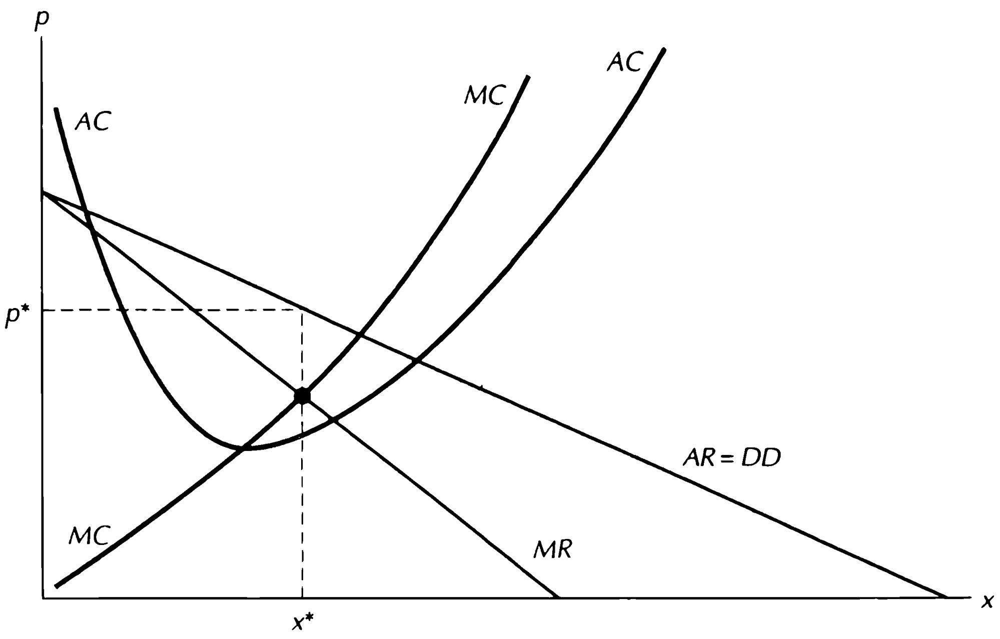



Figure 3.21 *The MC-, AC-, MR- and AR-curves for a pure monopolist.*

the *MR*-curve). One can use Figure 3.21 to show, for example, that if manufacturing costs increase so that the *MC*-curve rises, then output will decline and price will rise.

## EXERCISES

**3.15** Show that the function $f(x) = x^3 + x + 1$ has the essential properties of a cost function. Carefully graph its corresponding average cost function and marginal cost function on the same coordinate axes and compare your answer with Figure 3.18.

**3.16** What happens to a competitive firm whose cost function exhibits *decreasing* marginal cost everywhere? Construct a concrete cost function of this type and carry out the search for the profit-maximizing output.

**3.17** *a*) Which rectangle in Figure 3.19 has an area equal to the firm's optimal revenue at $x = x^*$?

*b*) Using the fact that $AC(x) = C(x)/x$ and therefore $C(x) = AC(x) \cdot x$, find the rectangle whose area gives the total cost of output $x^*$.

*c*) Which area in Figure 3.19 represents the firm's optimal profit?

**3.18** Prove that the point elasticity is $-1$ exactly at the midpoint of the linear demand in Figure 3.20.

**3.19** Compute the point elasticity for the demand function (12) and conclude why (12) is called the constant elasticity demand.

**3.20** What happens to $x^*$ and $p^*$ if the demand curve rises in Figure 3.21?

**3.21** Indicate carefully the rectangle in Figure 3.21 whose area gives the monopolist's profit.

**3.22** For $F(p) = a - bp$ and $C(x) = kx^2$, calculate explicitly the formula for the optimal output and its price.

CHAPTER 4

# One-Variable Calculus: Chain Rule

Many economic situations involve chains of relationships between economic variables: variable $A$ affects variable $B$ which in turn affects variable $C$. For example, in a model of a firm, the amount of input used determines the amount of output produced, and the amount of output produced determines the firm's revenue. Revenue is a direct function of output and an indirect or composite function of input. This chapter presents the **Chain Rule**, which describes the derivative of a composite function in terms of the derivatives of its component functions, so that if the effect of a change in input on output is known and the effect of a change in output on revenue is known, the effect of a change in input on revenue can be computed.

Section 4.2 focuses on invertible functions. Such functions correspond to relationships between economic variables, say $A$ and $B$, in which sometimes we want to understand the effect of $A$ on $B$ and other times we are more interested in how $B$ affects $A$. For example, economists are usually concerned about how a price increase affects demand, but they sometimes focus on how a change in demand affects prices. There is, of course, a close relationship between the derivative of a function and the derivative of its inverse function; if we know one, we can deduce the other. We will use the concept of an inverse function and its derivative in the next chapter when we study the logarithmic function as the inverse of the exponential function.

Finally, at the end of this chapter, these mathematical results are used to compute the derivative of the function $f(x) = x^{m/n}$. This function arises naturally in many economic models, such as Cobb-Douglas utility and production functions.

## 4.1 COMPOSITE FUNCTIONS AND THE CHAIN RULE

### Composite Functions

Section 2.4 described the rules for computing the derivative of a function that is formed by taking the sum, difference, product, or quotient of two other functions. This section presents and then applies the Chain Rule — the formula for differentiating a function which is formed by taking the *composition* of two other functions. If $g$ and $h$ are two functions on $\mathbf{R}^1$, the function formed by first applying function $g$ to any number $x$ and then applying function $h$ to the result $g(x)$ is called the **composition** of functions $g$ and $h$ and is written as

$$f(x) = h(g(x)) \quad \text{or} \quad f(x) = (h \circ g)(x).$$

The function $f$ is called the **composite** of functions $h$ and $g$; we say that "$f$ is $h$ composed with $g$" and the "$f$ is $g$ followed by $h$."

*Example 4.1* For example, if $g(x) = x^2$ and $h(x) = x+4$, then $(h \circ g)(x) = x^2+4$. If the order of composition is reversed in this case, then $(g \circ h)(x) = (x + 4)^2$. Note that $h \circ g \neq g \circ h$.

When we compose two functions, we are taking a function of a function. For example, if the second function raises $x$ to the power $k$, that is, if $h(x) = x^k$, then $h(g(x)) = (g(x))^k$ raises $g(x)$ to the power $k$. This is the most common composite function one meets in the calculus of polynomials and rational functions. However, when dealing with exponential, logarithmic, and trigonometric functions, one regularly deals with general composite functions.

*Example 4.2* The functions which describe a firm's behavior, such as its profit function $\Pi$, are usually written as functions of a firm's *output* $y$. If one wants to study the dependence of a firm's profit on the amount of labor *input* $L$ it uses, one must compose the profit function with the firm's production function $y = f(L)$, the function which tells how much output $y$ the firm can obtain from $L$ units of labor input. The result is a function

$$\mathcal{P}(L) \equiv \Pi(f(L)) = (\Pi \circ f)(L).$$

For example, if

$$\Pi(y) = -y^4 + 6y^2 - 5 \quad \text{and} \quad f(L) = 5L^{2/3}, \tag{1}$$

then

$$\begin{aligned} \mathcal{P}(L) &= \Pi(f(L) \\ &= -(5L^{2/3})^4 + 6(5L^{2/3})^2 - 5 \\ &= -625L^{8/3} + 150L^{4/3} - 5. \end{aligned} \tag{2}$$

Note that a different letter is used to denote profit as a function of $L$ than to denote profit as a function of $y$, simply because they are different functions.

*Example 4.3* Composite functions arise naturally in **dynamic models** — models in which the variables vary over time. For example, let $x = F(p)$ denote the market demand function for a commodity in terms of its price. Suppose that because of inflation or external events, the commodity's price changes over time according to the function $p = p(t)$. Then, the commodity's demand will also vary over time, in accordance with the composite function

$$\mathcal{F}(t) \equiv F(p(t)).$$

When working with a composite function $f(x) = h(g(x))$, it is natural to call the first function one applies ($g$ in this case) the **inside function** and the second function one applies ($h$ in this case) the **outside function**. For example, in the composition $(x^2 + 3x + 2)^7$, the inside function is $g(x) = x^2 + 3x + 2$ and the outside function is $h(z) = z^7$.

## Differentiating Composite Functions: The Chain Rule

In Section 2.4, we introduced the *Power Rule*, which is the rule for taking the derivative of a composite function in which the outside function is $h(z) = z^k$ for some exponent $k$:

$$\text{Power Rule:} \qquad \frac{d}{dx}(g(x))^k = k(g(x))^{k-1} \cdot g'(x). \tag{3}$$

In words, the derivative of a function to the $k$th power is $k$ times the function to the $(k - 1)$th power times the derivative of the function. Since the derivative of $h(z) = z^k$ is $h'(z) = kz^{k-1}$, we can think of (3) as the derivative of the outside function $h$ (evaluated at the inside function $g$) times the derivative of the inside function $g$; in symbols,

$$\frac{d}{dx}(h(g(x))) = h'(g(x)) \cdot g'(x). \tag{4}$$

Formula (4) is precisely the formula for differentiating a general composite $h \circ g$ of two functions $h$ and $g$. In this general form, it is called the **Chain Rule**. It is often quickly summarized as "the derivative of the outside times the derivative of the inside," but one must remember that the derivative of the outside function is evaluated at the inside function.

*Example 4.4* Let's apply the Chain Rule (4) to compute the derivative of the composite function $\mathcal{P} = \Pi \circ f$, given by (1) in Example 4.2. The outside function is

$$\Pi(\;) = -(\;)^4 + 6(\;)^2 - 5;$$

the derivative of the outside function is

$$\Pi'(\;) = -4(\;)^3 + 12(\;),$$

and the derivative evaluated at the inside function $f(L) = 5L^{2/3}$ is

$$\Pi'(f(L)) = -4(5L^{2/3})^3 + 12(5L^{2/3}).$$

On the other hand, the derivative of the inside function $f$ is

$$f'(L) = \frac{10}{3}L^{-1/3}.$$

Multiplying these two expressions according to the Chain Rule (3) yields

$$\begin{aligned} \mathcal{P}'(L) &= \frac{d}{dL}(\Pi(f(L))) = \Pi'(f(L)) \cdot f'(L) \\ &= \left(-4(5L^{2/3})^3 + 12(5L^{2/3})\right) \cdot \left(\frac{10}{3}L^{-1/3}\right), \end{aligned}$$

which after simplifying equals

$$(-4 \cdot 125L^2 + 60L^{2/3}) \cdot \left(\frac{10}{3}L^{-1/3}\right) = -\frac{5000}{3}L^{5/3} + 200L^{1/3}.$$

Note that this agrees with what we compute by directly taking the derivative of the expression (2) for the composite function $\mathcal{P}(L)$:

$$\left(-625L^{8/3} + 150L^{4/3} - 5\right)' = -\frac{5000}{3}L^{5/3} + 200L^{1/3}.$$

*Example 4.5* To see how the Chain Rule works with functions other than generalized polynomials, consider the trigonometric functions sine $x$ and cosine $x$, which are usually abbreviated as $\sin x$ and $\cos x$. At this point you need only know that the derivative of the function $\sin x$ is the function $\cos x$. To compute the derivative of the composite function

$$f(x) = \sin(x^3 + 4x),$$

note that $f$ is a composite of $(x^3 + 4x)$, the inside function, and $\sin z$, the outside function. The derivative of the outside function is

$$\frac{d}{dz}\sin(\ ) = \cos(\ ).$$

The derivative of the outside function, evaluated at the inside function, is

$$\cos(x^3 + 4x).$$

The derivative of the inside function $(x^3 + 4x)$ is $(3x^2 + 4)$. By the Chain Rule, the derivative of $\sin(x^3 + 4x)$ is

$$\frac{d}{dx}(\sin(x^3 + 4x)) = \cos(x^3 + 4x) \cdot (3x^2 + 4).$$

**Notation** In addition to the phrase "derivative of the outside times derivative of the inside," there is one other convenient device for remembering and using the Chain Rule. Continue to write the inside function as $g(x)$ and the outside function as $h(z)$. Then, the Chain Rule can be written as

$$\frac{d(h \circ g)}{dx}(x) = \frac{dh}{dz}(g(x)) \cdot \frac{dg}{dx}(x). \tag{5}$$

To derive a new mnemonic device for the Chain Rule, we will allow ourselves three abuses of notation. First, write the inside function $g$ as $z = g(x)$, since $g(x)$ will be used as the argument of the outside function $h(z)$. Second, temporarily ignoring the above warning about always using different letters for the different functions, write $h(x)$ for $h(g(x))$. Finally, ignoring the fact that $h'(z)$ is evaluated at $z = g(x)$, write (5) as

$$\frac{dh}{dx} = \frac{dh}{dz} \cdot \frac{dz}{dx},$$

which is a deceptively simple-looking formula.

## EXERCISES

**4.1** For each of the following pairs of functions $g$ and $h$, write out the composite functions $g \circ h$ and $h \circ g$ in as simple a form as possible. In each case, describe the domain of the composite.

*a)* $g(x) = x^2 + 4$, $\quad h(z) = 5z - 1$;

*b)* $g(x) = x^3$, $\quad h(z) = (z - 1)(z + 1)$;

*c)* $g(x) = (x - 1)/(x + 1)$, $\quad h(z) = (z + 1)/(1 - z)$;

*d)* $g(x) = 4x + 2$, $\quad h(z) = \frac{1}{4}(z - 2)$;

*e)* $g(x) = 1/x$, $\quad h(z) = z^2 + 1$.

**4.2** For each of the following composite functions, what are the inside and outside functions? *a)* $\sqrt{3x^2 + 1}$, *b)* $(1/x)^2 + 5(1/x) + 4$, *c)* $\cos(2x - 7)$, *d)* $3^{4t+1}$.

**4.3** Use the Chain Rule to compute the derivative of all the composite functions in Exercise 4.1 from the derivatives of the two component functions. Then, compute each derivative directly, using your expression for the composite function, simplify, and compare your answers.

**4.4** Repeat the calculations of the previous exercise for the composite functions in Exercise 4.2.

**4.5** Given that the derivative of $\sin x$ is $\cos x$, the derivative of $\exp(x)$ is $\exp(x)$ itself, and the derivative of $\log x$ is $1/x$, use the Chain Rule to calculate the derivatives of the following composite functions:

| | | | |
|---|---|---|---|
| *a*) $\sin(x^4)$, | *b*) $\sin(1/x)$, | *c*) $\sqrt{\sin x}$, | *d*) $\sin\sqrt{x}$, |
| *e*) $\exp(x^2 + 3x)$, | *f*) $\exp(1/x)$, | *g*) $\log(x^2 + 4)$, | *h*) $\sin((x^2 + 4)^2)$. |

**4.6** A firm computes that at the present moment its output is increasing at a rate of 2 units per hour and that its marginal cost is 12. At what rate is its cost increasing per hour? Explain your answer.

## 4.2 INVERSE FUNCTIONS AND THEIR DERIVATIVES

The Chain Rule is one of the most useful theorems in calculus, both in analyzing applications and in deriving other principles of calculus. As an illustration, it will be used in this section to derive the formula for the derivative of the inverse of a function when the derivative of the original function is known.

### Definition and Examples of the Inverse of a Function

Consider the demand relationship between the market price $p$ and the amount $x$ that consumers are willing to consume at that price. Economists sometimes find it convenient to think of this relationship as defining $x$ as a function of $p$, for example when computing elasticities, and sometimes as defining $p$ as a function of $x$, for example when computing marginal revenue in the process of studying the profit-maximizing output. The former function is called a demand function, and the latter an inverse demand function. For example, if the demand function is given by the linear function

$$x = 3 - 2p, \tag{6}$$

then the inverse demand function is obtained by solving (6) for $p$ in terms of $x$:

$$p = \frac{1}{2}(3 - x). \tag{7}$$

This same inverse relationship exists between the function in Example 2.3, which converts degrees Centigrade to degrees Fahrenheit:

$$F = \frac{9}{5}C + 32, \tag{8}$$

and the function which converts degrees Fahrenheit to degrees Centigrade:

$$C = \frac{5}{9}(F - 32). \tag{9}$$

We say that the function $p \longmapsto 3 - 2p$ in (6) is the *inverse* of the function $x \longmapsto \frac{1}{2}(3 - x)$ in (7), and vice versa. Similarly, the functions $C \longmapsto \frac{9}{5}C + 32$ and $F \longmapsto \frac{5}{9}(F - 32)$ in (8) and (9) are *inverses* of each other. More formally, for any given function $f: E_1 \longrightarrow \mathbf{R}^1$, where $E_1$, the domain of $f$, is a subset of $\mathbf{R}^1$, we say the function $g: E_2 \longrightarrow \mathbf{R}^1$ is an **inverse** of $f$ if

$$\begin{aligned} g(f(x)) &= x \text{ for all } x \text{ in the domain } E_1 \text{ of } f \text{ and} \\ f(g(z)) &= z \text{ for all } z \text{ in the domain } E_2 \text{ of } g. \end{aligned} \tag{10}$$

*Example 4.6* To see that the functions described by expressions (6) and (7) are inverses of each other, form their composition by substituting the expression (7) for $p$ into (6):

$$x = 3 - 2\left(\frac{1}{2}(3 - x)\right) = 3 - (3 - x) = x.$$

*Example 4.7* Other examples of functions and their inverses are:

$$\begin{aligned} f(x) &= 2x && \text{and} && g(y) = \tfrac{1}{2}y, \\ f(x) &= x^2 && \text{and} && g(y) = \sqrt{y} \qquad \text{for } x, y \geq 0, \\ f(x) &= x^3 && \text{and} && g(y) = y^{1/3}, \\ f(x) &= \frac{x-1}{x+1} && \text{and} && g(y) = \frac{1+y}{1-y}, \\ f(x) &= \frac{1}{x} && \text{and} && g(y) = \frac{1}{y}. \end{aligned}$$

Note that $1/x$ is its own inverse.

Suppose the function $f$ has an inverse $g$. If $f$ assigns the point $y_0$ to the point $x_0$, then $g$ assigns the point $x_0$ to the point $y_0$. In symbols,

$$f(x_0) = y_0 \iff g(y_0) = x_0.$$

If $f$ assigns the same point $y_0$ to a point $z_0 \neq x_0$, that is, $f(z_0) = y_0$ too, then $g$ would also need to assign $z_0$ to $y_0$; that is, $g(y_0) = z_0$. But then $g$ would not be a well-defined function at $y_0$ since it would assign two different numbers $x_0$ and $z_0$

to $y_0$. In order for $f$ to have an inverse $g$, $f$ cannot assign the same point to two different points in its domain; in symbols

$$x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2), \tag{11}$$

or equivalently

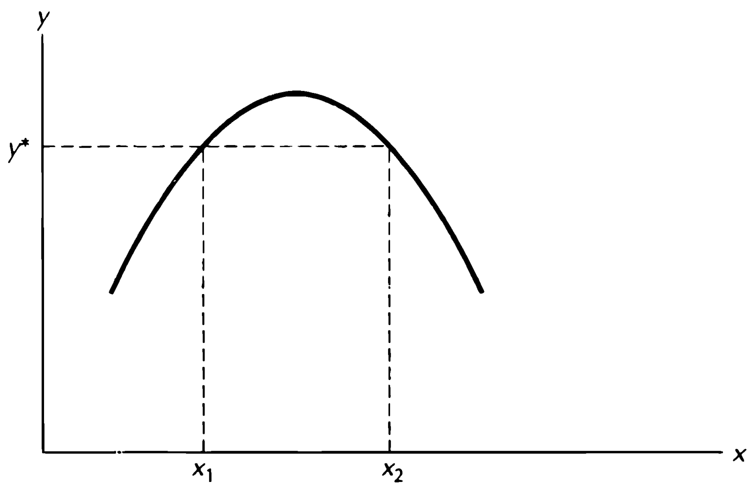
$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2. \tag{12}$$

A function $f$ that satisfies (11) or (12) on a set $E$ is said to be **one-to-one** or **injective** on $E$.

To summarize the previous paragraph, in order for a function to be invertible, it must be one-to-one. Conversely, if a function $f$ is one-to-one on a set $E$, there is a well-defined function $g: f(E) \to \mathbf{R}^1$ which sends each point $y$ in the image of $f$ back to the (unique) point to which $f$ assigned it. If $f$ is given by a formula which expresses $y$ in terms of $x$, one finds a formula for its inverse $g$ by rewriting the formula for $f$ to express $x$ in terms of $y$. If this process determines a unique $x$ for every $y$, the new formula defines the inverse $g$ of $f$.

**Notation** If $f$ is invertible on its domain, then its inverse is uniquely defined. We often write $f^{-1}$ for the inverse function of $f$.

It is easy to look at the graph of a function $f$ defined on a interval $E$ of $\mathbf{R}^1$ and determine whether or not $f$ is one-to-one on $E$. As Figure 4.1 illustrates, the graph of $f$ cannot turn around; that is, it cannot have any local maxima or minima on $E$. It must be monotonically increasing or monotonically decreasing on $E$. The function whose graph is pictured in Figure 4.1 is not one-to-one because two points $x_1$ and $x_2$ map to the same point $y^*$.




*A function is not one-to-one in an interval containing a local max or min.* **Figure 4.1**

*Example 4.8* Consider the function $f(x) = x^2$. As a function defined on all of $\mathbf{R}^1$, $f$ is not one-to-one since it sends both $x = -2$ and $x = +2$ to the point $y = 4$. Its inverse $g$ would have to send $y = 4$ back to one of the two, say to $x = +2$. But then, $g(f(-2)) = g(4) = +2$ and $g$ does not satisfy the definition

(10) of an inverse. However, if we restrict the domain of $f$ to be the nonnegative numbers $[0, \infty)$, as we did in Example 4.7, then the restricted $f$ *is* one-to-one and therefore it has a well-defined inverse $g(y) = \sqrt{y}$. The domain of $g$ is also the interval $[0, \infty)$. See Figure 4.3.

*Example 4.9* The function $x^3 - 3x$, whose graph is pictured in Figure 3.2, is not one-to-one on $\mathbf{R}^1$, since $x = -\sqrt{3},\ 0, +\sqrt{3}$ all map to $y = 0$. As further evidence, $f$ has two local extrema, so it is not a monotone function. However, since $f$ is monotone for $x > 1$, its restriction to $(1, \infty)$ is invertible.

The following theorem summarizes the discussion thus far.

**Theorem 4.1** A function $f$ defined on an interval $E$ in $\mathbf{R}^1$ has a well-defined inverse on the interval $f(E)$ if and only if $f$ is monotonically increasing on all of $E$ or monotonically decreasing on all of $E$.

For differentiable functions, Theorem 3.2 gives a calculus criterion for a function to be monotonically increasing or decreasing. Combining that result with Theorem 4.1 leads naturally to the following theorem.

**Theorem 4.2** A $C^1$ function $f$ defined on an interval $E$ in $\mathbf{R}^1$ is one-to-one and therefore invertible on $E$ if $f'(x) > 0$ for all $x \in E$ or $f'(x) < 0$ for all $x \in E$.

From a geometric point of view, if $f$ sends $x_0$ to $y_0$, so that the point $(x_0, y_0)$ is on the graph of $f$, then $f^{-1}$ sends $y_0$ back to $x_0$ and therefore the point $(y_0, x_0)$ is on its graph. For any point $(a, b)$ on the graph of $f$, the point $(b, a)$ is on the graph

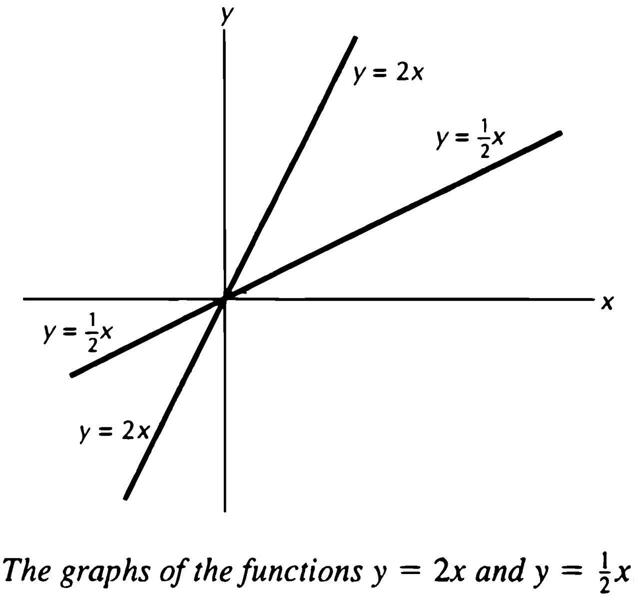
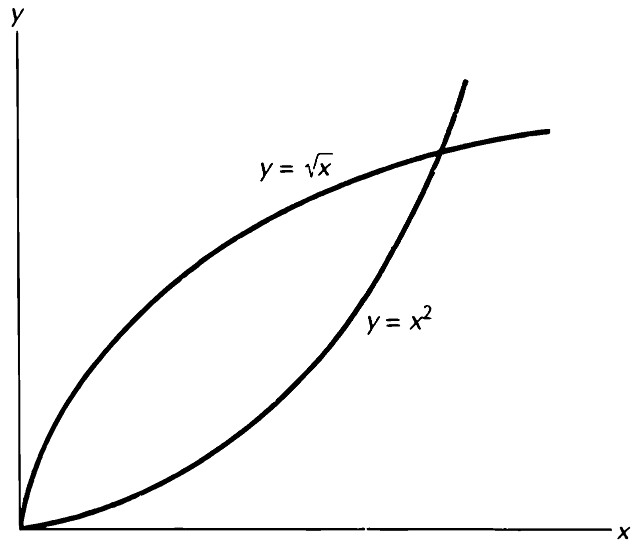



**Figure 4.2** *The graphs of the functions $y = 2x$ and $y = \frac{1}{2}x$.*




*The graphs of the functions* $y = x^2$ *and* $y = \sqrt{x}$ *for* $x, y \geq 0$.

**Figure 4.3**

of $f^{-1}$. This means that the graph of $f^{-1}$ is simply the reflection of the graph of $f$ across the diagonal line $\{x = y\}$. Figures 4.2 and 4.3 illustrate this phenomenon for the first two pairs of functions in Example 4.7.

### The Derivative of the Inverse Function

Since there is such a close relationship between the graph of an invertible function $f$ and the graph of its inverse $f^{-1}$, it's not surprising that there is a close relationship between their derivatives. In particular, if $f$ is $C^1$ so that its graph has a smoothly varying tangent line, the graph of $f^{-1}$ will also have a smoothly varying tangent line; that is, $f^{-1}$ will be $C^1$ too. The following theorem combines this observation with Theorem 4.2 to give a rather complete picture for the existence and differentiability of the inverse of a $C^1$ function.

**Theorem 4.3** **(Inverse Function Theorem)** Let $f$ be a $C^1$ function defined on the interval $I$ in $\mathbf{R}^1$. If $f'(x) \neq 0$ for all $x \in I$, then:

(*a*) $f$ is invertible on $I$,
(*b*) its inverse $g$ is a $C^1$ function on the interval $f(I)$, and
(*c*) for all $z$ in the domain of the inverse function $g$,

$$g'(z) = \frac{1}{f'(g(z))}. \tag{13}$$

*Proof* The existence of $f^{-1}$ follows from Theorem 4.1. Since the graph of $f^{-1}$ is the reflection of the graph of $f$ across the diagonal line $\{y = x\}$, the graph of $f^{-1}$ will have a well-defined tangent line everywhere, i.e., be differentiable, if the graph of $f$ does. Assuming that $g = f^{-1}$ is differentiable, we compute $g'$ by first writing the inverse relation as in (10):

$$f(g(z)) = z. \tag{14}$$

Now, take the derivative of both sides of (14) with respect to $z$, using the Chain Rule on the left side:

$$f'(g(z)) \cdot g'(z) = 1,$$

or

$$g'(z) = \frac{1}{f'(g(z))}. \quad \blacksquare$$

*Example 4.10* The inverse of $y = f(x) \equiv mx$ is $x = g(y) = (1/m)y$. Note that

$$g'(y) = \frac{1}{m} = \frac{1}{f'(x)}.$$

*Example 4.11* Let's work with the fourth set of functions in Example 4.7. Start with

$$f(x) = \frac{x-1}{x+1} \quad \text{and} \quad x = 2.$$

Since $f(2) = 1/3$, the inverse $g$ of $f$ sends $1/3$ to 2. Since $f'(x) = 2/(x+1)^2$, $f'(2) = 2/9$. By Theorem 4.3,

$$g'\left(\frac{1}{3}\right) = \frac{1}{f'(2)} = \frac{1}{2/9} = \frac{9}{2}.$$

We can check this answer by computing directly that

$$g(y) = \frac{1+y}{1-y}, \quad g'(y) = \frac{2}{(1-y)^2}, \quad \text{and} \quad g'\left(\frac{1}{3}\right) = \frac{2}{4/9} = \frac{9}{2}.$$

## The Derivative of $x^{m/n}$

In Theorem 2.3 and Exercise 2.14, we proved that the derivative of $x^k$ is $kx^{k-1}$ for any integer $k$. In Theorem 2.4, we stated without proof that this formula holds for any number $k$. In this section, we will use Theorem 4.3 and the Chain Rule to show that this formula holds for *any rational number* $k = m/n$.

**Theorem 4.4** For any positive integer $n$,

$$\left(x^{1/n}\right)' = \frac{1}{n} x^{(1/n)-1}. \tag{15}$$

*Proof* The inverse of $y = x^{1/n}$ is $x = y^n$. By Theorem 4.3,

$$\begin{aligned}\left(x^{1/n}\right)' &= \frac{1}{(y^n)'}, \quad \text{evaluated at } y = x^{1/n},\\ &= \frac{1}{ny^{n-1}}, \quad \text{evaluated at } y = x^{1/n},\\ &= \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{(1/n)-1}. \quad \blacksquare\end{aligned}$$

**Theorem 4.5** For any positive integers $m$ and $n$,

$$\left(x^{m/n}\right)' = \frac{m}{n}x^{(m/n)-1}. \tag{16}$$

*Proof* Since $x^{m/n} = \left(x^{1/n}\right)^m$, we can apply the Chain Rule directly:

$$\begin{aligned}\left(x^{m/n}\right)' &= m\left(x^{1/n}\right)^{m-1} \cdot \left(x^{1/n}\right)', \quad \text{(by the Chain Rule)}\\ &= mx^{(m-1)/n} \cdot \frac{1}{n}x^{(1-n)/n}, \quad \text{(by Theorem 4.4)}\\ &= \frac{m}{n}x^{(m-n)/n} = \frac{m}{n}x^{(m/n)-1} \quad \text{(simplifying)}. \quad \blacksquare\end{aligned}$$

Having proved that the derivative of $x^k$ is $kx^{k-1}$ for all rational numbers $k$, we can extend this result to all *real* numbers $k$ by approximating any irrational exponent by a sequence of rational numbers, applying the formula to each rational number in the sequence, and then using a limiting process.

## EXERCISES

**4.7** Substitute (6) into (7), (8) into (9), and (9) into (8) and verify that the criterion (10) of an inverse function is satisfied.

**4.8** Calculate an expression for the inverse of each of the following functions, specifying the domains carefully: *a)* $3x + 6$, *b)* $1/(x + 1)$, *c)* $x^{2/3}$, *d)* $x^2 + x + 2$. [Hint: Use the quadratic formula for *d.*]

**4.9** For each of the functions $f$ in the previous exercise, use Theorem 4.3 to compute the derivative of its inverse function at the point $f(1)$. Check your answer by directly taking the derivative of the inverse functions calculated in the previous exercise.

**4.10** Apply the Quotient Rule to the results of Theorems 4.4 and 4.5 to derive the corresponding results for *negative* exponents.

# CHAPTER 5

# Exponents and Logarithms

In the last three chapters, we dealt exclusively with relationships expressed by polynomial functions or by quotients of polynomial functions. However, in many economics models, the function which naturally models the growth of a given economic or financial variable over time has the independent variable $t$ appearing as an *exponent*; for example, $f(t) = 2^t$. These exponential functions occur naturally, for example, as models for the amount of money in an interest-paying savings account or for the amount of debt in a fixed-rate mortgage account after $t$ years.

This chapter focuses on exponential functions and their derivatives. It also describes the inverse of the exponential function — the logarithm, which can turn multiplicative relationships between economic variables into additive relationships that are easier to work with. This chapter closes with applications of exponentials and logarithms to problems of present value, annuities, and optimal holding time.

## 5.1 EXPONENTIAL FUNCTIONS

When first studying calculus, one works with a rather limited collection of functional forms: polynomials and rational functions and their generalizations to fractional and negative exponents — all functions constructed by applying the usual arithmetic operations to the monomials $ax^k$. We now enlarge the class of functions under study by including those functions in which the variable $x$ appears as an *exponent*. These functions are naturally called **exponential functions**.

A simple example is $f(x) = 2^x$, a function whose domain is all the real numbers. Recall that:

(1) if $x$ is a positive integer, $2^x$ means "multiply 2 by itself $x$ times";
(2) if $x = 0$, $2^0 = 1$, by definition;
(3) if $x = 1/n$, $2^{1/n} = \sqrt[n]{2}$, the $n$th root of 2;
(4) if $x = m/n$, $2^{m/n} = (\sqrt[n]{2})^m$, the $m$th power of the $n$th root of 2; and
(5) if $x$ is a negative number, $2^x$ means $1/2^{|x|}$, the reciprocal of $2^{|x|}$

In these cases, the number 2 is called the **base** of the exponential function.

To understand this exponential function better, let's draw its graph. Since we do not know how to take the derivative of $2^x$ yet — $(2^x)'$ is certainly not $x2^{x-1}$ —

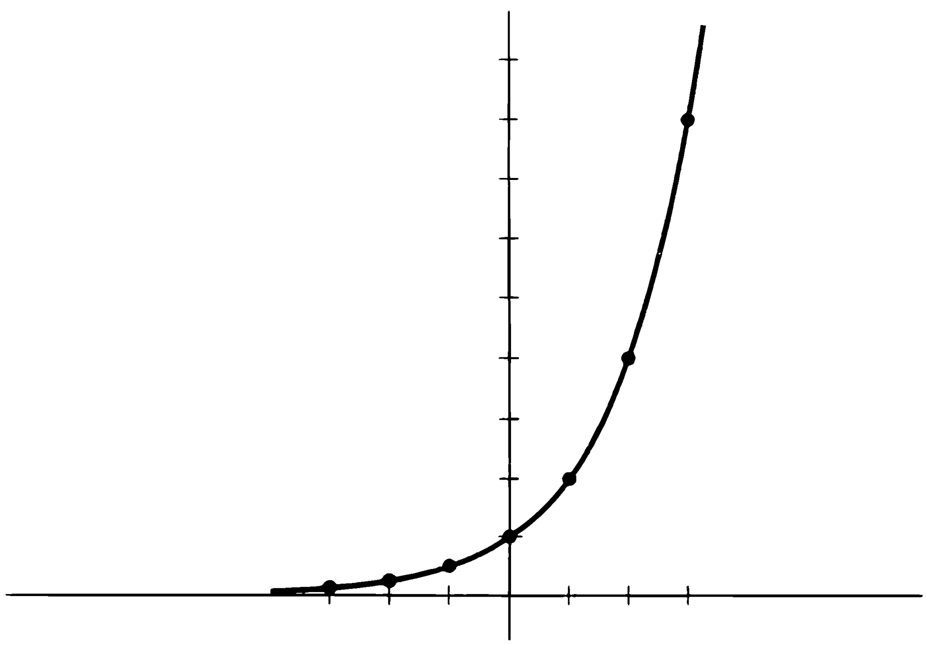
| $x$ | $2^x$ |
|---|---|
| $-3$ | $1/8$ |
| $-2$ | $1/4$ |
| $-1$ | $1/2$ |
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |

we will have to plot points. We compute values of $2^x$ in Table 5.1 and draw the corresponding graph in Figure 5.1.

Note that the graph has the negative $x$-axis as a horizontal asymptote, but unlike any rational function, the graph approaches this asymptote in only one direction. In the other direction, the graph increases very steeply. In fact, it increases more rapidly than *any* polynomial — "exponentially fast."

In Figure 5.2, the graphs of $f_1(x) = 2^x$, $f_2(x) = 3^x$, and $f_3(x) = 10^x$ are sketched. Note that the graphs are rather similar; the larger the base, the more quickly the graph becomes asymptotic to the $x$-axis in one direction and steep in the other direction.

The three bases in Figure 5.2 are greater than 1. The graph of $y = b^x$ is a bit different if the base $b$ lies between 0 and 1. Consider $h(x) = (1/2)^x$ as an example. Table 5.2 presents a list of values of $(x, y)$ in the graph of $h$ for small integers $x$. Note that the entries in the $y$-column of Table 5.2 are the same as the entries in the $y$-column of Table 5.1, but in reverse order, because $(1/2)^x = 2^{-x}$. This means that the graph of $h(x) = (1/2)^x$ is simply the reflection of the graph of $f(x) = 2^x$ in the $y$-axis, as pictured in Figure 5.3. The graphs of $(1/3)^x$ and $(1/10)^x$ look similar to that of $(1/2)^x$.



*The graph of* $y = 2^x$.

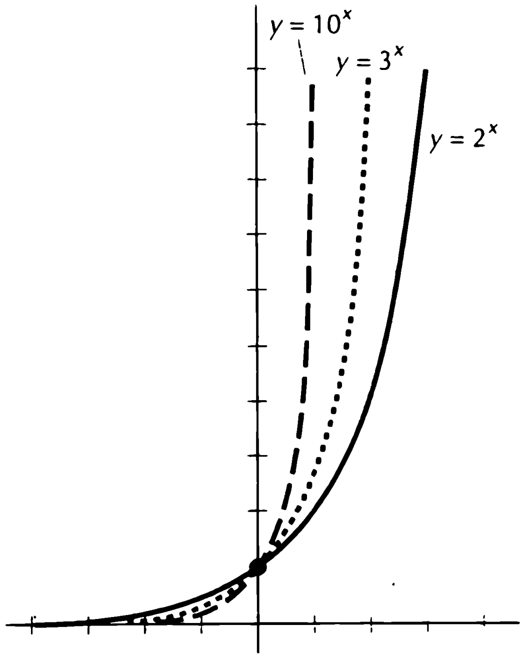



**Figure 5.2** *The graphs of $f_1(x) = 2^x$, $f_2(x) = 3^x$, and $f_3(x) = 10^x$.*

**Table 5.2**

| $x$ | $(1/2)^x$ |
|---|---|
| −3 | 8 |
| −2 | 4 |
| −1 | 2 |
| 0 | 1 |
| 1 | 1/2 |
| 2 | 1/4 |
| 3 | 1/8 |

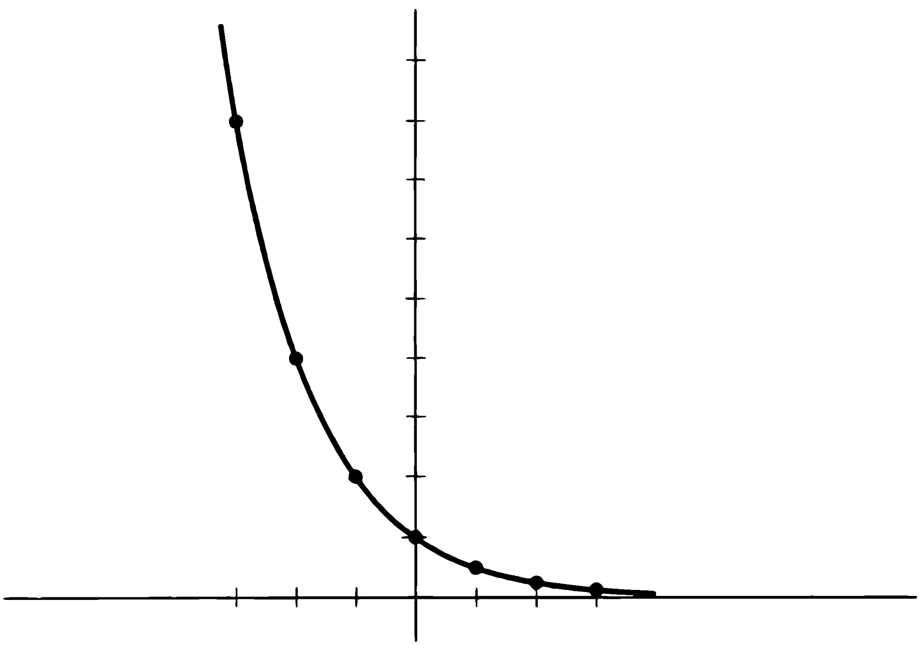


**Figure 5.3** *The graph of $y = (1/2)^x$.*

Negative bases are not allowed for the exponential function. For example, the function $k(x) = (-2)^x$ would take on positive values for $x$ an even integer and negative values for $x$ an odd integer; yet it is never zero in between. Furthermore, since you cannot take the square root of a negative number, the function $(-2)^x$ is not even defined for $x = 1/2$ or, more generally, whenever $x$ is a fraction $p/q$ and $q$ is an even integer. So, we can only work with exponential functions $a^x$, where $a$ is a number greater than 0.

### EXERCISES

**5.1** Evaluate each of the following:

$$2^3, \quad 2^{-3}, \quad 8^{1/3}, \quad 8^{2/3}, \quad 8^{-2/3}, \quad \pi^0, \quad 64^{-5/6}, \quad 625^{3/4}, \quad 25^{-5/2}.$$

**5.2** Sketch the graph of: *a)* $y = 5^x$; *b)* $y = .2^x$; *c)* $y = 3(5^x)$; *d)* $y = 1^x$.

## 5.2 THE NUMBER $e$

Figure 5.2 presented graphs of exponential functions with bases 2, 3, and 10, respectively. We now introduce a number which is the most important base for an exponential function, the irrational number $e$. To motivate the definition of $e$, consider the most basic economic situation — the growth of the investment in a savings account. Suppose that at the beginning of the year, we deposit \$$A$ into a savings account which pays interest at a simple annual interest rate $r$. If we will let the account grow without deposits or withdrawals, after one year the account will grow to $A + rA = A(1 + r)$ dollars. Similarly, the amount in the account in any one year is $(1 + r)$ times the previous year's amount. After two years, there will be

$$A(1 + r)(1 + r) = A(1 + r)^2$$

dollars in the account. After $t$ years, there will be $A(1 + r)^t$ dollars in the account.

Next, suppose that the bank compounds interest four times a year; at the end of each quarter, it pays interest at $r/4$ times the current principal. After one quarter of a year, the account contains $A + \frac{r}{4}A$ dollars. After one year, that is, after four compoundings, there will be $A\left(1 + \frac{r}{4}\right)^4$ dollars in the account. After $t$ years, the account will grow to $A\left(1 + \frac{r}{4}\right)^{4t}$ dollars.

More generally, if interest is compounded $n$ times a year, there will be $A\left(1 + \frac{r}{n}\right)$ dollars in the account after the first compounding period, $A\left(1 + \frac{r}{n}\right)^n$ dollars in the account after the first year, and $A\left(1 + \frac{r}{n}\right)^{nt}$ dollars in the account after $t$ years.

Many banks compound interest daily: others advertise that they compound interest *continuously*. By what factor does money in the bank grow in one year at interest rate $r$ if interest is compounded so frequently, that is, if $n$ is very large? Mathematically, we are asking, "What is the limit of $(1 + \frac{r}{n})^n$ as $n \to \infty$?" To simplify this calculation, let's begin with a 100 percent annual interest rate; that is, $r = 1$. Some countries, like Israel, Argentina, and Russia, have experienced interest rates of 100 percent and higher in recent years.

We compute $(1 + \frac{1}{n})^n$ with a calculator for various values of $n$ and list the results in Table 5.3.

**Table 5.3**

| $n$ | $\left(1 + \frac{1}{n}\right)^n$ |
|---:|---|
| 1 | 2.0 |
| 2 | 2.25 |
| 4 | 2.4414 |
| 10 | 2.59374 |
| 100 | 2.704814 |
| 1,000 | 2.7169239 |
| 10,000 | 2.7181459 |
| 100,000 | 2.71826824 |
| 10,000,000 | 2.718281693 |

One sees in Table 5 3 that the sequence $(1 + \frac{1}{n})^n$ is an increasing sequence in $n$ and converges to a number a little bigger than 2.7. The limit turns out to be an irrational number, in that it cannot be written as a fraction or as a repeating decimal. The letter $e$ is reserved to denote this number; formally,

$$e \equiv \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n. \tag{1}$$

To seven decimal places, $e = 2.7182818 \cdots$.

This number $e$ plays the same fundamental role in finance and in economics that the number $\pi$ plays in geometry. In particular, the function $f(x) = e^x$ is called *the* **exponential function** and is frequently written as $\exp(x)$. Since $2 < e < 3$, the graph of $\exp(x) = e^x$ is shaped like the graphs in Figure 5.2.

Next, we reconsider the general interest rate $r$ and ask: What is the limit of the sequence

$$\left(1 + \frac{r}{n}\right)^n$$

in terms of $e$? A simple change of variables answers this question. Fix $r > 0$ for the rest of this discussion. Let $m \equiv n/r$; so $n = mr$. As $n$ gets larger and goes to infinity, so does $m$. (Remember $r$ is fixed.) Since $r/n = 1/m$,

$$\left(1 + \frac{r}{n}\right)^n = \left(1 + \frac{1}{m}\right)^{mr} = \left(\left(1 + \frac{1}{m}\right)^m\right)^r$$

by straightforward substitution. Letting $n \to \infty$, we find

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n &= \lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^m\right)^r \\ &= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^r \\ &= e^r.\end{aligned}$$

In the second step, we used the fact that $x^r$ is a continuous function of $x$, so that if $\{x_m\}_{m=1}^{\infty}$ is a sequence of numbers which converges to $x_0$, then the sequence of powers $\{x_m^r\}$ converges to $x_0^r$; that is

$$\left(\lim_{m\to\infty} x_m\right)^r = \lim_{m\to\infty}(x_m^r).$$

If we let the account grow for $t$ years, then

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} &= \lim_{n\to\infty}\left(\left(1+\frac{r}{n}\right)^n\right)^t \\ &= \left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n\right)^t \\ &= (e^r)^t = e^{rt}.\end{aligned}$$

The following theorem summarizes these simple limit computations.

**Theorem 5.1** As $n \to \infty$, the sequence $\left(1+\frac{1}{n}\right)^n$ converges to a limit denoted by the symbol $e$. Furthermore,

$$\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k.$$

If one deposits $A$ dollars in an account which pays annual interest at rate $r$ compounded continuously, then after $t$ years the account will grow to $Ae^{rt}$ dollars.

Note the advantages of frequent compounding. At $r = 1$, that is, at a 100 percent interest rate, $A$ dollars will double to $2A$ dollars in a year with no compounding. However, if interest is compounded continuously, then the $A$ dollars will grow to $eA$ dollars with $e > 2.7$; the account nearly triples in size.

## 5.3 LOGARITHMS

Consider a general exponential function, $y = a^x$, with base $a > 1$. Such an exponential function is a strictly increasing function:

$$x_1 > x_2 \implies a^{x_1} > a^{x_2}.$$

In words, the more times you multiply $a$ by itself, the bigger it gets. As we pointed out in Theorem 4.1, strictly increasing functions have natural inverses. Recall that the inverse of the function $y = f(x)$ is the function obtained by solving $y = f(x)$ for $x$ in terms of $y$. For example, for $a > 0$, the inverse of the increasing linear function $f(x) = ax + b$ is the linear function $g(y) = (1/a)(y - b)$, which is computed by solving the equation $y = ax + b$ for $x$ in terms of $y$:

$$y = ax + b \iff x = \frac{1}{a}(y - b). \tag{2}$$

In a sense, the inverse $g$ of $f$ undoes the operation of $f$, so that

$$g(f(x)) = x.$$

See Section 4.2 for a detailed discussion of the inverse of a function.

We cannot compute the inverse of the increasing exponential function $f(x) = a^x$ explicitly because we can't solve $y = a^x$ for $x$ in terms of $y$, as we did in (2). However, this inverse function is important enough that we give it a name. We call it the **base $a$ logarithm** and write

$$y = \log_a(z) \iff a^y = z.$$

The **logarithm** of $z$, by definition, is the power to which one must raise $a$ to yield $z$. It follows immediately from this definition that

$$a^{\log_a(z)} = z \quad \text{and} \quad \log_a(a^z) = z. \tag{3}$$

We often write $\log_a(z)$ without parentheses, as $\log_a z$.

### Base 10 Logarithms

Let's first work with base $a = 10$. The logarithmic function for base 10 is such a commonly used logarithm that it is usually written as $y = \text{Log}\,x$ with an uppercase L:

$$y = \text{Log}\,z \iff 10^y = z.$$

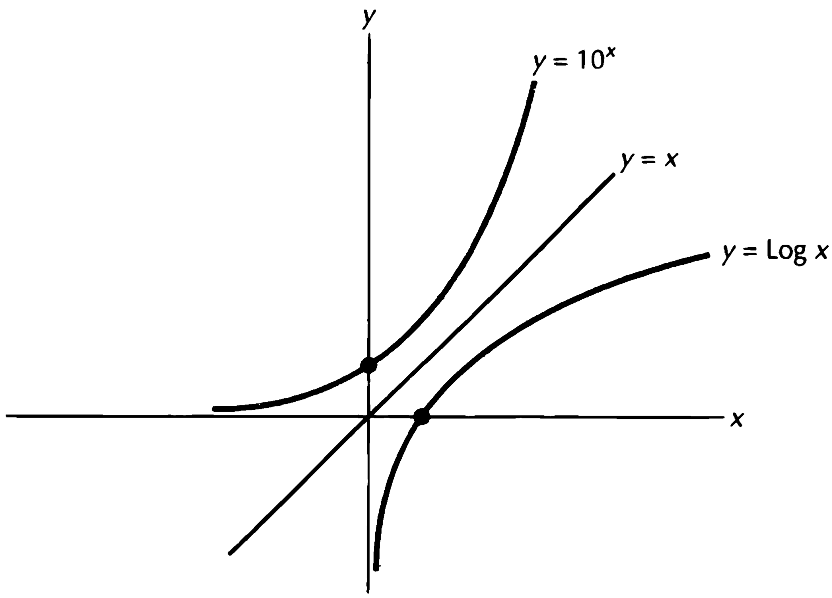
*Example 5.1* For example, the Log of 1000 is that power of 10 which yields 1000. Since $10^3 = 1000$, $\text{Log}\,1000 = 3$. The Log of 0.01 is $-2$, since $10^{-2} = 0.01$. Here are a few more values of $\text{Log}\,z$:

$$\begin{aligned} \text{Log}\,10 &= 1 && \text{since } 10^1 &&= 10, \\ \text{Log}\,100{,}000 &= 5 && \text{since } 10^5 &&= 100{,}000, \\ \text{Log}\,1 &= 0 && \text{since } 10^0 &&= 1, \\ \text{Log}\,625 &= 2.79588\cdots && \text{since } 10^{2.79588\cdots} &&= 625. \end{aligned}$$

For most values of $z$, you'll have to use a calculator or table of logarithms to evaluate $\text{Log}\,z$.

One forms the graph of the inverse function $f^{-1}$ by reversing the roles of the horizontal and vertical axes in the graph of $f$. In other words, the graph of the inverse of a function $y = f(x)$ is the reflection of the graph of $f$ across the diagonal $\{x = y\}$, because $(y, z)$ is a point on the graph of $f^{-1}$ if and only if $(z, y)$ is a point on the graph of $f$. In Figure 5.4, we have drawn the graph of $y = 10^x$ and reflected it across the diagonal $\{x = y\}$ to draw the graph of $y = \text{Log}\,x$.

Since the negative "$x$-axis" is a horizontal asymptote for the graph of $y = 10^x$, the negative "$y$-axis" is a vertical asymptote for the graph of $y = \text{Log}\,x$. Since $10^x$ grows very quickly, $\text{Log}\,x$ grows very slowly. At $x = 1000$, $\text{Log}\,x$ is just at $y = 3$; at $x$ equals a million, $\text{Log}\,x$ has just climbed to $y = 6$. Finally, since for *every* $x$, $10^x$ is a positive number, $\text{Log}\,x$ is only defined for $x > 0$. Its domain is $\mathbf{R}_{++}$, the set of strictly positive numbers.




*The graph of $y = \text{Log}\,x$ is the reflection of the graph of $y = 10^x$ across the diagonal $\{y = x\}$.* **Figure 5.4**

## Base *e* Logarithms

Since *the* exponential function $\exp(x) = e^x$ has all the properties that $10^x$ has, it also has an inverse. Its inverse works the same way that $\operatorname{Log} x$ does. Mirroring the fundamental role that $e$ plays in applications, the inverse of $e^x$ is called the **natural logarithm** function and is written as $\ln x$. Formally,

$$\ln x = y \iff e^y = x;$$

$\ln x$ is the power to which one must raise $e$ to get $x$. As we saw in general in (3), this definition can also be summarized by the equations

$$e^{\ln x} = x \quad \text{and} \quad \ln e^x = x. \tag{4}$$

The graph of $e^x$ and its reflection across the diagonal, the graph of $\ln x$, are similar to the graphs of $10^x$ and $\operatorname{Log} x$ in Figure 5.4.

*Example 5.2* Let's work out some examples. The natural log of 10 is the power of $e$ that gives 10. Since $e$ is a little less than 3 and $3^2 = 9$, $e^2$ will be a bit less than 9. We have to raise $e$ to a power bigger than 2 to obtain 10. Since $3^3 = 27$, $e^3$ will be a little less than 27. Thus, we would expect that $\ln 10$ to lie between 2 and 3 and somewhat closer to 2. Using a calculator, we find that the answer to four decimal places is $\ln 10 = 2.3026$.

We list a few more examples. Cover the right-hand side of this table and try to estimate these natural logarithms.

$$\begin{aligned}
\ln e &= 1 && \text{since } e^1 = e;\\
\ln 1 &= 0 && \text{since } e^0 = 1;\\
\ln 0.1 &= -2.3025\cdots && \text{since } e^{-2.3025\cdots} = 0.1;\\
\ln 40 &= 3.688\cdots && \text{since } e^{3.688\cdots} = 40;\\
\ln 2 &= 0.6931\cdots && \text{since } e^{0.6931\cdots} = 2.
\end{aligned}$$

### EXERCISES

**5.3** First estimate the following logarithms without a calculator. Then, use your calculator to compute an answer correct to four decimal places:

*a*) $\operatorname{Log} 500$, *b*) $\operatorname{Log} 5$, *c*) $\operatorname{Log} 1234$, *d*) $\operatorname{Log} e$,
*e*) $\ln 30$, *f*) $\ln 100$, *g*) $\ln 3$, *h*) $\ln \pi$.

**5.4** Give the exact values of the following logarithms without using a calculator:

| | | |
|---|---|---|
| *a*) Log 10, | *b*) Log 0.001, | *c*) Log(billion), |
| *d*) $\log_2 8$, | *e*) $\log_6 36$, | *f*) $\log_5 0.2$, |
| *g*) $\ln(e^2)$, | *h*) $\ln \sqrt{e}$, | *i*) ln 1. |

## 5.4 PROPERTIES OF EXP AND LOG

Exponential functions have the following five basic properties:

(1) $a^r \cdot a^s = a^{r+s}$,
(2) $a^{-r} = 1/a^r$,
(3) $a^r/a^s = a^{r-s}$,
(4) $(a^r)^s = a^{rs}$, and
(5) $a^0 = 1$.

Properties 1, 3, and 4 are straightforward when $r$ and $s$ are positive integers. The definitions that $a^{-n} = 1/a^n$, $a^0 = 1$, $a^{1/n}$ is the $n$th root of $a$, and $a^{m/n} = (a^{1/n})^m$ are all specifically designed so that the above five rules would hold for *all real* numbers $r$ and $s$.

These five properties of exponential functions are mirrored by five corresponding properties of the logarithmic functions:

(1) $\log(r \cdot s) = \log r + \log s$,
(2) $\log(1/s) = -\log s$,
(3) $\log(r/s) = \log r - \log s$,
(4) $\log r^s = s \log r$, and
(5) $\log 1 = 0$.

The fifth property of logs follows directly from the fifth property of $a^x$ and the fact that $a^x$ and $\log_a$ are inverses of each other. To prove the other four properties, let $u = \log_a r$ and $v = \log_a s$, so that $r = a^u$ and $s = a^v$. Then, using the fact that $\log_a(a^x) = x$, we find:

(1) $\log(r \cdot s) = \log(a^u \cdot a^v) = \log(a^{u+v}) = u + v = \log r + \log s$,
(2) $\log(1/s) = \log(1/a^v) = \log(a^{-v}) = -v = -\log s$,
(3) $\log(r/s) = \log(a^u/a^v) = \log(a^{u-v}) = u - v = \log r - \log s$,
(4) $\log r^s = \log(a^u)^s = \log a^{us} = us = s \cdot \log r$.

Logarithms are especially useful in bringing a variable $x$ that occurs as an exponent back down to the base line where it can be more easily manipulated.

*Example 5.3* To solve the equation $2^{5x} = 10$ for $x$, we take the Log of both sides:

$$\text{Log}\, 2^{5x} = \text{Log}\, 10 \quad \text{or} \quad 5x \cdot \text{Log}\, 2 = 1.$$

It follows that

$$x = \frac{1}{5\ \text{Log}\, 2} \approx .6644.$$

We could have used ln instead of Log in this calculation.

*Example 5.4* Suppose we want to find out how long it takes $A$ dollars deposited in a saving account to double when the annual interest rate is $r$ compounded continuously. We want to solve the equation

$$2A = Ae^{rt} \tag{5}$$

for the unknown $t$. We first divide both sides of (5) by $A$. This eliminates $A$ from the calculation — a fact consistent with our intuition that the doubling time should be independent of the amount of money under consideration. To bring the variable $t$ down to where we can work with it, take the natural log of both sides of the equation $2 = e^{rt}$:

$$\begin{aligned} \ln 2 &= \ln e^{rt} \\ &= rt, \end{aligned} \tag{6}$$

using (4). Solving (6) for $t$ yields the fact that the doubling time is $t = (\ln 2)/r$.

Since $\ln 2 \approx 0.69$, this rule says that to estimate the doubling time for interest rate $r$, just divide the interest rate into 69. For example, the doubling time at 10 percent interest is $69/10 = 6.9$ years; the doubling time at 8 percent interest is $69/8 = 8.625$ years. This calculation also tells us that it would take 8.625 years for the price level to double if the inflation rate stays constant at 8 percent.

As we discussed in Section 3.6, economists studying the relationship between the price $p$ and the quantity $q$ demanded of some good will often choose to work with the two-parameter family of **constant elasticity demand functions,** $q = kp^{\varepsilon}$, where $k$ and $\varepsilon$ are parameters which depend on the good under study. The parameter $\varepsilon$ is the most interesting of the two since it equals the elasticity $(p/q)(dq/dp)$. Taking the log of both sides of $q = kp^{\varepsilon}$ yields:

$$\ln q = \ln kp^{\varepsilon} = \ln k + \varepsilon \ln p. \tag{7}$$

In logarithmic coordinates, demand is now a *linear* function whose slope is the elasticity $\varepsilon$.

### EXERCISES

**5.5** Solve the following equations for $x$:

$$a)\ 2e^{6x} = 18;\quad b) e^{x^2} = 1;\quad c) 2^x = e^5;$$
$$d)\ 2^{x-2} = 5;\quad e) \ln x^2 = 5;\quad f) \ln x^{5/2} - 0.5 \ln x = \ln 25.$$

**5.6** Derive a formula for the amount of time that it takes money to triple in a bank account that pays interest at rate $r$ compounded continuously.

**5.7** How quickly will $500 grow to $600 if the interest rate is 5 percent compounded continuously?

## 5.5 DERIVATIVES OF EXP AND LOG

To work effectively with exponential and logarithmic functions, we need to compute and use their derivatives. The natural logarithmic and exponential functions have particularly simple derivatives, as the statement of the following theorem indicates.

**Theorem 5.2** The functions $e^x$ and $\ln x$ are continuous functions on their domains and have continuous derivatives of every order. Their first derivatives are given by

$$a)\quad (e^x)' = e^x,$$

$$b)\quad (\ln x)' = \frac{1}{x}.$$

If $u(x)$ is a differentiable function, then

$$c)\quad \left(e^{u(x)}\right)' = \left(e^{u(x)}\right) \cdot u'(x),$$

$$d)\quad (\ln u(x))' = \frac{u'(x)}{u(x)} \qquad \text{if } u(x) > 0.$$

We will prove this theorem in stages. That the exponential map is continuous should be intuitively clear from the graph in Figure 5.4; its graph has no jumps or discontinuities. Since the graph of $\ln x$ is just the reflection of the graph of $e^x$ across the diagonal $\{x = y\}$, the graph of $\ln x$ has no discontinuities either, and so the function $\ln x$ is continuous for all $x$ in the set $\mathbf{R}_{++}$ of positive numbers.

It turns out to be easier to compute the derivative of the natural logarithm first.

**Lemma 5.1** Given that $y = \ln x$ is a continuous function on $\mathbf{R}_{++}$, it is also differentiable and its derivative is given by

$$(\ln x)' = \frac{1}{x}.$$

*Proof* We start, of course, with the difference quotient that defines the derivative, and we then simplify it using the basic properties of the logarithm. Fix $x > 0$.

$$\frac{\ln(x+h) - \ln x}{h} = \frac{1}{h}\ln\left(\frac{x+h}{x}\right) = \ln\left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$$
$$= \ln\left(1 + \frac{1/x}{1/h}\right)^{\frac{1}{h}}.$$

Now, let $m = 1/h$. As $h \to 0$, $m \to \infty$. Continuing our calculation with $m = 1/h$, we find

$$\lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{m \to \infty} \ln\left(1 + \frac{1/x}{m}\right)^m$$
$$= \ln \lim_{m \to \infty}\left(1 + \frac{1/x}{m}\right)^m$$
$$= \ln e^{1/x} = \frac{1}{x}.$$

Therefore, $(\ln x)' = 1/x$. The fact that we can interchange ln and lim in the above string of equalities follows from the fact that $y = \ln x$ is a continuous function: $x_m \to x_0$ implies that $\ln x_m \to \ln x_0$; or equivalently,

$$\lim_m(\ln x_m) = \ln\left(\lim_m x_m\right). \quad \blacksquare$$

The other three conclusions of Theorem 5.2 follow immediately from the Chain Rule, as we now prove.

**Lemma 5.2** If $h(x)$ is a differentiable and positive function, then

$$\frac{d}{dx}(\ln h(x)) = \frac{h'(x)}{h(x)}.$$

*Proof* We simply apply the Chain Rule to the composite function $f(x) = \ln h(x)$. The derivative of $f$ is the derivative of the *outside* function $\ln y$—which is $1/y$—evaluated at the inside function $h(x)$—so it's $1/h(x)$—times the derivative $h'(x)$ of the inside function $h$:

$$(\ln h(x))' = \frac{1}{h(x)} \cdot h'(x) = \frac{h'(x)}{h(x)}. \quad \blacksquare$$

We can now easily evaluate the derivative of the exponential function $y = e^x$, using the fact that it is the inverse of $\ln x$.

**Lemma 5.3** $(e^x)' = e^x$.

*Proof* Use the definition of $\ln x$ in (4) to write $\ln e^x = x$. Taking the derivative of both sides of this equation and using the previous lemma, we compute

$$(\ln e^x)' = \frac{1}{e^x} \cdot (e^x)' = 1.$$

It follows that

$$(e^x)' = e^x. \quad \blacksquare$$

Finally, to prove part *c* of Theorem 5.2, we simply apply the Chain Rule to the composite function $y = e^{u(x)}$. The outside function is $e^z$, whose derivative is also $e^z$. Its derivative evaluated at the inside function is $e^{u(x)}$. Multiplying this by the derivative of the inside function $u(x)$, we conclude that

$$\left(e^{u(x)}\right)' = e^{u(x)}\, u'(x).$$

*Example 5.5* Using Theorem 5.2, we compute the following derivatives:

*a)* $\left(e^{5x}\right)' = 5e^{5x}$, *b)* $\left(Ae^{kx}\right)' = Ake^{kx}$,

*c)* $\left(5e^{x^2}\right)' = 10xe^{x^2}$, *d)* $(e^x \ln x)' = e^x \ln x + \dfrac{e^x}{x}$,

*e)* $\left(\ln x^2\right)' = \dfrac{1}{x^2} \cdot 2x = \dfrac{2}{x}$, *f)* $\left((\ln x)^2\right)' = \dfrac{2\ln x}{x}$,

*g)* $\left(xe^{3-x}\right)' = e^{3-x} - xe^{3-x} = (1-x)e^{3-x}$, *h)* $\left(\ln(x^2+3x+1)\right)' = \dfrac{2x+3}{x^2+3x+1}$.

*Example 5.6* The density function for the standard **normal distribution** is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Let's use calculus to sketch the graph of its core function

$$g(x) = e^{-x^2/2}.$$

We first note that $g$ is always positive, so its graph lies above the $x$-axis everywhere. Its first derivative is

$$g'(x) = -xe^{-x^2/2}.$$

Since $e^{-x^2/2}$ is always positive, $g'(x) = 0$ if and only if $x = 0$. Since $g(0) = 1$, the only candidate for max or min of $g$ is the point $(0, 1)$. Furthermore, $g'(x) > 0$ if and only if $x < 0$, and $g'(x) < 0$ if and only if $x > 0$; so $g$ is increasing for $x < 0$ and decreasing for $x > 0$. This tells us that the critical point $(0, 1)$ must be a max, in fact, a global max.

So far, we know that the graph of $g$ stays above the $x$-axis all the time, increases until it reaches the point $(0, 1)$ on the $y$-axis, and then decreases to the right of the $y$-axis. Let's use the second derivative to fine-tune this picture:

$$g''(x) = \left(-xe^{-x^2/2}\right)' = x^2e^{-x^2/2} - e^{-x^2/2} = \left(x^2 - 1\right)e^{-x^2/2}.$$

Since $e^{-x^2/2} > 0$, $g''(x)$ has the same sign as $(x^2 - 1)$. In particular,

$$g''(0) < 0, \qquad \text{and} \qquad g''(x) = 0 \Longleftrightarrow x = \pm 1. \tag{8}$$

The first inequality in (8) verifies that the critical point $(0, 1)$ is indeed a local max of $g$. Using the second part of (8), we note that

$$\begin{aligned} -\infty < x < -1 \quad &\Longrightarrow \quad g''(x) > 0, \\ -1 < x < +1 \quad &\Longrightarrow \quad g''(x) < 0, \\ 1 < x < +\infty \quad &\Longrightarrow \quad g''(x) > 0; \end{aligned}$$

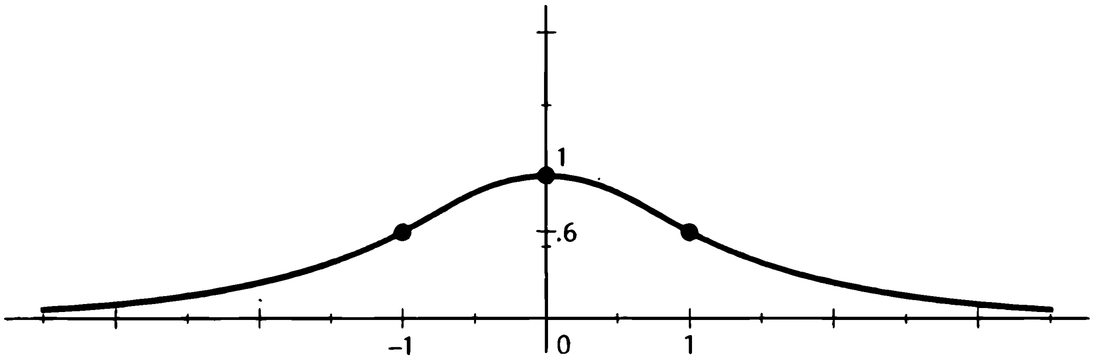
this implies that $g$ is concave up on $(-\infty, -1)$ and on $(1, \infty)$ and concave down on $(-1, +1)$. The second order critical points occur at the points $(-1, e^{-1/2})$ and $(1, e^{-1/2})$. Putting all this information together, we sketch the graph of $g$ in Figure 5.5.

The graph of $g$ is the graph of the usual bell-shaped probability distribution. Since $f$ is simply $g$ times $(2\pi)^{-1/2} \approx .39$, the graph of $f$ will be similar to the graph of $g$ but closer to the $x$-axis.

We now use equation $b$ in Example 5.5 to compute the derivative of the general exponential function $y = b^x$.

**Theorem 5.3** For any fixed positive base $b$,

$$(b^x)' = (\ln b)(b^x). \tag{9}$$




*The graph of* $e^{-x^2/2}$.

Figure 5.5

*Proof* Since $b = e^{\ln b}$, then $b^x = \left(e^{\ln b}\right)^x = e^{(\ln b)x}$. By equation *b* in Example 5.5,

$$(b^x)' = \left(e^{(\ln b)\cdot x}\right)' = (\ln b)(e^{(\ln b)\cdot x}) = (\ln b)(b^x). \quad \blacksquare$$

*Example 5.7* $(10^x)' = (\ln 10)(10^x)$.

Note that $(b^x)' = b^x$ if and only if $\ln b = 1$, that is, if and only if $b = e$. In fact, *the* exponential functions $y = ke^x$ are the only functions which are equal to their derivatives throughout their domains. This fact gives another justification for $e$ being considered the ***natural*** base for exponential functions.

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**EXERCISES**

**5.8** Compute the first and second derivatives of each of the following functions:
*a)* $xe^{3x}$, *b)* $e^{x^2+3x-2}$, *c)* $\ln(x^4+2)^2$, *d)* $\frac{x}{e^x}$, *e)* $\frac{x}{\ln x}$, *f)* $\frac{\ln x}{x}$.

**5.9** Use calculus to sketch the graph of each of the following functions:
*a)* $xe^x$, *b)* $xe^{-x}$, *c)* $\cosh(x) \equiv (e^x + e^{-x})/2$.

**5.10** Use the equation $10^{\operatorname{Log} x} = x$, Example 5.7, and the method of the proof of Lemma 5.3 to derive a formula for the derivative of $y = \operatorname{Log} x$.

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## 5.6 APPLICATIONS

### Present Value

Many economic problems entail comparing amounts of money at different points of time in the same computation. For example, the benefit/cost analysis of the construction of a dam must compare in the same equation this year's cost of construction, future years' costs of maintaining the dam, and future years' monetary benefits from the use of the dam. The simplest way to deal with such comparisons is to use the concept of *present value* to bring all money figures back to the present.

If we put $A$ dollars into an account which compounds interest continuously at rate $r$, then after $t$ years there will be

$$B = Ae^{rt} \tag{10}$$

dollars in the account, by Theorem 5.1. Conversely, in order to generate $B$ dollars $t$ years from now in an account which compounds interest continuously at rate $r$, we would have to invest $A = Be^{-rt}$ dollars in the account now, solving (10) for $A$ in terms of $B$. We call $Be^{-rt}$ the **present value** (PV) of $B$ dollars $t$ years from now (at interest rate $r$).

Present value can also be defined using *annual* compounding instead of continuous compounding. In an account which compounds interest annually at rate $r$, a deposit of $A$ dollars now will yield $B = A(1 + r)^t$ dollars $t$ years from now. Conversely, in this framework, the present value of $B$ dollars $t$ years from now is $B/(1 + r)^t = B(1 + r)^{-t}$ dollars. Strictly speaking, this latter framework only makes sense for integer $t$'s. For this reason and because the exponential map $e^{rt}$ is usually easier to work with than $(1 + r)^t$, we will use the continuous compounding version of present value.

Present value can also be defined for *flows* of payments. At interest rate $r$, the present value of the flow — $B_1$ dollars $t_1$ years from now, $B_2$ dollars $t_2$ years from now, ..., $B_n$ dollars $t_n$ years from now — is

$$\text{PV} = B_1e^{-rt_1} + B_2e^{-rt_2} + \cdots + B_ne^{-rt_n}. \tag{11}$$

## Annuities

An **annuity** is a sequence of equal payments at regular intervals over a specified period of time. The present value of an annuity that pays $A$ dollars at the end of each of the next $N$ years, assuming a constant interest rate $r$ compounded continuously, is

$$\begin{aligned}\text{PV} &= Ae^{-r\cdot 1} + Ae^{-r\cdot 2} + \cdots + Ae^{-r\cdot N} \\ &= A\left(e^{-r} + e^{-r\cdot 2} + \cdots + e^{-r\cdot N}\right).\end{aligned} \tag{12}$$

Since $(a + \cdots + a^n)(1 - a) = a - a^{n+1}$, as one can easily check,

$$a + \cdots + a^n = \frac{a(1 - a^n)}{1 - a}. \tag{13}$$

Substituting $a = e^{-r}$ and $n = N$ from (12) yields a present value for the annuity of

$$\text{PV} = A \cdot \frac{e^{-r}\left(1 - e^{-rN}\right)}{1 - e^{-r}} = \frac{A\left(1 - e^{-rN}\right)}{e^r - 1}. \tag{14}$$

To calculate the present value of an annuity which pays $A$ dollars a year *forever*, we let $N \to \infty$ in (14):

$$\mathrm{PV} = \frac{A}{e^r - 1}, \tag{15}$$

since $e^{-rN} \to 0$ as $N \to \infty$.

It is sometimes convenient to calculate the present value of an annuity using *annual* compounding instead of continuous compounding. In this case, equation (12) becomes

$$\mathrm{PV} = \frac{A}{1+r} + \cdots + \frac{A}{(1+r)^N}.$$

Apply equation (13) with $a = 1/(1+r)$ and $n = N$:

$$\mathrm{PV} = A \cdot \frac{1/(1+r)}{r/(1+r)}\left(1 - \left(\frac{1}{1+r}\right)^N\right) = \frac{A}{r}\left(1 - \left(\frac{1}{1+r}\right)^N\right) \tag{16}$$

To calculate the present value of an annuity which pays $A$ dollars a year forever at interest rate $r$ compounded annually, we let $N \to \infty$ in (16):

$$\mathrm{PV} = \frac{A}{r}. \tag{17}$$

The intuition for (17) is straightforward; in order to generate a perpetual flow of $A$ dollars a year from a savings account which pays interest annually at rate $r$, one must deposit $A/r$ dollars into the account initially.

## Optimal Holding Time

Suppose that you own some real estate the market value of which will be $V(t)$ dollars $t$ years from now. If the interest rate remains constant at $r$ over this period, the corresponding time stream of present values is $V(t)e^{-rt}$. Economic theory suggests that the optimal time $t_0$ to sell this property is at the maximum value of this time stream of present value. The first order conditions for this maximization problem are

$$(V(t)e^{-rt})' = V'(t)e^{-rt} - rV(t)e^{-rt} = 0,$$

or

$$\frac{V'(t)}{V(t)} = r \quad \text{at } t = \text{the optimal selling time } t_0. \tag{18}$$

Condition (18) is a natural condition for the **optimal holding time**. The left-hand side of (18) gives the rate of change of $V$ divided by the amount of $V$ — a quantity called the **percent rate of change** or simply the **growth rate**. The right-hand side gives the interest rate, which is the percent rate of change of money in the bank. As long as the value of the real estate is growing more rapidly than money in the bank, one should hold on to the real estate. As soon as money in the bank has a higher growth rate, one would do better by selling the property and banking the proceeds at interest rate $r$. The point at which this switch takes place is given by (18), where the percent rates of change are equal.

This principle of optimal holding time holds in a variety of circumstances, for example, when a wine dealer is trying to decide when to sell a case of wine that is appreciating in value or when a forestry company is trying to decide how long to let the trees grow before cutting them down for sale.

*Example 5.8* You own real estate the market value of which $t$ years from now is given by the function $V(t) = 10{,}000e^{\sqrt{t}}$. Assuming that the interest rate for the foreseeable future will remain at 6 percent, the optimal selling time is given by maximizing the present value

$$F(t) = 10{,}000e^{\sqrt{t}}e^{-.06t} = 10{,}000e^{\sqrt{t}-.06t}.$$

The first order condition for this maximization problem is

$$0 = F'(t) = 10{,}000e^{\sqrt{t}-.06t}\left(\frac{1}{2\sqrt{t}} - .06\right),$$

which holds if and only if

$$\frac{1}{2\sqrt{t_0}} = .06 \quad \text{or} \quad t_0 = \left(\frac{1}{.12}\right)^2 \approx 69.44.$$

Since $F'(t)$ is positive for $0 < t < t_0$ and negative for $t > t_0$, $t_0 \approx 69.44$ is indeed the *global* max of the present value and is the optimal selling time of the real estate.

## Logarithmic Derivative

Since the logarithmic operator turns exponentiation into multiplication, multiplication into addition, and division into subtraction, it can often simplify the computation of the derivative of a complex function, because, by Lemma 5.2,

$$(\ln u(x))' = \frac{u'(x)}{u(x)},$$

and therefore

$$u'(x) = (\ln u(x))' \cdot u(x). \tag{19}$$

If $\ln u(x)$ is easier to work with than $u(x)$ itself, one can compute $u'$ more easily using (19) than by computing it directly.

*Example 5.9* Let's use this idea to compute the derivative of

$$y = \frac{\sqrt[4]{x^2 - 1}}{x^2 + 1}. \tag{20}$$

The natural log of this function is

$$\ln\left(\frac{\sqrt[4]{x^2 - 1}}{x^2 + 1}\right) = \frac{1}{4}\ln(x^2 - 1) - \ln(x^2 + 1). \tag{21}$$

It is much simpler to compute the derivative of (21) than it is to compute the derivative of the quotient (20):

$$\begin{aligned}\frac{d}{dx}\ln\left(\frac{\sqrt[4]{x^2 - 1}}{x^2 + 1}\right) &= \frac{1}{4}\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1} \\ &= \frac{-3x^3 + 5x}{2(x^2 - 1)(x^2 + 1)}.\end{aligned}$$

Now, use (19) to compute $y'$:

$$\begin{aligned}\left(\frac{\sqrt[4]{x^2 - 1}}{x^2 + 1}\right)' &= \frac{-3x^3 + 5x}{2(x^2 - 1)(x^2 + 1)} \cdot \frac{\sqrt[4]{x^2 - 1}}{x^2 + 1} \\ &= \frac{-3x^3 + 5x}{2(x^2 - 1)^{3/4}(x^2 + 1)^2}.\end{aligned}$$

*Example 5.10* A favorite calculus problem, which can only be solved by this method, is the computation of the derivative of $g(x) = x^x$. Since

$$(\ln x^x)' = (x \ln x)' = \ln x + 1,$$

the derivative of $x^x$ is $(\ln x + 1) \cdot x^x$, by (19).

Occasionally, scientists prefer to study a given function $y = f(x)$ by comparing $\ln y$ and $\ln x$, that is, by graphing $f$ on log-log graph paper. See, for example, our discussion of constant elasticity demand functions in (7). In this case, they are working with the change of variables

$$Y = \ln y \quad \text{and} \quad X = \ln x.$$

Since $X = \ln x$, $x = e^X$ and $dx/dX = e^X = x$. In $XY$-coordinates, $f$ becomes

$$Y = \ln f(x) = \ln f(e^X) \equiv F(X).$$

Now, the slope of the graph of $Y = F(X)$, that is, of the graph of $f$ in log-log coordinates, is given by

$$\begin{aligned} \frac{dF(X)}{dX} &= \frac{dF(x(X))}{dx} \cdot \frac{dx}{dX} \qquad \text{(by the Chain Rule)} \\ &= \frac{d}{dx}(\ln f(x)) \cdot \frac{dx}{dX} = \frac{f'(x)}{f(x)} \cdot x. \end{aligned} \tag{22}$$

The difference approximation of the last term in (22) is

$$\frac{df(x)}{dx} \cdot \frac{x}{f(x)} \approx \frac{\Delta f}{\Delta x} \cdot \frac{x}{f(x)} = \frac{\Delta f}{f(x)} \Big/ \frac{\Delta x}{x},$$

the percent change of $f$ relative to the percent change of $x$. This is the quotient we have been calling the (point) **elasticity** of $f$ with respect to $x$, especially if $f$ is a demand function and $x$ represents price or income.

This discussion shows that the slope of the graph of $f$ in log-log coordinates is the (point) elasticity of $f$:

$$\varepsilon = \frac{f'(x) \cdot x}{f(x)}.$$

In view of this discussion, economists sometimes write this elasticity as

$$\varepsilon = \frac{d(\ln f)}{d(\ln x)}.$$

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## EXERCISES

**5.11** At 10 percent annual interest rate, which of the following has the largest present value:

*a*) \$215 two years from now,

*b*) \$100 after each of the next two years, or

*c*) \$100 now and \$95 two years from now?

**5.12** Assuming a 10 percent interest rate compounded continuously, what is the present value of an annuity that pays \$500 a year *a*) for the next five years, *b*) forever?

**5.13** Suppose that you own a rare book whose value at time $t$ years from now will be $B(t) = 2^{\sqrt{t}}$ dollars. Assuming a constant interest rate of 5 percent, when is the best time to sell the book and invest the proceeds?

**5.14** A wine dealer owns a case of fine wine that can be sold for $Ke^{\sqrt{t}}$ dollars $t$ years from now. If there are no storage costs and the interest rate is $r$, when should the dealer sell the wine?

**5.15** The value of a parcel of land bought for speculation is increasing according to the formula $V = 2000\,e^{t^{1/4}}$. If the interest rate is 10 percent, how long should the parcel be held to maximize present value?

**5.16** Use the logarithmic derivative method to compute the derivative of each of the following functions: *a*) $\sqrt{(x^2 + 1)/(x^2 + 4)}$, *b*) $(x^2)^{x^2}$.

**5.17** Use the above discussion to prove that the elasticity of the product of two functions is the sum of the elasticities.

PART II

# Linear Algebra

CHAPTER 6

# Introduction to Linear Algebra

The analysis of many economic models reduces to the study of systems of equations. Furthermore, some of the most frequently studied economic models are linear models. In the next few chapters, we will study the simplest possible systems of equations — linear systems.

## 6.1 LINEAR SYSTEMS

Typical linear equations are

$$x_1 + 2x_2 = 3 \qquad \text{and} \qquad 2x_1 - 3x_2 = 8.$$

They are called linear because their graphs are straight lines. In general, an equation is **linear** if it has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

The letters $a_1, \dots, a_n$, and $b$ stand for fixed numbers, such as 2, $-3$, and 8 in the second equation. These are called **parameters**. The letters $x_1, \dots, x_n$ stand for **variables**. The key feature of the general form of a linear equation is that each term of the equation contains at most one variable, and that variable appears only to the first power rather than to the second, third, or some other power.

There are several reasons why it is natural to begin with systems of linear equations. These are the most elementary equations that can arise. Linear algebra, the study of such systems, is one of the simpler branches of mathematics. It requires no calculus and, at least in the beginning, very little familiarity with functions. It builds on techniques learned in high school, such as the solution of two linear equations in two unknowns via substitution or elimination of variables. It also builds on the simple geometry of the plane and the cube, which is easy to visualize. Linear equations describe geometric objects such as lines and planes. In fact, linear algebra is a simple way to translate the insights of planar and cubical geometry to higher dimensions.

Linear systems have the added advantage that we can often calculate exact solutions to the equations. By contrast, solutions of nonlinear systems often cannot be calculated explicitly, and we can only hope to discover indirectly some of the properties of these solutions. Equally important for linear systems, the precise relationship between the solution of the linear system and various parameters determining the system (the $a_i$'s and $b$ in the equation above) can be easily described.

Of course, linearity is a simplifying assumption. The real world is nonlinear. Calculus exploits the manageability of linear systems to study nonlinear systems. The fundamental idea of calculus is that we can learn much about the behavior of a nonlinear system of equations by studying suitably chosen linear approximations to the system. For example, the best linear approximation to the graph of a nonlinear function at any point on its graph is the tangent line to the graph at that point. We can learn much about the behavior of a function near any point by examining the slope of the tangent line. Whether the function is increasing or decreasing can be determined by seeing whether the tangent line is rising or falling. The first important exercise in the study of the calculus is to learn how to calculate this slope — the *derivative* of the function. For a more prosaic example of the importance of linear approximations, consider that few people disagree with the proposition that the earth is roughly spherical, and yet in constructing homes, skyscrapers, and even cities, we assume that the earth is flat and obtain some rather impressive results using Euclidean planar geometry. Once again we are taking advantage of an effective linear approximation to a nonlinear phenomenon.

Since a primary goal of multivariable calculus is to provide a mechanism for approximating complicated nonlinear systems by simpler linear ones, it makes sense to begin by squeezing out all the information we can about linear systems — the task we take up in the next six chapters.

A final reason for looking at linear systems first is that some of the most frequently studied economic models are linear. We sketch five such models here. As we develop our theory of linear systems, we will often refer back to these models and call attention to the insights which linear theory offers. References for further study of these topics can be found in the notes at the end of the chapter.

## 6.2 EXAMPLES OF LINEAR MODELS

### Example 1: Tax Benefits of Charitable Contributions

A company earns before-tax profits of $100,000. It has agreed to contribute 10 percent of its after-tax profits to the Red Cross Relief Fund. It must pay a state tax of 5 percent of its profits (after the Red Cross donation) and a federal tax of 40 percent of its profits (after the donation and state taxes are paid). How much does the company pay in state taxes, federal taxes, and Red Cross donation?

Without a model to structure our analysis, this problem is rather difficult because each of the three payments must take into consideration the other payments. However, after we write out the (linear) equations which describe the various deductions, we can understand more clearly the relationships between these payments and then solve in a straightforward manner for the payment amounts.

Let $C$, $S$, and $F$ represent the amounts of the charitable contribution, state tax, and federal tax, respectively. After-tax profits are \$100,000 − $(S + F)$; so $C = 0.10 \cdot (100{,}000 - (S + F))$. We write this as

$$C + 0.1S + 0.1F = 10{,}000,$$

putting all the variables on one side. The statement that the state tax is 5 percent of the profits net of the donation becomes the equation $S = 0.05 \cdot (100{,}000 - C)$, or

$$0.05C + S = 5{,}000.$$

Federal taxes are 40 percent of the profit after deducting $C$ and $S$; this relation is expressed by the equation $F = 0.40 \cdot [100{,}000 - (C + S)]$, or

$$0.4C + 0.4S + F = 40{,}000.$$

We can summarize the payments to be made by the system of linear equations

$$\begin{aligned} C + 0.1S + 0.1F &= 10{,}000 \\ 0.05C + S \phantom{+ 0.1F} &= 5{,}000 \\ 0.4C + 0.4S + F &= 40{,}000. \end{aligned} \qquad (1)$$

There are a number of ways to solve this system. For example, you can solve the middle equation for $S$ in terms of $C$, substitute this relation into the first and third equations in (1), and then easily solve the resulting system of two equations in two unknowns to compute

$$C = 5{,}956, \quad S = 4{,}702, \quad \text{and} \quad F = 35{,}737,$$

rounded to the nearest dollar. The next chapter is devoted to the solution of such systems of linear equations. For the moment, note that the firm's after-tax and after-contribution profits are \$53,605.

We can use this linear model to compute (Exercise 6.1) that the firm would have had after-tax profits of \$57,000, if it had not made the Red Cross donation. So, the \$5,956 donation really only cost it \$3,395 (= \$57,000 − \$53,605). Later, we will develop a formula for $C$, $S$, and $T$ in terms of unspecified before-tax profits $P$ and even, in Chapter 26, in terms of the tax rates and contribution percentages.

## Example 2: Linear Models of Production

Linear models of production are perhaps the simplest production models to describe. Here we will describe the simplest of the linear models. We will suppose that our economy has $n + 1$ goods. Each of goods 1 through $n$ is produced by one **production process**. There is also one commodity, labor (good 0), which is not produced by any process and which each process uses in production. A production process is simply a list of amounts of goods: so much of good 1, so much of good 2, and so on. These quantities are the amounts of input needed to produce one unit of the process's output. For example, the making of one car requires so much steel, so much plastic, so much labor, so much electricity, and so forth. In fact, some production processes, such as those for steel or automobiles, use some of their own output to aid in subsequent production.

The simplicity of the linear production model is due to two facts. First, in these models, the amounts of inputs needed to produce two automobiles are exactly twice those required for the production of one automobile. Three cars require 3 times as much of the inputs, and so on. In the jargon of microeconomics, each production process exhibits **constant returns to scale**. The production of 2, 3, or $k$ cars requires 2, 3, or $k$ times the amounts of inputs required for the production of 1 car. Second, in these models there is only one way to produce a car. There is no way to substitute electricity for labor in the production of cars. Output cannot be increased by using more of any one factor alone; more of all the factors is needed, and always in the same proportions. This simplifies the analysis of production problems, because the optimal input mix for the production of, say, 1000 cars, does not have to be computed. It is simply 1000 times the optimal input mix required for the production of 1 car.

Before undertaking an abstract analysis, we will work out an example to illustrate the key features of the model. Consider the economy of an organic farm which produces two goods: corn and fertilizer. Corn is produced using corn (to plant) and fertilizer. Fertilizer is made from old corn stalks (and perhaps by feeding the corn to cows, who then produce useful end products). Suppose that the production of 1 ton of corn requires as inputs 0.1 ton of corn and 0.8 ton of fertilizer. The production of 1 ton of fertilizer requires no fertilizer and 0.5 ton of corn.

We can describe each of the two production processes by pairs of numbers $(a, b)$, where $a$ represents the corn input and $b$ represents the fertilizer input. The corn production process is described by the pair of numbers (0.1, 0.8). The fertilizer production process is described by the pair of numbers (0.5, 0).

The most important question to ask of this model is: What can be produced for consumption? Corn is used both in the production of corn and in the production of fertilizer. Fertilizer is used in the production of corn. Is there any way of running both processes so as to leave some corn and some fertilizer for individual consumption? If so, what combinations of corn and fertilizer for consumption are feasible?

Answers to these questions can be found by examining a particular system of linear equations. Suppose the two production processes are run so as to produce $x_C$ tons of corn and $x_F$ tons of fertilizer. The amount of corn actually used in the production of corn is $0.1x_C$ — the amount of corn needed per ton of corn output times the number of tons to be produced. Similarly, the amount of corn used in the production of fertilizer is $0.5x_F$. The amount of corn left over for consumption will be the total amount produced minus the amounts used for production of corn and fertilizer: $x_C - 0.1x_C - 0.5x_F$, or $0.9x_C - 0.5x_F$ tons. The amount of fertilizer needed in production is $0.8x_C$ tons. Thus the amount left over for consumption is $x_F - 0.8x_C$ tons.

Suppose we want our farm to produce for consumption 4 tons of corn and 2 tons of fertilizer. How much total production of corn and fertilizer will be required? Put another way, how much corn and fertilizer will the farm have to produce in order to have 4 tons of corn and 2 tons of fertilizer left over for consumers? We can answer this question by solving the pair of linear equations

$$\begin{aligned} 0.9x_C - 0.5x_F &= 4, \\ -0.8x_C + \quad x_F &= 2. \end{aligned}$$

This system is easily solved. Solve the second equation for $x_F$ in terms of $x_C$:

$$x_F = 0.8x_C + 2. \tag{2}$$

Substitute this expression for $x_F$ into the first equation:

$$0.9x_C - 0.5(0.8x_C + 2) = 4$$

and solve for $x_C$:

$$0.5x_C = 5, \quad \text{so} \quad x_C = 10.$$

Finally, substitute $x_C = 10$ back into (2) to compute

$$x_F = 0.8 \cdot 10 + 2 = 10.$$

In the general case, the production process for good $j$ can be described by a set of **input-output coefficients** $\{a_{0j}, a_{1j}, \ldots, a_{nj}\}$, where $a_{ij}$ denotes the input of good $i$ needed to output one unit of good $j$. Keep in mind that the first subscript stands for the input good and the second stands for the output good. The production of $x_j$ units of good $j$ requires $a_{0j}x_j$ units of good 0, $a_{1j}x_j$ units of good 1, and so on.

Total output of good $i$ must be allocated between production activities and consumption. Denote by $c_i$ the consumer demand for good $i$. This demand is given **exogenously**, which is to say that it is not solved for in the model. Let $c_0$ be the consumer's supply of labor. Since good 0 (labor) is supplied by consumers rather than demanded by consumers, $c_0$ will be a negative number. An $n$-tuple $(c_0, c_1, \ldots, c_n)$ is said to be an **admissible** $n$-tuple of consumer demands if $c_0$ is negative, while all the other $c_i$'s are nonnegative. We want each process to produce an output that is sufficient to meet both consumer demand and the input requirements of the $n$ industries. For our simple linear economy, this is the law of supply and demand: output produced must be used in production or in consumption. Let $x_j$ denote the amount of output produced by process $j$. If process $j$ produces $x_j$ units of output, it will need $a_{ij}x_j$ units of good $i$. Adding these terms up over all the industries gives the demand for good $i$: $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + c_i$. The law of supply and demand then requires

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + c_i.$$

It is convenient to rearrange this equation to say that consumer demand must equal gross output less the amount of the good needed as an input for the production processes. For good 1, this says

$$(1 - a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_{1n} = c_1.$$

The analogous equation for good $i$ is

$$-a_{i1}x_1 - \cdots - a_{ii-1} + (1 - a_{ii})x_i - a_{ii+1}x_{i+1} - \cdots - a_{in}x_n = c_i.$$

The corresponding law of supply and demand for labor says

$$-a_{01}x_1 - \cdots - a_{0n}x_n = c_0.$$

This leads to the following system of $n + 1$ equations in $n$ unknowns, which summarizes the equilibrium output levels for the entire $n$-industry economy:

$$\begin{aligned}
(1 - a_{11})x_1 \quad & - a_{12}x_2 - \cdots & - a_{1n}x_n &= c_1 \\
-a_{21}x_1 + (1 & - a_{22})x_2 - \cdots & - a_{2n}x_n &= c_2 \\
\vdots \qquad & \qquad \vdots & \vdots \quad & \\
-a_{n1}x_1 \quad & - a_{n2}x_2 - \cdots & + (1 - a_{nn})x_n &= c_n \\
-a_{01}x_1 \quad & - a_{02}x_2 - \cdots & - a_{0n}x_n &= c_0.
\end{aligned} \tag{3}$$

This linear system is called an **open Leontief system** after Wassily Leontief, who first studied this type of system in the 1930s and later won a Nobel Prize in economics for his work. It is said to be **open** because the demand $c_0, \ldots, c_n$ is exogenously given, while the supply of goods is endogenously determined, that is, is determined by the equations under study. In this system of equations, the

$a_{ij}$'s and the $c_i$'s are given and we must solve for the $x_i$'s, the gross outputs of the industries.

There are a number of algebraic questions associated with these equations whose answers are important for obtaining the economic insights the interindustry model has to offer. For example, what sets of input-output coefficients yield a nonnegative solution of system (3) for some admissible $n$-tuple of consumer demands? What set of output $n$-tuples will achieve a specified admissible $n$-tuple of consumer demands? What set of admissible $n$-tuples of consumer demands can be obtained from some given set of input-output coefficients?

We have seen how this model sets up in terms of a system of linear equations. But many insights into the workings of the Leontief model can best be understood by studying the geometry of the model. We will study linear systems from the geometric point of view in Chapter 27.

## Example 3: Markov Models of Employment

Aggregate unemployment rates do not tell the whole story of unemployment. In order to target appropriate incomes policies it is necessary to see exactly who is unemployed. For example, is most unemployment due to a few people who are unemployed for long periods of time, or is it due to many people, each of whom is only briefly unemployed? Questions like these can be answered by data about the duration of unemployment and the transition between employment and unemployment. Markov models are the probability models commonly used in these studies.

If an individual is not employed in a given week, in the next week he or she may either find a job or remain unemployed. With some chance, say probability $p$, the individual will find a job, and therefore with probability $1 - p$ that individual will remain unemployed. Similarly, if an individual is employed in a given week, we let $q$ be the probability that he or she will remain employed and therefore $1 - q$ the probability of becoming unemployed. The probabilities $p$, $q$, $1 - p$, and $1 - q$ are called **transition probabilities**. In order to keep this model simple, we will assume that the chances of finding a job are independent of how many weeks the job seeker has been unemployed and that the chances of leaving a job are also independent of the number of weeks worked. Then the random process of leaving jobs and finding new ones is said to be a **Markov process**. The two possibilities, employed and unemployed, are the **states** of the process.

The transition probabilities can lead to a description of the pattern of unemployment over time. For example, suppose that there are $x$ males of working age who are currently employed, and $y$ who are currently unemployed. How will these numbers change next week? Of the $x$ males currently employed, on average $qx$ will remain employed and $(1 - q)x$ will become unemployed. Of the $y$ males currently unemployed, on average $py$ will become employed while $(1 - p)y$ will remain unemployed. Summing up, the average number employed next week will be $qx + py$; and the average number unemployed will be $(1 - q)x + (1 - p)y$. If changes in the size of the labor force are ignored, the week-by-week dynamics of average unemployment are described by the linear equations

$$\begin{aligned} x_{t+1} &= qx_t + py_t \\ y_{t+1} &= (1-q)x_t + (1-p)y_t, \end{aligned} \tag{4}$$

where $x_t$ and $y_t$ are the average numbers of employed and unemployed, respectively, in week $t$. This system of equations is an example of a linear system of **difference equations.**

Macroeconomist Robert Hall estimated the transition probabilities for various segments of the U.S. population in 1966. For white males the corresponding system (4) of equations is

$$\begin{aligned} x_{t+1} &= .998x_t + .136y_t \\ y_{t+1} &= .002x_t + .864y_t. \end{aligned} \tag{5}$$

For black males, the system is

$$\begin{aligned} x_{t+1} &= .996x_t + .102y_t \\ y_{t+1} &= .004x_t + .898y_t. \end{aligned} \tag{6}$$

In the above three systems of equations, note that for any pair of numbers $x_t$ and $y_t$,

$$x_{t+1} + y_{t+1} = x_t + y_t.$$

In particular, if we start out with data in percentages, so that $x_0$ and $y_0$ sum to 1, then $x_t$ and $y_t$ will sum to 1 for all $t$. To see this, just add the two equations in (4). Furthermore, it is easy to see that if $x_t$ and $y_t$ are nonnegative numbers, then $x_{t+1}$ and $y_{t+1}$ will be also. Thus, if the initial data we plug into the equation at time 0 is a distribution of the population, the data at each time $t$ will also be a distribution.

There are two questions that are typically asked of Markov processes. First, will $x_t$ and $y_t$ ever be constant over time? That is, is there a distribution of the population between the two states that will replicate itself in the dynamics of equation (4)? In other words, is there a nonnegative $(x, y)$ pair with

$$\begin{aligned} x &= qx + py \\ y &= (1-q)x + (1-p)y \\ 1 &= x + y. \end{aligned} \tag{7}$$

Such a pair, if it exists, is called a **stationary distribution,** or a **steady state** of (4). Once such a distribution occurs, it will continue to recur for all time (unless $p$ or $q$ changes).

The second question is contingent on the existence of a stationary distribution. Will the system, starting from any initial distribution of states, converge to a steady state distribution? If so, the system is said to be **globally stable**. Both of these questions can be answered using techniques of linear algebra.

The first two equations of equation system (7) can be rewritten as

$$\begin{aligned} 0 &= (q-1)x + py \\ 0 &= (1-q)x - py. \end{aligned} \tag{8}$$

However, there is really only one distinct equation in (8), since the second equation is just the negative of the first equation and therefore can be discarded. Combining the first equation in system (8) with the remaining equation in (7), we conclude that candidates for steady states will be solutions to the system of equations

$$\begin{aligned} (q-1)x + py &= 0 \\ x + y &= 1. \end{aligned} \tag{9}$$

(We also have the nonnegativity constraint, but this will not be a problem.) To solve system (9), multiply the second equation through by $-p$ and add the result to the first equation. The resulting equation will contain no $y$'s and can easily be solved for $x$. Then, use either of the equations in (9) to solve for the corresponding $y$. The resulting solution is

$$x = \frac{p}{1+p-q} \quad \text{and} \quad y = \frac{1-q}{1+p-q}.$$

Applying this formula to Hall's data gives a steady state unemployment rate of 1.4 percent for white males and 3.77 percent for black males. The stability question asks: when is there a tendency to move toward these rates? This analysis is harder than anything else we have done so far, but it still involves linear techniques.

Note that we have seen two different linear systems in the Markov model: system (4) which describes the dynamics of the population distribution, and system (9) which describes the long-run steady state equilibrium.

### Example 4: IS-LM Analysis

IS-LM analysis is Sir John Hicks's interpretation of the basic elements of John Maynard Keynes' classic work, the *General Theory of Employment, Interest, and Money*. We examine a simple example of IS-LM analysis: a linear model of a closed economy such as one can find in any undergraduate macroeconomics text.

Consider an economy with no imports, exports, or other leakages. In such an economy, the value of total production equals total spending, which in turn equals total national income, all of which we denote by the variable $Y$. From the expenditure side, total spending $Y$ can be decomposed into the spending $C$

by consumers (consumption) plus the spending $I$ by firms (investment) plus the spending $G$ by government:

$$Y = C + I + G.$$

On the consumer side, consumer spending $C$ is proportional to total income $Y$: $C = bY$, with $0 < b < 1$. The parameter $b$ is called the **marginal propensity to consume**, while $s = 1 - b$ is called the **marginal propensity to save**. On the firms' side, investment $I$ is a decreasing function of the interest rate $r$. In its simplest linear form, we write this relationship as

$$I = I^o - ar.$$

The parameter $a$ is called the **marginal efficiency of capital.**

Putting these relations together gives the **IS schedule**, the relationship between national income and interest rates consistent with savings and investment behavior

$$Y = bY + (I^o - ar) + G,$$

which we write as

$$sY + ar = I^o + G, \tag{10}$$

where $s = 1-b, a, I^o$, and $G$ are positive parameters. This IS equation is sometimes said to describe the real side of the economy, since it summarizes consumption, investment, and savings decisions.

On the other hand, the **LM equation** is determined by the money market equilibrium condition that money supply $M_s$ equals money demand $M_d$. The money supply $M_s$ is determined outside the system. Money demand $M_d$ is assumed to have two components: the **transactions** or **precautionary demand** $M_{dt}$ and the **speculative demand** $M_{ds}$. The transactions demand derives from the fact that most transactions are denominated in money. Thus, as national income rises, so does the demand for funds. We write this relationship as

$$M_{dt} = mY.$$

The speculative demand comes from the portfolio management problem faced by an investor in the economy. The investor must decide whether to hold bonds or money. Money is more liquid but returns no interest, while bonds pay at rate $r$. It is usually argued that the speculative demand for money varies inversely with the interest rate (directly with the price of bonds). The simplest such relationship is the linear one

$$M_{ds} = M^o - hr.$$

The LM curve is the relationship between national income and interest rates given by the condition that money supply equals total money demand:

$$M_s = mY + M^o - hr,$$

or

$$mY - hr = M_s - M^o.$$

The parameters $m$, $h$, and $M^o$ are all positive.

Equilibrium in this simple model will occur when both the IS equation (production equilibrium) and the LM equation (monetary equilibrium) are simultaneously satisfied. Equilibrium national income $Y$ and interest rates $r$ are solutions to the system of equations

$$\begin{aligned} sY + ar &= I^o + G \\ mY - hr &= M_s - M^o. \end{aligned} \tag{11}$$

Algebraic questions come into play in examining how solutions $(Y, r)$ depend upon the policy parameters $M_s$ and $G$ and on the behavioral parameters $a$, $h$, $I^o$, $m$, $M^o$, and $s$. The comparative statics of the model — the determination of the relationship between parameters and solutions of the equations — is an algebraic problem on which the tools of linear algebra shed much light.

The importance of studying the linear version of the IS-LM model in addition to the general nonlinear version of the model cannot be underestimated. First, the intuition of the model is most easily seen in its linear form. Second, study of the linear model can suggest what to look for in more general models. Finally, the comparative statics of nonlinear models — the exploration of how solutions to the system change as the parameters describing the system change — is uncovered by approximating the nonlinear model with a linear one, and then studying the linear approximation. These three reasons for focusing on linear models for the study of nonlinear phenomena will recur frequently. The simplicity of linear models commends such models as a first step in the construction of more complex models, and the more complex models are frequently studied by examining carefully chosen linear approximations.

### Example 5: Investment and Arbitrage

In the simple neoclassical model of consumer choice, a consumer decides how much of each of the $n$ well-specified goods to consume today. In order to extend this model to the study of investment decisions, we must add two new ingredients — time and uncertainty. Suppose that there are $A$ investment assets, which our investor may buy at the beginning of an investment period and sell at the end of the period. To bring uncertainty into this discussion, assume that $S$ different financial climates are possible during the coming period. We call these conditions **states of nature**. Exactly one of these $S$ states will occur; of course, no one knows which one. An asset will have different returns in different states of nature. Let $v_i$ be the current value of one unit of asset $i$, and let $y_{si}$ be the value of one unit of asset $i$ at the end of the investment period, including dividends paid, if state $s$ occurs. Then, the **realized return** or **payoff** on the $i$th asset in state $s$ is

$$R_{si} = \frac{y_{si}}{v_i}.$$

This is the amount the investor will receive per dollar invested in asset $i$ should state $s$ occur. (The realized return can be thought of as 1 plus the *rate of return.*)

Let $n_i$ denote the number of units or **shares** of asset $i$ held. The share amounts $n_i$ can have either sign. A positive $n_i$ indicates a **long position** and thus entitles the investor to *receive* $y_{si}n_i$ if state $s$ occurs. A negative $n_i$ indicates a **short position**; the investor, in effect, borrows $n_i$ shares of asset $i$ and promises to *pay back* $y_{si}n_i$ at the end of the period if state $s$ occurs. In this case, the investment in asset $i$ has a positive rate of return only if $y_{si} < v_i$, that is, if it is cheaper to pay back the borrowed shares that it was to borrow them.

If the investor has wealth $w_0$ available for investment purposes, the investor's budget constraint is

$$n_1 v_1 + \cdots + n_A v_A = w_0.$$

If state $s$ occurs, the return to the investor of purchasing $n_i$ shares of asset $i$ for $i = 1, \ldots, A$ is

$$R_s = \frac{y_{s1}n_1 + y_{s2}n_2 + \cdots + y_{sA}n_A}{w_0} = \sum_{i=1}^{A} \frac{y_{si}n_i}{w_0}. \tag{12}$$

We usually normalize by letting

$$x_i = \frac{n_i v_i}{w_0}$$

represent the fraction of the investor's wealth held in asset $i$. Budget constraint (12) becomes simply

$$x_1 + x_2 + \cdots + x_A = 1.$$

The $A$-tuple $(x_1, x_2, \ldots, x_A)$ is called a **portfolio** and the $x_i$'s are called **portfolio weights**. If state s occurs, the return to the investor of portfolio $(x_1, \ldots, x_A)$ is

$$R_s = \frac{\sum_{i=1}^{A} y_{si}n_i}{w_0} = \sum_{i=1}^{A} \frac{y_{si}}{v_i} \cdot \frac{n_i v_i}{w_0} = \sum_{i=1}^{A} R_{si}x_i$$

by the definitions of $R_{si}$ and $x_i$.

At this point, we introduce some of the vocabulary of finance theory. A portfolio $(x_1, \ldots, x_A)$ is called **riskless** if it provides the same return in every state of nature:

$$\sum_{i=1}^{A} R_{1i}x_i = \sum_{i=1}^{A} R_{2i}x_i = \cdots = \sum_{i=1}^{A} R_{Si}x_i.$$

A nonzero $A$-tuple $(x_1, \ldots, x_A)$ is called an **arbitrage portfolio** if

$$x_1 + \cdots + x_A = 0 \qquad \text{(instead of 1).}$$

In such a "portfolio," the money received from the short sales is used in the purchase of the long positions. Notice that, in an arbitrage portfolio, $n_1v_1 + \cdots + n_Av_A = 0$, so that the portfolio costs nothing.

A portfolio $(x_1, \ldots, x_A)$ is called **duplicable** if there is a different portfolio $(w_1, \ldots, w_A)$ with exactly the same returns in every state:

$$\sum_{i=1}^{A} R_{si}x_i = \sum_{i=1}^{A} R_{si}w_i \qquad \text{for each } s = 1, \ldots, S.$$

A state $s^*$ is called **insurable** if there is a portfolio $(x_1, \ldots, x_A)$ which has a positive return if state $s^*$ occurs and zero return if any other state occurs:

$$\sum_{i=1}^{A} R_{s^*i}x_i > 0$$

$$\sum_{i=1}^{A} R_{si}x_i = 0 \qquad \text{for all } s \neq s^*.$$

The name is appropriate because the given portfolio can provide insurance against the occurrence of state $s^*$.

It is sometimes convenient to assign a price to each of the $s$ states of nature. An $S$-tuple $(p_1, \ldots, p_S)$ is called a **state price vector** if

$$\begin{aligned} p_1y_{11} + p_2y_{21} + \cdots + p_Sy_{S1} &= v_1 \\ p_1y_{12} + p_2y_{22} + \cdots + p_Sy_{S2} &= v_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ p_1y_{1A} + p_2y_{2A} + \cdots + p_Sy_{SA} &= v_A, \end{aligned} \tag{13}$$

or equivalently,

$$\begin{aligned} p_1R_{11} + p_2R_{21} + \cdots + p_SR_{S1} &= 1 \\ p_1R_{12} + p_2R_{22} + \cdots + p_SR_{S2} &= 1 \\ \vdots \qquad \vdots \qquad & \vdots \\ p_1R_{1A} + p_2R_{2A} + \cdots + p_SR_{SA} &= 1. \end{aligned} \tag{14}$$

Systems (13) and (14) state that the current price $v_j$ of asset $j$ is equal to a weighted sum of its returns in each state of nature, with the same weights for each $j$. The weight $p_s$ for state $s$ is a kind of price for wealth in state $s$ and is often called a **state price**. If we can price states, then the price of each asset is just the value at the state prices of the returns in each state. This is the content of the linear system of equations (13).

Since all the equations in this application are linear, it is not surprising that techniques of linear algebra can answer questions about the existence and characterization of riskless, duplicable, and arbitrage portfolios and of insurable states and state prices.

*Example 6.1* Suppose that there are two assets and three possible states. If state 1 occurs, asset 1 returns $R_{11} = 1$ and asset 2 returns $R_{12} = 3$. If state 2 occurs, $R_{21} = 2$ and $R_{22} = 2$. If state 3 occurs, $R_{31} = 3$ and $R_{32} = 1$. If both assets have the same current value and if the investor buys $n_1 = 3$ shares of asset 1 and $n_2 = 1$ share of asset 2, the corresponding portfolio is $(\frac{3}{4}, \frac{1}{4})$ and the returns are

$$R_{11} \cdot \tfrac{3}{4} + R_{12} \cdot \tfrac{1}{4} = \tfrac{3}{2} \qquad \text{in state 1,}$$

$$R_{21} \cdot \tfrac{3}{4} + R_{22} \cdot \tfrac{1}{4} = 2 \qquad \text{in state 2,}$$

$$R_{31} \cdot \tfrac{3}{4} + R_{32} \cdot \tfrac{1}{4} = \tfrac{5}{2} \qquad \text{in state 3.}$$

Portfolio $(\frac{1}{2}, \frac{1}{2})$ is a riskfree portfolio since it yields a return of 2 in all three states (check). The 3-tuple $(\frac{1}{8}, \frac{1}{2}, \frac{1}{8})$ is a pricing system for this economy (check). As we will see in Section 7.4, there are no duplicable portfolios and no insurable states.

These five examples illustrate the important role that linear models play in economics and indeed in all the social sciences. We conclude this chapter by mentioning three other instances where economists use linear algebra. First, many of the elementary techniques of econometrics, such as (generalized) least-squares estimation, rely heavily on linear systems of equations. Second, linear programming, the optimization of a linear function on a set defined by a system of linear equalities and inequalities, is a fundamental economic technique. As such, a number of textbooks are devoted entirely to it and it is the total subject matter of graduate courses in mathematics and engineering, as well as economics. Finally, we will rely on linear algebra techniques when we study the generalization of the second derivative test in calculus to maximization problems which involve (nonlinear) functions of more than one variable.

## EXERCISES

**6.1** Suppose that the firm in Example 1 did not make any charitable contribution. Write out and solve the system of equations which describe its state and federal taxes. What is the net cost of its \$5956 charitable contribution?

**6.2** In Missouri, federal income taxes are deducted from state taxes. Write out and solve the system of equations which describes the state and federal taxes and charitable contribution of the firm in Example 1 if it were based in Missouri.

**6.3** The economy on the island of Bacchus produces only grapes and wine. The production of 1 pound of grapes requires 1/2 pound of grapes, 1 laborer, and no wine. The production of 1 liter of wine requires 1/2 pound of grapes, 1 laborer, and 1/4 liter of wine. The island has 10 laborers who all together demand 1 pound of grapes and 3 liters of wine for their own consumption. Write out the input-output system for the economy of this island. Can you solve it?

**6.4** Suppose that the production of a pound of grapes now requires 7/8 liter of wine. If none of the other input-output coefficients change, write out the new systems for the outputs.

**6.5** Suppose that 10 percent of white males of working age and 20 percent of black males of working age are unemployed in 1966. According to Hall's model, what will the corresponding unemployment rates be in 1967?

**6.6** For the Markov employment model, Hall gives $p = .106$ and $q = .993$ for black females, and $p = .151$ and $q = .997$ for white females. Write out the Markov systems of difference equations for these two situations. Compute the stationary distributions.

**6.7** Consider the IS-LM model of Example 4 with no fiscal policy ($G = 0$). Suppose that $M_s = M^o$; that is, the intercept of the LM curve is 0. Suppose that $I^o = 1000$, $s = 0.2$, $h = 1500$, $a = 2000$, and $m = 0.16$. Write out the explicit IS-LM system of equations. Solve them for the equilibrium GNP $Y$ and the interest rate $r$.

**6.8** Carry out the two checks at the end of Example 5.

## NOTES

We present some references for the linear models discussed in this section. For the intellectual origins of input-output models, see F. Quesnay (1694–1774), *Tableau Economique*, and W. Leontief (1906– ), *The Structure of the American Economy: 1919–1929* (Cambridge, Mass.: Harvard University Press, 1941). For a good modern treatment, see Chapters 8 and 9 of D. Gale, *The Theory of Linear Economic Models* (New York: McGraw-Hill, 1960). Gale's book is also an excellent reference for the techniques and economic applications of linear programming.

The estimates for the transition probabilities for the employment of various segments in the United States in 1966 are from Robert Hall, "Turnover in the labor force," *Brookings Papers on Economic Activity* 3, 1972, p. 709.

Most undergraduate macroeconomic texts treat IS-LM models. See, for example, Chapter 4 of R. Hall and J. Taylor's *Macroeconomics*, 4th ed. (New York: Norton, 1993.) Keynes's classic in this area is J. M. Keynes (1883–1946), *The General Theory of Employment, Interest, and Money* (New York: Harcourt, Brace, 1936). For Hicks's interpretation of this theory, see J. R. Hicks (1904–1989), "Mr. Keynes and the 'Classics': a Suggested Interpretation," *Econometrica* April 1937, 147–159. Among the good expositions of modern portfolio theory is Jonathan Ingersoll *Theory of Financial Decision Making* (Rowman & Littlefield, 1987).

CHAPTER 7

# Systems of Linear Equations

As was discussed in the last chapter, systems of linear equations arise in two ways in economic theory. Some economics models have a natural linear structure, like the five examples in the last chapter. On the other hand, when the relationships among the variables under consideration are described by a system of *nonlinear* equations, one takes the derivative of these equations to convert them to an approximating *linear* system. Theorems of calculus tell us that by studying the properties of this latter linear system, we can learn a lot about the underlying nonlinear system.

In this chapter we begin the study of systems of linear equations by describing techniques for solving such systems. The preferred solution technique — Gaussian elimination — answers the fundamental questions about a given linear system: does a solution exist, and if so, how many solutions are there?

An implicit system is one in which the equations that describe the economic relationships under study have the exogenous and endogenous variables mixed in with each other on the same side of the equal signs. This chapter closes with a discussion of the Linear Implicit Function Theorem, which tells how to use linear algebra techniques to quantify the effect of a change in the exogenous variables on the endogenous ones in a linear implicit system.

## 7.1 GAUSSIAN AND GAUSS-JORDAN ELIMINATION

We begin our study of linear phenomena by considering the problem of solving linear systems of equations, such as

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 - x_3 &= 0. \end{aligned} \tag{1}$$

The general linear system of $m$ equations in $n$ unknowns can be written

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\quad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{2}$$

In this system, the $a_{ij}$'s and $b_i$'s are given real numbers; $a_{ij}$ is the **coefficient** of the unknown $x_j$ in the $i$th equation. A **solution** of system (2) is an $n$-tuple of real numbers $x_1, x_2, \ldots, x_n$ which satisfies each of the $m$ equations in (2). For example, $x_1 = 2, x_2 = 1$ solves the first system in (1), and $x_1 = 5, x_2 = 0, x_3 = 0$ solves the second.

For a linear system such as (2), we are interested in the following three questions:

(1) Does a solution exist?
(2) How many solutions are there?
(3) Is there an efficient algorithm that computes actual solutions?

There are essentially three ways of solving such systems:

(1) substitution,
(2) elimination of variables, and
(3) matrix methods.

## Substitution

Substitution is the method usually taught in beginning algebra classes. To use this method, solve one equation of system (2) for one variable, say $x_n$, in terms of the other variables in that equation. Substitute this expression for $x_n$ into the other $m - 1$ equations. The result is a new system of $m - 1$ equations in the $n - 1$ unknowns $x_1, \ldots, x_{n-1}$. Continue this process by solving one equation in the new system for $x_{n-1}$ and substituting this expression into the other $m - 2$ equations to obtain a system of $m - 2$ equations in the $n - 2$ variables $x_1, \ldots, x_{n-2}$. Proceed until you reach a system with just a single equation, a situation which is easily solved. Finally, use the earlier expressions of one variable in terms of the others to find all the $x_i$'s.

This sounds complicated but it really is straightforward. We used substitution to solve the input-output system in Section 6.2. Let us see how it works on a *three*-good input-output model.

*Example 7.1* The production process for a three-good economy is summarized by the input-output table:

| | | |
|---|---|---|
| 0 | 0.4 | 0.3 |
| 0.2 | 0.12 | 0.14 |
| 0.5 | 0.2 | 0.05 |

**Table 7.1**

Recall from the last chapter that the entries in the second column of Table 7.1 declare that it takes 0.4 unit of good 1, 0.12 unit of good 2, and 0.2 unit of good 3 to produce 1 unit of good 2. We ignore the labor component in this example. Suppose that there is an exogenous demand for 130 units of good 1, 74 units of good 2, and 95 units of good 3. How much will the economy have to produce to meet this demand?

Let $x_i$ denote the amount of good $i$ produced. As we described last chapter, "supply equals demand" leads to the following system of equations:

$$\begin{aligned} x_1 &= 0\ x_1 + 0.4\ x_2 + 0.3\ x_3 + 130 \\ x_2 &= 0.2x_1 + 0.12x_2 + 0.14x_3 + \ 74 \\ x_3 &= 0.5x_1 + 0.2\ x_2 + 0.05x_3 + \ 95, \end{aligned}$$

which can be rewritten as the system

$$\begin{aligned} x_1 - 0.4\ x_2 - 0.3\ x_3 &= 130 \\ -0.2x_1 + 0.88x_2 - 0.14x_3 &= \ 74 \\ -0.5x_1 - 0.2\ x_2 + 0.95x_3 &= \ 95. \end{aligned} \tag{3}$$

We write (3a), (3b) and (3c) for the three equations in system (3) in the order given, and similarly for following systems. Solving equation (3a) for $x_1$ in terms of $x_2$ and $x_3$ yields

$$x_1 = 0.4x_2 + 0.3x_3 + 130. \tag{4}$$

Substitute (4) into equations (3b) and (3c):

$$\begin{aligned} -0.2(0.4x_2 + 0.3x_3 + 130) + 0.88x_2 - 0.14x_3 &= 74 \\ -0.5(0.4x_2 + 0.3x_3 + 130) - 0.2\ x_2 + 0.95x_3 &= 95, \end{aligned}$$

which simplifies to

$$\begin{aligned} 0.8x_2 - 0.2x_3 &= 100 \\ -0.4x_2 + 0.8x_3 &= 160. \end{aligned} \tag{5}$$

Now, use substitution to solve subsystem (5) by solving the first equation (5a) for $x_2$ in terms of $x_3$:

$$x_2 = 125 + 0.25x_3, \tag{6}$$

and plugging this expression into the second equation (5b):

$$-0.4(125 + 0.25x_3) + 0.8x_3 = 160,$$

or

$$x_3 = 300.$$

Substitute $x_3 = 300$ into (6) to compute that $x_2 = 200$. (Check.) Finally, substitute $x_2 = 200$ and $x_3 = 300$ into (4) to compute that

$$x_1 = 0.4 \cdot 200 + 0.3 \cdot 300 + 130 = 300.$$

Therefore, this economy needs to produce 300 units of good one, 200 units of good two, and 300 units of good three to meet the exogenous demands.

As this example shows, the substitution method is straightforward, but it can be cumbersome. Furthermore, it does not provide much insight into the nature of the general solution to systems like (3). It is not a method around which one can build a general theory of linear systems. However, it is the most direct method for solving certain systems with a special, very simple form. As such, it will play a role in the general solution technique we now develop.

## Elimination of Variables

The method which is most conducive to theoretical analysis is *elimination of variables*, another technique that should be familiar from high school algebra. First, consider the simple system

$$\begin{aligned} x_1 - 2x_2 &= 8 \\ 3x_1 + x_2 &= 3. \end{aligned} \tag{7}$$

We can "eliminate" the variable $x_1$ from this system by multiplying equation (7a) by $-3$ to obtain $-3x_1 + 6x_2 = -24$ and adding this new equation to (7b). The result is

$$7x_2 = -21, \qquad \text{or} \qquad x_2 = -3.$$

To find $x_1$, we substitute $x_2 = -3$ back into (7b) or (7a) to compute that $x_1 = 2$. We chose to multiply equation (7a) by $-3$ precisely so that when we added the new equation to equation (7b), we would "eliminate" $x_1$ from the system.

To solve a general system of $m$ equations by elimination of variables, use the coefficient of $x_1$ in the first equation to eliminate the $x_1$ term from all the equations *below* it. To do this, add proper multiples of the first equation to each of the succeeding equations. Now disregard the first equation and eliminate the next variable — usually $x_2$ — from the last $m - 1$ equations just as before, that is, by adding proper multiples of the second equation to each of the succeeding equations. If the second equation does not contain an $x_2$ but a lower equation does, you will have to interchange the order of these two equations before proceeding. Continue eliminating variables until you reach the last equation. The resulting simplified system can then easily be solved by substitution.

Let us try this method on the system (3) arising from the three-good input-output Table 7.1:

$$\begin{aligned} x_1 - 0.4\ x_2 - 0.3\ x_3 &= 130 \\ -0.2x_1 + 0.88x_2 - 0.14x_3 &= \ \ 74 \\ -0.5x_1 - 0.2\ x_2 + 0.95x_3 &= \ \ 95 \end{aligned} \tag{8}$$

We first try to eliminate $x_1$ from the last two equations by adding to each of these equations a proper multiple of the first equation. To eliminate the $-0.2x_1$-term in (8b), we multiply (8a) by 0.2 and add this new equation to (8b). The result is the following calculation:

$$\begin{array}{rl} 0.2x_1 - 0.08x_2 - 0.06x_3 &= \ \ 26 \\ + \ -0.2x_1 + 0.88x_2 - 0.14x_3 &= \ \ 74 \\ \hline + \ \ 0.8x_2 - \ \ 0.2x_3 &= 100. \end{array}$$

Similarly, by adding 0.5 times (8a) to (8c), we obtain a new third equation

$$-0.4x_2 + 0.8x_3 = 160.$$

Our system (8) has been transformed to the simpler system

$$\begin{aligned} 1x_1 - 0.4x_2 - 0.3x_3 &= 130 \\ + 0.8x_2 - 0.2x_3 &= 100 \\ - 0.4x_2 + 0.8x_3 &= 160. \end{aligned} \tag{9}$$

In transforming system (8) to system (9), we used only one operation: we added a multiple of one equation to another. This operation is reversible. For example, we can recover (8) from (9) by adding $-0.2$ times (9a) to (9b) to obtain (8b) and then by adding $-0.5$ times (9a) to (9c) to obtain (8c). (We continue to write (9a) to denote the first equation in system (9).) There are two other reversible operations one often uses to transform a system of equations: 1) multiplying both sides of an equation by a nonzero scalar and 2) interchanging two equations. These three operations are called the **elementary equation operations**. Since equals are always added to or subtracted from equals or multiplied by the same scalar, the set of $x_i$'s which solve the original system will also solve the transformed system. In fact, since these three operations are reversible, any solution of the transformed system will also be a solution of the original system. Consequently, both systems will have the exact same set of solutions. We call two systems of linear equations **equivalent** if any solution of one system is also a solution of the other.

**Fact** If one system of linear equations is derived from another by elementary equation operations, then both systems have the same solutions; that is, the systems are equivalent.

Let us return to our elimination procedure and continue working on system (9). Having eliminated $x_1$ from the last two equations, we now want to eliminate $x_2$ from the last equation. We apply the elimination process to the system of two equations (9b) and (9c) in two unknowns. Multiply (9b) by 1/2 and add this new equation to (9c) to obtain the new system:

$$\begin{aligned} 1x_1 - 0.4x_2 - 0.3x_3 &= 130 \\ + 0.8x_2 - 0.2x_3 &= 100 \\ + 0.7x_3 &= 210. \end{aligned} \tag{10}$$

Since each equation in (10) has one fewer variable than the previous one, this system is particularly amenable to solution by substitution. Thus, $x_3 = 300$ from (10c). Substituting $x_3 = 300$ into (10b) gives $x_2 = 200$. Finally, substituting these two values into (10a) yields $x_1 = 300$. The method used in this paragraph is usually called **back substitution**.

This method of reducing a given system of equations by adding a multiple of one equation to another or by interchanging equations until one reaches a system of the form (10) and then solving (10) via back substitution is called **Gaussian elimination**. The important characteristic of system (10) is that each equation contains fewer variables than the previous equation.

At each stage of the Gaussian elimination process, we want to change some coefficient of our linear system to 0 by adding a multiple of an *earlier* equation to the given one. For example, if you want to use the coefficient $a_{3k}$ in the third equation to eliminate the coefficient $a_{5k}$ in the fifth equation, we add $(-a_{5k}/a_{3k})$ times the third equation to the fifth equation, to get a new fifth equation whose $k$th coefficient is 0. The coefficient $a_{3k}$ is then called a **pivot**, and we say that we "pivot on $a_{3k}$ to eliminate $a_{5k}$." At each stage of the elimination procedure, we use a pivot to eliminate all coefficients directly below it. For example, in transforming system (8) to system (9), the coefficient 1 in equation (8a) is the pivot; in transforming system (9) to system (10), the coefficient 0.8 in equation (9b) is the pivot.

Note that 0 can never be a pivot in this process. If you want to eliminate $x_k$ from a subsystem of equations and if the coefficient of $x_k$ is zero in the first equation of this subsystem and nonzero in a subsequent equation, you will have to reverse the order of these two equations before proceeding.

We did not use the operation of transforming an equation by simply multiplying it by a nonzero scalar. There is a variant of Gaussian elimination, called **Gauss-Jordan elimination**, which uses all three elementary equation operations. This method starts like Gaussian elimination, e.g., by transforming (8) to (10). After reaching system (10), multiply each equation in (10) by a scalar so that the first nonzero coefficient is 1:

$$\begin{aligned} x_1 - 0.4x_2 - 0.3\,x_3 &= 130 \\ x_2 - 0.25x_3 &= 125 \\ x_3 &= 300. \end{aligned} \tag{11}$$

Now, instead of using back substitution, use Gaussian elimination methods from the *bottom* equation to the top to eliminate all but the first term on the left-hand side in each equation in (11). For example, add 0.25 times equation (11c) to equation (11b) to eliminate the coefficient of $x_3$ in (11b) and obtain $x_2 = 200$. Then, add 0.3 times (11c) to (11a) and 0.4 times (11b) to (11a) to obtain the new system:

$$\begin{aligned} x_1 \qquad\qquad &= 300 \\ x_2 \qquad &= 200 \\ x_3 &= 300, \end{aligned} \tag{12}$$

which needs no further work to see the solution. Gauss-Jordan elimination is particularly useful in developing the theory of linear systems; Gaussian elimination is usually more efficient in solving actual linear systems.

Earlier we mentioned a third method for solving linear systems, namely matrix methods. We will study these methods in the next two chapters, when we discuss matrix inversion and Cramer's rule. For now, it suffices to note that all the intuition behind these more advanced methods derives from Gaussian elimination. The understanding of this technique will provide a solid base on which to build your knowledge of linear algebra.

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## EXERCISES

**7.1** Which of the following equations are linear?

*a*) $3x_1 - 4x_2 + 5x_3 = 6$; *b*) $x_1x_2x_3 = -2$; *c*) $x^2 + 6y = 1$;
*d*) $(x + y)(x - z) = -7$; *e*) $x + 3^{1/2}z = 4$; *f*) $x + 3z^{1/2} = -4$.

**7.2** Solve the following systems by substitution, Gaussian elimination, and Gauss-Jordan elimination:

*a*)
$$\begin{aligned} x - 3y + 6z &= -1 \\ 2x - 5y + 10z &= 0 \\ 3x - 8y + 17z &= 1; \end{aligned}$$

*b*)
$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 12x_1 + 2x_2 - 3x_3 &= 5 \\ 3x_1 + 4x_2 + x_3 &= -4. \end{aligned}$$

**7.3** Solve the following systems by Gauss-Jordan elimination. Note that the third system requires an equation interchange.

$a)$ $\begin{aligned} 3x + 3y &= 4 \\ x - y &= 10; \end{aligned}$ $b)$ $\begin{aligned} 4x + 2y - 3z &= 1 \\ 6x + 3y - 5z &= 0 \\ x + y + 2z &= 9; \end{aligned}$ $c)$ $\begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x - 4y + 3z &= 0. \end{aligned}$

**7.4** Formalize the three elementary equation operations using the abstract notation of system (2), and for each operation, write out the operation which reverses its effect.

**7.5** Solve the IS-LM system in Exercise 6.7 by substitution.

**7.6** Consider the general IS-LM model with no fiscal policy in Chapter 6. Suppose that $M_s = M^o$; that is, the intercept of the LM-curve is 0.

*a*) Use substitution to solve this system for $Y$ and $r$ in terms of the other parameters.

*b*) How does the equilibrium GNP depend on the marginal propensity to save?

*c*) How does the equilibrium interest rate depend on the marginal propensity to save?

**7.7** Use Gaussian elimination to solve

$$\begin{cases} 3x + 3y = 4 \\ -x - y = 10. \end{cases}$$

What happens and why?

**7.8** Solve the general system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

What assumptions do you have to make about the coefficients $a_{ij}$ in order to find a solution?

## 7.2 ELEMENTARY ROW OPERATIONS

The focus of our concern in the last section was on the coefficients $a_{ij}$ and $b_i$ of the systems with which we worked. In fact, it was a little inefficient to rewrite the $x_i$'s, the plus signs, and the equal signs each time we transformed a system. It makes sense to simplify the representation of linear system (2) by writing two rectangular arrays of its coefficients, called **matrices**. The first array is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

which is called the **coefficient matrix** of (2). When we add on a column corresponding to the right-hand side in system (2), we obtain the matrix

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix},$$

which is called the **augmented matrix** of (2). The rows of $\hat{A}$ correspond naturally to the equations of (2). For example,

$$\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -2 & 8 \\ 3 & 1 & 3 \end{pmatrix}$$

are the coefficient matrix and the augmented matrix of system (7). For accounting purposes, it is often helpful to draw a vertical line just before the last column of the augmented matrix, where the = signs would naturally appear, e.g.,

$$\left(\begin{array}{cc|c} 1 & -2 & 8 \\ 3 & 1 & 3 \end{array}\right).$$

Our three elementary equation operations now become **elementary row operations**:

(1) interchange two rows of a matrix,
(2) change a row by adding to it a multiple of another row, and
(3) multiply each element in a row by the same nonzero number.

The new augmented matrix will represent a system of linear equations which is equivalent to the system represented by the old augmented matrix.

To see this equivalence, first observe that each elementary row operation can be reversed. Clearly the interchanging of two rows or the multiplication of a row by a nonzero scalar can be reversed. Suppose we consider the row operation in which $k$ times the second row of the augmented matrix $\hat{A}$ is added to the first row of $\hat{A}$. The new augmented matrix is

$$B = \left(\begin{array}{ccc|c} a_{11} + ka_{21} & \cdots & a_{1n} + ka_{2n} & b_1 + kb_2 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right).$$

However, if we start with $B$ and add $-k$ times the second row to the first row, we will recover $\hat{A}$. Thus the row operation can be reversed. Since elementary row operations correspond to the three operations of adding a multiple of one equation to another equation, multiplying both sides of an equation by the same scalar, and changing the order of the equations, any solution to the original system of equations will be a solution to the transformed system. Since these operations are reversible, any solution to the transformed system of equations will also be a solution to the original system. Consequently the systems represented by matrices $\hat{A}$ and $B$ have *identical solution sets*; they are equivalent.

The goal of performing row operations is to end up with a matrix that looks much like (10). The nice feature about the augmented matrix representing (10)

$$\left(\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ 0 & 0.8 & -0.2 & 100 \\ 0 & 0 & 0.7 & 210 \end{array}\right) \tag{13}$$

is that each row *begins* with more zeros than does the previous row. Such a matrix is said to be in *row echelon form.*

**Definition** A row of a matrix is said to have $k$ **leading zeros** if the first $k$ elements of the row are all zeros and the $(k + 1)$th element of the row is not zero. With this terminology, a matrix is in **row echelon form** if each row has more leading zeros than the row preceding it.

The first row of the augmented matrix (13) has no leading zeros. The second row has one, and the third row has two. Since each row has more leading zeros than the previous row, matrix (13) is in row echelon form. Let's look at some more concrete examples.

*Example 7.2* The matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 6 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2 & 3 \\ 0 & 6 \\ 0 & 0 \end{pmatrix}$$

are in row echelon form. If a matrix in row echelon form has a row containing only zeros, then all the subsequent rows must contain only zeros.

*Example 7.3* The matrices

$$\begin{pmatrix} 1 & 5 & 2 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 7 \\ 9 & 0 \\ 0 & 2 \end{pmatrix}$$

are not in row echelon form.

*Example 7.4* The matrix whose diagonal elements ($a_{ii}$'s) are 1s and whose off-diagonal elements ($a_{ij}$'s with $i$ not equal to $j$) are all 0s is in row echelon form. This matrix arises frequently throughout linear algebra, and is called the **identity matrix** when the number of rows is the same as the number of columns:

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

*Example 7.5* The matrix each of whose elements is 0 is called the **zero matrix** and is in row echelon form:

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

The usefulness of row echelon form can be seen by considering the system of equations (8). The augmented matrix associated with (8) is

$$\left(\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ -0.2 & 0.88 & -0.14 & 74 \\ -0.5 & -0.2 & 0.95 & 95 \end{array}\right),$$

and through various row operations we reduced it to

$$\left(\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ 0 & 0.8 & -0.2 & 100 \\ 0 & 0 & 0.7 & 210 \end{array}\right). \tag{14}$$

This last matrix is in row echelon form and the corresponding system can be easily solved by substitution. Simply rewrite it in equation form and solve it from bottom to top as we did for system (10).

Because of this connection with Gaussian elimination, it is natural that the first nonzero entry in each row of a matrix in row echelon form be called a **pivot**.

The row echelon form is the goal in the Gaussian elimination process. In Gauss-Jordan elimination, one wants to use row operations to reduce the matrix even further. First, multiply each row of the row echelon form by the reciprocal of the pivot in that row and create a new matrix all of whose pivots are 1s. Then, use these new pivots (starting with the 1 in the last row) to turn each nonzero entry *above* it (in the same column) into a zero.

For example, multiply the second row of (14) by $1/0.8$ and the third row of (14) by $1/0.7$ to achieve the matrix

$$\left(\begin{array}{rrr|r} 1 & -0.4 & -0.3 & 130 \\ 0 & 1 & -0.25 & 125 \\ 0 & 0 & 1 & 300 \end{array}\right)$$

Then, use the pivot in row 3 to turn the entries $-0.25$ and $-0.3$ above it into zeros — first by adding 0.25 times row 3 to row 2 and then by adding 0.3 times row 3 to row 1. The result is

$$\left(\begin{array}{rrr|r} 1 & -0.4 & 0 & 220 \\ 0 & 1 & 0 & 200 \\ 0 & 0 & 1 & 300 \end{array}\right)$$

Finally, use the pivot in row 2 to eliminate the nonzero entry above it by adding 0.4 times row 2 to row 1 to get the matrix

$$\left(\begin{array}{rrr|r} 1 & 0 & 0 & 300 \\ 0 & 1 & 0 & 200 \\ 0 & 0 & 1 & 300 \end{array}\right) \tag{15}$$

Notice that this is the augmented matrix for system (12) and that one can read the solution right off the last column of this matrix:

$$x_1 = 300, \qquad x_2 = 200, \qquad x_3 = 300.$$

We say that matrix (15) is in *reduced row echelon form.*

**Definition** A row echelon matrix in which each pivot is a 1 and in which each column containing a pivot contains no other nonzero entries is said to be in **reduced row echelon form.**

The matrices in Examples 7.4 and 7.5 above are in reduced row echelon form. Note that in transforming a matrix to row echelon form we work from top left to bottom right. To achieve the reduced row echelon form, we continue in the same way but in the other direction, from bottom right to top left.

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## EXERCISES

**7.9** Describe the row operations involved in going from equations (8) to (10).

**7.10** Put the matrices in Examples 7.2 and 7.3 in reduced row echelon form.

**7.11** Write the three systems in Exercise 7.3 in matrix form. Then use row operations to find their corresponding row echelon and reduced row echelon forms and to find the solution.

**7.12** Use Gauss-Jordan elimination in matrix form to solve the system

$$\begin{aligned} w + x + 3y - 2z &= 0 \\ 2w + 3x + 7y - 2z &= 9 \\ 3w + 5x + 13y - 9z &= 1 \\ -2w + x \qquad - z &= 0. \end{aligned}$$

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## 7.3 SYSTEMS WITH MANY OR NO SOLUTIONS

As we will study in more detail later, the locus of all points $(x_1, x_2)$ which satisfy the linear equation $a_{11}x_1 + a_{12}x_2 = b_1$ is a straight line in the plane. Therefore, the solution $(x_1, x_2)$ of the two linear equations in two unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \tag{16}$$

is a point which lies on both lines of (16) in the Cartesian plane. Solving system (16) is equivalent to finding where the two lines given by (16) cross. In general, two lines in the plane will be nonparallel and will cross in exactly one point. However, the lines given by (16) can be parallel to each other. In this case, they will either coincide or they will never cross. If they coincide, every point on either line is a solution to (16); and (16) has *infinitely* many solutions. An example is the system

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 6. \end{aligned}$$

In the case where the two parallel lines do not cross, the corresponding system has *no* solution, as the example

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

illustrates. Therefore, it follows from geometric considerations that two linear equations in two unknowns can have one solution, no solution, or infinitely many solutions. We will see later in this chapter that this principle holds for every system of *m linear* equations in *n* unknowns.

So far we have worked with examples of systems in which there are exactly as many equations as there are unknowns. As we saw in the input-output model and the Markov model in Chapter 6, systems in which the number of equations differs from the number of unknowns arise naturally.

For example, let us look for a state price system for the investment model in Example 6.1. Substitution of the state returns $R_{si}$ from Example 6.1 into equations (14) in Chapter 6 for the state prices leads to the system

$$\begin{aligned} p_1 + 2p_2 + 3p_3 &= 1 \\ 3p_1 + 2p_2 + p_3 &= 1, \end{aligned} \tag{17}$$

whose augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right).$$

Adding $-3$ times the first row to the second yields the row echelon matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \end{array}\right).$$

To obtain the *reduced* row echelon form, multiply the last equation by $-1/4$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0.5 \end{array}\right).$$

Then, add $-2$ times the new last row to the first row to eliminate the 2 in the first row above the pivot. The result is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0.5 \end{array}\right),$$

the reduced row echelon matrix, which corresponds to the system

$$\begin{aligned} p_1 \qquad - p_3 &= 0 \\ p_2 + 2p_3 &= 0.5. \end{aligned}$$

If we write this system as

$$\begin{aligned} p_1 &= p_3 \\ p_2 &= 0.5 - 2p_3, \end{aligned} \tag{18}$$

we notice that there is no single solution to (18); for *any* value of $p_3$, system (18) determines corresponding values of $p_1$ and $p_2$. Since system (18) has multiple solutions, so does system (17). For any choice of $p_3$, (18) determines values of $p_1$ and $p_2$ which solve system (17). For example,

$$p_3 = \frac{1}{8}, \quad p_1 = \frac{1}{8}, \quad p_2 = \frac{1}{4};$$

$$p_3 = \frac{1}{6}, \quad p_1 = \frac{1}{6}, \quad p_2 = \frac{1}{6};$$

$$p_3 = \frac{1}{5}, \quad p_1 = \frac{1}{5}, \quad p_2 = \frac{1}{10}.$$

(Check that these three are truly solutions of (17).)

As an example of a system with *no solutions*, consider an investment model with state returns

$$\begin{array}{lll} R_{11} = 1 & R_{12} = 3 & R_{13} = 2 \\ R_{21} = 3 & R_{22} = 1 & R_{23} = 3. \end{array}$$

Once again by equation (14) in Chapter 6, the corresponding system of equations for a state price vector $(p_1, p_2)$ is

$$\begin{aligned} 1p_1 + 3p_2 &= 1 \\ 3p_1 + 1p_2 &= 1 \\ 2p_1 + 3p_2 &= 1. \end{aligned} \tag{19}$$

In system (19), note that the only $p_1, p_2$ pair that solves the first two equations is $x_1 = 0.25, x_2 = 0.25$. Since this pair does not satisfy the third equation, (19) has no solution. When we reduce the augmented matrix of (19) to row echelon form, we obtain

$$\begin{pmatrix} 1 & 3 & | & 1 \\ 0 & -8 & | & -2 \\ 0 & 0 & | & -0.25 \end{pmatrix}.$$

The last row corresponds to the equation

$$0p_1 + 0p_2 = -.25. \tag{20}$$

The left-hand side of this equation is always 0 and thus can never equal $-0.25$. So there is no $p_1, p_2$ pair which solves this equation. Note that if we replace the ast equation in (19) by

$$2p_1 + 2p_2 = 1,$$

hen the new system has the unique solution

$$p_1 = 0.25, \qquad p_2 = 0.25,$$

nd the row echelon form of the augmented matrix becomes

$$\begin{pmatrix} 1 & 3 & | & 1 \\ 0 & -8 & | & -2 \\ 0 & 0 & | & 0 \end{pmatrix},$$

hich contains no contradictions. (Exercise: check all these computations.)

These examples raise the following questions about systems of linear equa-
ons.

(1) When does a particular system of linear equations have a solution?
(2) How many solutions does it have? How do we compute them?
(3) What conditions on the coefficient matrix will guarantee the existence of *at least* one solution for any choice of $b_i$'s on the right-hand side of (2)?
(4) What conditions on the coefficient matrix will guarantee the existence of *at most* one solution for any choice of $b_i$'s?
(5) What conditions on the coefficient matrix will guarantee the existence of a *unique* solution for any choice of $b_i$'s?

The answers to these questions can be found by studying augmented matrices in educed row echelon form. Gauss-Jordan elimination, which achieves the reduced ow echelon form, works the same way whether or not the number of equations quals the number of unknowns. Let us recall this procedure one more time. Beginning with an augmented matrix, use row operations to achieve a row echelon natrix $B$ in which the first nonzero entry in each row (that is, the pivot) is a 1. Then use these pivots (starting with the one in the last row) to turn each nonzero ntry above it (in the same column) into zero.

For example, if the last pivot is in row $h$ and column $k$ and if $a_{jk}$ is a nonzero entry in row $j$ and (the same) column $k$ with $j < h$, one adds $-a_{jk}$ times row $h$ to ow $j$ to achieve a new $a'_{jk}$ equal to zero. One continues until the pivot ($= 1$) is he only nonzero entry in column $k$. One then moves on to the pivot in row $h - 1$ and uses row operations until it too is the only nonzero entry in its column. These operations will not change the new column $k$ since all the entries above row $h$ in column $k$ are zero. The end result of this process is a row echelon matrix in which each pivot is a 1 and each column which contains a pivot contains no other nonzero entries, that is, a reduced row echelon matrix.

For another example, consider an investment model with three assets and four states. Suppose that shares of the three assets have the following current values:

$$v_1 = 38, \qquad v_2 = 98, \qquad v_3 = 153.$$

As in Section 6.2, we write $y_{si}$ for the value of a share of asset $i$ one year from now if state $s$ occurs. Suppose the $y_{si}$'s have the following values:

$$\begin{array}{lll} y_{11} = 1 & y_{12} = 2 & y_{13} = 3 \\ y_{21} = 4 & y_{22} = 12 & y_{23} = 18 \\ y_{31} = 17 & y_{32} = 46 & y_{33} = 69 \\ y_{41} = 4 & y_{42} = 10 & y_{43} = 17. \end{array}$$

By (13) in Chapter 6, the state prices $p_1, p_2, p_3, p_4$ for this model satisfy the system

$$\begin{aligned} 1p_1 + 4p_2 + 17p_3 + 4p_4 &= 38 \\ 2p_1 + 12p_2 + 46p_3 + 10p_4 &= 98 \\ 3p_1 + 18p_2 + 69p_3 + 17p_4 &= 153. \end{aligned}$$

Its augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 4 & 17 & 4 & 38 \\ 2 & 12 & 46 & 10 & 98 \\ 3 & 18 & 69 & 17 & 153 \end{array}\right),$$

with corresponding row echelon form

$$\left(\begin{array}{cccc|c} 1 & 4 & 17 & 4 & 38 \\ 0 & 4 & 12 & 2 & 22 \\ 0 & 0 & 0 & 2 & 6 \end{array}\right).$$

Divide the second row by 4 and the third row by 2 to obtain

$$\left(\begin{array}{cccc|c} 1 & 4 & 17 & 4 & 38 \\ 0 & 1 & 3 & 0.5 & 5.5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

Work first with the pivot in the third row to change column 4 from

$$\begin{pmatrix} 4 \\ 0.5 \\ 1 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

by adding $-0.5$ times row 3 to row 2 and then add $-4$ times row 3 to row 1:

$$\left(\begin{array}{cccc|c} 1 & 4 & 17 & 0 & 26 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

Now working with the pivot in row 2 and column 2, add $-4$ times row 2 to row 1 to change the 4 in row 1 to a 0:

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 10 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

(Note that because the new column 4 contains only 0s in the appropriate places, it was not affected by our work on column 2.) This matrix is now in reduced row echelon form. One can read off the final solution of the linear system from the reduced row echelon form matrix. For example, the linear system corresponding to the previous matrix is

$$\begin{aligned} p_1 \quad + 5p_3 \quad &= 10 \\ p_2 + 3p_3 \quad &= 4 \\ p_4 &= 3, \end{aligned}$$

which we can rewrite as

$$\begin{aligned} p_1 &= 10 - 5p_3 \\ p_2 &= 4 - 3p_3 \\ p_4 &= 3. \end{aligned}$$

Observe that $p_4$ is unambiguously determined, but not the other variables. The variable $p_3$ is free to take on any value. Once a value for $p_3$ has been selected, the values of variables $p_1$ and $p_2$ are determined by the above equations. This is another system, like system (17), with infinitely many solutions, and all these solutions can be read right off the reduced row echelon matrix.

For example, if we choose $p_3 = 1$, we obtain the price system

$$p_1 = 5, \quad p_2 = 1, \quad p_3 = 1, \quad p_4 = 3.$$

If we choose $p_3 = 0.5$, we obtain the price system

$$p_1 = 7.5, \quad p_2 = 2.5, \quad p_3 = 0.5, \quad p_4 = 3.$$

As a final example, consider the following schematic matrix in which the stars ($*$) represent nonzero pivots and the $w$'s may be either zero or nonzero:

$$\begin{pmatrix} * & w & w & w & w & w & w & | & w \\ 0 & 0 & 0 & * & w & w & w & | & w \\ 0 & 0 & 0 & 0 & * & w & w & | & w \\ 0 & 0 & 0 & 0 & 0 & 0 & * & | & w \end{pmatrix}$$

This matrix is in row echelon form. The corresponding reduced row echelon form is

$$\begin{pmatrix} 1 & w & w & 0 & 0 & w & 0 & | & w \\ 0 & 0 & 0 & 1 & 0 & w & 0 & | & w \\ 0 & 0 & 0 & 0 & 1 & w & 0 & | & w \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & | & w \end{pmatrix}$$

The final solution will have the form

$$\begin{aligned} x_1 &= a_1 - a_2x_2 - a_3x_3 - a_4x_6, \\ x_4 &= b_1 \qquad\qquad\qquad\quad - b_2x_6, \\ x_5 &= c_1 \qquad\qquad\qquad\quad - c_2x_6, \\ x_7 &= d_1. \end{aligned}$$

Here $x_7$ is the only variable which is unambiguously determined. The variables $x_2$, $x_3$, and $x_6$ are free to take on any values; once values have been selected for these three variables, then values for $x_1$, $x_4$, and $x_5$ are automatically determined.

Some more vocabulary is helpful here. If the $j$th column of the row echelon matrix $\hat{B}$ contains a pivot, we call $x_j$ a **basic variable.** If the $j$th column of $\hat{B}$ does not contain a pivot, we call $x_j$ a **free** or **nonbasic variable.** In this terminology, Gauss-Jordan elimination determines a solution of the system in which each basic variable is either unambiguously determined or a linear expression of the free variables. The free variables are free to take on any value. Once one chooses values for the free variables, values for the basic variables are determined.

As in the example above, the free variables are often placed on the right-hand side of the equations to emphasize that their values are not determined by the system; rather, they act as parameters in determining values for the basic variables.

In a given problem which variables are free and which are basic may depend on the order of the operations used in the Gaussian elimination process and on the order in which the variables are indexed.

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## EXERCISES

**7.13** Reduce the following matrices to row echelon and reduced row echelon forms:

*a)* $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$, *b)* $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{pmatrix}$, *c)* $\begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$

**7.14** Solve the system of equations $\begin{cases} -4x + 6y + 4z = 4 \\ 2x - y + z = 1. \end{cases}$

**7.15** Use Gauss-Jordan elimination to determine for what values of the parameter $k$ the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - kx_2 &= 1 \end{aligned}$$

has no solutions, one solution, and more than one solution.

**7.16** Use Gauss-Jordan elimination to solve the following four systems of linear equations. Which variables are free and which are basic in each solution?

$$a)\quad \begin{aligned} w + 2x + y - z &= 1 \\ 3w - x - y + 2z &= 3 \\ -x + y - z &= 1 \\ 2w + 3x + 3y - 3z &= 3; \end{aligned} \qquad b)\quad \begin{aligned} w - x + 3y - z &= 0 \\ w + 4x - y + z &= 3 \\ 3w + 7x + y + z &= 6 \\ 3w + 2x + 5y - z &= 3; \end{aligned}$$

$$c)\quad \begin{aligned} w + 2x + 3y - z &= 1 \\ -w + x + 2y + 3z &= 2 \\ 3w - x + y + 2z &= 2 \\ 2w + 3x - y + z &= 1; \end{aligned} \qquad d)\quad \begin{aligned} w + x - y + 2z &= 3 \\ 2w + 2x - 2y + 4z &= 6 \\ -3w - 3x + 3y - 6z &= -9 \\ -2w - 2x + 2y - 4z &= -6. \end{aligned}$$

**7.17** *a*) Use the flexibility of the free variable to find *positive integers* which satisfy the system

$$\begin{aligned} x + y + z &= 13 \\ x + 5y + 10z &= 61. \end{aligned}$$

*b*) Suppose you hand a cashier a dollar bill for a 6-cent piece of candy and receive 16 coins as your change — all pennies, nickels, and dimes. How many coins of each type do you receive? [Hint: See part *a*.]

**7.18** For what values of the parameter $a$ does the following system of equations have a solution?

$$\begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= a. \end{aligned}$$

**7.19** From Chapter 6, the stationary distribution in the Markov model of unemployment satisfies the linear system

$$\begin{aligned} (q - 1)x + py &= 0 \\ (1 - q)x - py &= 0 \\ x + y &= 1. \end{aligned}$$

*a*) If $p$ and $q$ lie between 0 and 1, how many solutions does this system have? Why?

*b*) Ignoring the condition that $p$ and $q$ must be between and 0 and 1, find values of $p$ and $q$ so that this system has no solutions.

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## 7.4 RANK—THE FUNDAMENTAL CRITERION

We now answer the five basic questions about existence and uniqueness of solutions that were posed in Section 7.3. The main criterion involved in the answers to these questions is the rank of a matrix. First, note that we say a row of a matrix is nonzero if and only if it contains at least one nonzero entry.

**Definition** The **rank** of a matrix is the number of nonzero rows in its row echelon form.

Since we can reduce any matrix to several different row echelon matrices (if we interchange rows), we need to show that this definition of rank is independent of which row echelon matrix we compute. We will save this for Chapter 27, where we will also discuss the rank of a matrix from a different, more geometric point of view.

Let $A$ and $\hat{A}$ be the coefficient matrix and augmented matrix respectively of a system of linear equations. Let $B$ and $\hat{B}$ be their corresponding row echelon forms. One goes through the same steps in reducing $A$ to $B$ as in reducing $\hat{A}$ to $\hat{B}$ no matter what the last column of $\hat{A}$ is, because the choices of elementary row operations in going from $\hat{A}$ to $\hat{B}$ never involve the last column of the augmented matrix. In other words, $\hat{B}$ is itself an augmented matrix for $B$.

We first relate the rank of a coefficient matrix $A$ to the rank of a corresponding augmented matrix and to the number of rows and columns of $A$. Note that the rank of the augmented matrix must be at least as big as the rank of the coefficient matrix because if a row in the augmented matrix contains only zeros, then so does the corresponding row of the coefficient matrix. Furthermore, the definition of rank requires that the rank is less than or equal to the number of rows of the coefficient matrix. Since each nonzero row in the row echelon form contains exactly one pivot, the rank is equal to the number of pivots. Since each column of $A$ can have at most one pivot, the rank is also less than or equal to the number of columns of the coefficient matrix. Fact 7.1 summarizes the observations in this paragraph.

**Fact 7.1.** Let $A$ be the coefficient matrix and let $\hat{A}$ be the corresponding augmented matrix. Then,

(*a*) $\text{rank}A \leq \text{rank}\hat{A}$,
(*b*) $\text{rank}A \leq$ number of rows of $A$, and
(*c*) $\text{rank}A \leq$ number of columns of $A$.

The following fact relates the ranks of $A$ and of $\hat{A}$ to the existence of a solution of the system in question and gives us our first answer to Question 1 above.

**Fact 7.2.** A system of linear equations with coefficient matrix $A$ and augmented matrix $\hat{A}$ has a solution if and only if

$$\text{rank}\hat{A} = \text{rank}A.$$

*Proof* The proof of this statement follows easily from a careful consideration of the row echelon form $\hat{B}$ of $\hat{A}$. If rank$\hat{A}$ > rank$A$, then there is a zero row in the row echelon coefficient matrix $B$ which corresponds to a nonzero row in the corresponding row echelon augmented matrix $\hat{B}$. This translates into the equation

$$0x_1 + 0x_2 + \cdots + 0x_n = b' \tag{21}$$

with $b'$ nonzero. Consequently, the row echelon system has no solution and therefore the original system has no solution.

On the other hand, if the row echelon form of the augmented matrix contains no row corresponding to equation (21), that is, if rank$A$ = rank$\hat{A}$, then there is nothing to stop Gauss-Jordan elimination from finding a general solution to the original system. As the discussion in the last section indicates, one can easily read off the solution directly from the reduced row echelon form. Some basic variables will be uniquely determined; others will be linear expressions of the free variables. ■

If a system with a solution has free variables, then these variables can take on any value in the general solution of the system. Consequently, the original system has infinitely many solutions. If there are no free variables, then every variable is a basic variable. In this case, Gaussian or Gauss-Jordan elimination determines a unique value for every variable; that is, there is only one solution to the system. We can summarize these observations.

**Fact 7.3.** A linear system of equations must have either no solution, one solution, or infinitely many solutions. Thus, if a system has more than one solution, it has infinitely many.

Let us look carefully at the case where there are no free variables in the system under study. Since every variable must be a basic variable, each column contains exactly one pivot. Since each nonzero row contains a pivot too, there must be at least as many rows as columns. (There may be some all-zero rows at the bottom of the row echelon matrix.) This proves:

**Fact 7.4.** If a system has exactly one solution, then the coefficient matrix $A$ has at least as many rows as columns. In other words, a system with a unique solution must have at least as many equations as unknowns.

Fact 7.4 can be expressed another way.

**Fact 7.5.** If a system of linear equations has more unknowns than equations, it must have either no solution or infinitely many solutions.

Consider a system in which all the $b_j$'s on the right-hand side are 0:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Such a system is called **homogeneous.** As we shall see later, homogeneous systems play an especially important role in the study of linear equations. Any homogeneous system has at least one solution:

$$x_1 = x_2 = \cdots = x_n = 0.$$

The following statement is an immediate consequence of Fact 7.5.

**Fact 7.6.** A homogeneous system of linear equations which has more unknowns than equations must have infinitely many distinct solutions.

We now turn to the answers of Questions 3, 4, and 5 of the previous section. In many economic models, the $b_i$'s on the right-hand side of a system of linear equations can be considered as exogenous variables which vary from problem to problem. For each choice of $b_i$'s on the right-hand side, one solves the linear system to find the corresponding values of the endogenous variables $x_1, \ldots, x_n$. For example, in the input-output example in Chapter 6, for each choice of consumption amounts $c_1, \ldots, c_n, c_0$, one wants to compute the required outputs $x_1, \ldots, x_n$. In the linear IS-LM model of Chapter 6, for each choice of policy variables $G$ and $M_s$ and parameters $I^*$ and $M^*$, one wants to compute the corresponding equilibrium GNP $Y$ and interest rate $r$. Thus, it becomes especially important to understand what properties of a system will guarantee that it has at least one solution or, better yet, exactly one solution for *any* right-hand side (RHS) $b_1, b_2, \ldots, b_m$. Again the answers flow directly from a careful look at reduced row echelon matrices. First we answer Question 3.

**Fact 7.7.** A system of linear equations with coefficient matrix $A$ will have a solution for every choice of RHS $b_1, \ldots, b_m$ if and only if

$$\text{rank} A = \text{number of rows of} A.$$

*Proof* (If): If rank$A$ equals the number of rows of $A$, then the row echelon matrix $B$ of $A$ has no all-zero rows. Let $b_1, \ldots, b_m$ be a choice of RHS in system (2). Let $\hat{B}$ be the row echelon form of the corresponding augmented matrix. By the remarks at the beginning of this section, $\hat{B}$ is an augmented matrix for $B$, and hence $\hat{B}$ will have no all-zero rows either. Thus,

$$\operatorname{rank} A = \#\text{ rows of } A = \#\text{ rows of } \hat{A} = \operatorname{rank} \hat{A}.$$

By Fact 7.2, our system has a solution.

(Only If): If rank$A$ is less than the number of rows of $A$, then the last row, row $m$, in the row echelon matrix $B$ of $A$ will contain only zeros. Since $B$ is in row echelon form,

$$B = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Augment $B$ by a column of 1s to make $\hat{B}$:

$$\hat{B} = \left(\begin{array}{cccc|c} * & * & \cdots & * & 1 \\ 0 & * & \cdots & * & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

The system corresponding to $\hat{B}$ can have no solution because nothing satisfies the equation described by the last row of $\hat{B}$: $0 = 1$. Starting now with $\hat{B}$, reverse in turn each row operation that was applied in transforming $A$ to $B$. The result is an augmented matrix $\hat{A}$ whose coefficient matrix is our original matrix $A$. The systems of equations $\hat{A}$ and $\hat{B}$ are equivalent since one was obtained from the other by a sequence of row operations. Since the system corresponding to $\hat{B}$ has no solution, neither does the system corresponding to $\hat{A}$. Since $\hat{A}$ is an augmented matrix for $A$, we have found a right-hand side for which the system with coefficient matrix $A$ has no solution under the assumption that the rank of $A$ is less than the number of rows of $A$. This finishes the proof of Fact 7.7. ■

If a system of equations has fewer unknowns than equations, then the corresponding coefficient matrix has fewer columns than rows. Since the rank is less than or equal to the number of columns, which is less than the number of rows, Fact 7.7 ensures that there are RHSs for which the corresponding system has no solutions. We summarize this observation as Fact 7.8.

**Fact 7.8.** If a system of linear equations has more equations than unknowns, then there is a right-hand side such that the resulting system has *no* solutions.

Next we turn to Question 4 and state a condition that guarantees that our system will have at most one solution, that is, will never have infinitely many solutions, for *any* choice of RHS $b_1, \ldots, b_m$.

**Fact 7.9.** Any system of linear equations having $A$ as its coefficient matrix will have at most one solution for every choice of RHS $b_1, \ldots, b_m$ if and only if

$$\operatorname{rank} A = \text{number of columns of } A.$$

*Proof* (If): If rank$A$ equals the number of columns of $A$, then there are as many pivots in the reduced row echelon matrix $A''$ as there are columns in $A''$. Since each column can contain at most one pivot, there is a pivot in each column. So, every variable is a basic variable; there are no free variables. The reduced row echelon matrix $A''$ has the form

$$A'' = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If there is a solution for some given RHS $b_1, \ldots, b_m$, it will be unambiguously determined by $A''$; that is, the solution will be unique.

(Only If): On the other hand, if the rank is less than the number of columns, then there must be some free variables. Choose a RHS so that the system has a solution, for example, $b_1 = \cdots = b_m = 0$. Because the free variables can take on any value in solutions (as shown in the previous section), there are infinitely many solutions to the system. This proves the second half of Fact 7.9. ∎

Finally, we combine Facts 7.7 and 7.9 to characterize those coefficient matrices which have the property that for *any* RHS $b_1, \ldots, b_m$, the corresponding system of linear equations has exactly one solution. Such coefficient matrices are called **nonsingular**. They are the ones which will arise most frequently in our study of linear systems and other linear phenomena.

**Fact 7.10.** A coefficient matrix $A$ is nonsingular, that is, the corresponding linear system has one and only one solution for every choice of right-hand side $b_1, \ldots, b_m$, if and only if

$$\text{number of rows of } A = \text{number of columns of } A = \operatorname{rank} A.$$

Fact 7.10 is a straightforward consequence of Facts 7.7 and 7.9. It tells us that a necessary condition for a system to have a unique solution for every RHS is that there be exactly as many equations as unknowns. The corresponding coefficient matrix must have the same number of rows as columns. Such a matrix is called a **square matrix**.

The problem of determining whether a square matrix has **maximal rank** (that is, rank as in Fact 7.10) is a central one in linear algebra. Fortunately, there is an easily computed number which one can assign to any square matrix which determines whether or not this rank condition holds. This number is called the **determinant** of the matrix; it will be the subject of our discussion in Chapters 9 and 26.

Finally, Fact 7.11 summarizes our findings in this section for a system of $m$ equations in $n$ unknowns — a system whose coefficient matrix has $m$ rows and $n$ columns.

**Fact 7.11.** Consider the linear system of equations $A\mathbf{x} = \mathbf{b}$.

(*a*) If the number of equations < the number of unknowns, then:
  (*i*) $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions,
  (*ii*) for any given $\mathbf{b}, A\mathbf{x} = \mathbf{b}$ has 0 or infinitely many solutions, and
  (*iii*) if rank $A$ = number of equations, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for *every* RHS $\mathbf{b}$.
(*b*) If the number of equations > the number of unknowns, then:
  (*i*) $A\mathbf{x} = \mathbf{0}$ has one or infinitely many solutions,
  (*ii*) for any given $\mathbf{b}, A\mathbf{x} = \mathbf{b}$ has 0, 1, or infinitely many solutions, and
  (*iii*) if rank $A$ = number of unknowns, $A\mathbf{x} = \mathbf{b}$ has 0 or 1 solution for *every* RHS $\mathbf{b}$.
(*c*) If the number of equations = the number of unknowns, then:
  (*i*) $A\mathbf{x} = \mathbf{0}$ has one or infinitely many solutions,
  (*ii*) for any given $\mathbf{b}, A\mathbf{x} = \mathbf{b}$ has 0, 1, or infinitely many solutions, and
  (*iii*) if rank $A$ = number of unknowns = number of equations, $A\mathbf{x} = \mathbf{b}$ has exactly 1 solution for *every* RHS $\mathbf{b}$.

## Application to Portfolio Theory

We return to our discussion of investment in Example 5 of Section 6.2. There we called an $A$–tuple $(x_1, \ldots, x_A)$ such that $x_1 + \cdots + x_A = 1$ a **portfolio**, where $x_i$ denotes the fraction of the investor's wealth to be spent on asset $i$.

Suppose that there are $S$ states of nature and that $R_{si}$ denotes the return at the end of the investment period to a unit of asset $i$ when the period is characterized by state $s$. The return to portfolio $\mathbf{x}$ in state $s$ is $R_s = \sum_{i=1}^{A} R_{si}x_i$. A portfolio is called **riskless** if it provides the same return in every state of nature:

$$\sum_{i=1}^{A} R_{1i}x_i = \sum_{i=1}^{A} R_{2i}x_i = \cdots = \sum_{i=1}^{A} R_{Si}x_i.$$

A portfolio $(x_1, \ldots, x_A)$ is called **duplicable** if there is a different portfolio $(w_1, \ldots, w_A)$ with exactly the same returns in every state:

$$\sum_{i=1}^{A} R_{si}x_i = \sum_{i=1}^{A} R_{si}w_i \qquad \text{for each } s = 1, \ldots, S.$$

A state $s^*$ is called **insurable** if there is a portfolio $(x_1, \ldots, x_A)$ which has a positive return if state $s^*$ occurs and zero return if any other state occurs:

$$\sum_{i=1}^{A} R_{s^*i} x_i > 0$$

$$\sum_{i=1}^{A} R_{si} x_i = 0 \qquad \text{for all } s \neq s^*.$$

For any portfolio **x**, the return to **x** in each state is given by the $S$–tuple $(R_1, \ldots, R_S)$, where

$$\begin{matrix} R_{11}\,x_1 + & \cdots & + R_{1A}\,x_A & = R_1 \\ \vdots & \vdots & & \vdots \\ R_{S1}\,x_1 + & \cdots & + R_{SA}\,x_A & = R_S. \end{matrix} \tag{22}$$

Let $\mathcal{R}$ be the $S \times A$ coefficient matrix of the $R_{si}$'s:

$$\mathcal{R} = \begin{pmatrix} R_{11} & \cdots & R_{1A} \\ \vdots & \ddots & \vdots \\ R_{S1} & \cdots & R_{SA} \end{pmatrix}.$$

Suppose first that the matrix $\mathcal{R}$ has rank $S$ = the number of rows of $\mathcal{R}$. Then, by Fact 7.7, one can solve system (22) for any given $S$–tuple $(R_1, \ldots, R_S)$ of returns. In particular, if we take $R_1 = \cdots = R_S = b$ for some $b \neq 0$, the solution to (22), when properly normalized so that $x_1 + \cdots + x_A = 1$, will be a riskless asset. If we set $R_k = 1$ and $R_i = 0$ for $i \neq k$, the solution to (22), when properly normalized, will be an insurance portfolio for state $k$. So, if the rank of $\mathcal{R} = S$, then there is a riskless asset and every state is insurable.

We will argue in Section 28.2 that the converse holds too. If every state is insurable, then $\mathcal{R}$ must have rank $S$. In particular, if $A < S$, that is, if there are more states of nature than assets, then $\mathcal{R}$ cannot have rank $S$ and there must exist states that are not insurable.

Finally, there are duplicable portfolios if and only if equation (22) has multiple portfolio solutions for some right-hand sides. This situation occurs only if system (22) has free variables, that is, only if the rank of $\mathcal{R}$ is less than $A$.

*Example 7.6* In Example 6.1, we worked with the $3 \times 2$ state–return matrix

$$\mathcal{R} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix},$$

which has 2 columns and rank 2. We use Gaussian elimination to transform $\mathcal{R}$ to its row echelon form:

$$\begin{pmatrix} 1 & 3 & | & a \\ 2 & 2 & | & b \\ 3 & 1 & | & c \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 3 & | & a \\ 0 & -4 & | & b-2a \\ 0 & 0 & | & a+c-2b \end{pmatrix}.$$

Since $a + c - 2b = 0$ if $a = b = c = 1$, this market has a riskless asset. Since $a + c - 2b \neq 0$ if $(a, b, c)$ has exactly one nonzero component, there are no insurable states. Since $\mathcal{R}$ has no free variables, there are no duplicable portfolios.

## EXERCISES

**7.20** Compute the rank of each of the following matrices:

$$a)\ \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \quad b)\ \begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad c)\ \begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix},$$

$$d)\ \begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 1 & 9 & -6 & 4 & 9 \\ 1 & 3 & -8 & 4 & 2 \\ 2 & 15 & -13 & 11 & 16 \end{pmatrix}, \quad e)\ \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix}.$$

**7.21** The following five matrices are coefficient matrices of systems of linear equations. For each matrix, what can you say about the number of solutions of the corresponding system: *a*) when the right-hand side is $b_1 = \cdots = b_m = 0$, and *b*) for general RHS $b_1, \ldots, b_m$?

$$i)\ \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}, \quad ii)\ \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix}, \quad iii)\ \begin{pmatrix} 2 & 1 \\ 1 & 4 \\ 0 & 3 \end{pmatrix},$$

$$iv)\ \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad v)\ \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 0 & 7 & 6 \end{pmatrix}.$$

**7.22** Repeat Exercise 7.21 for the five matrices in Exercise 7.20.

**7.23** Which coefficient matrix in Exercise 7.16 satisfies the conditions of Fact 7.10, that is, is nonsingular?

**7.24** Show that a square matrix $A$ is nonsingular if and only if its row echelon forms have no zeros on the diagonal.

## 7.5 THE LINEAR IMPLICIT FUNCTION THEOREM

The situation described in Fact 7.10 arises frequently in mathematical models, as we discussed last section. The $b_i$'s on the RHS of (2) represent some externally determined parameters, while the linear equations themselves represent some equilibrium condition which determines the internal variables $x_1, \ldots, x_n$. Ideally, there should be a unique equilibrium for each choice of the parameters $b_1, \ldots, b_m$. Fact 7.10 tells us exactly when this ideal situation occurs: the number of equations must equal the number of unknowns and the coefficient matrix must have maximal rank.

In this view, consider once again the IS-LM model described in Chapter 6:

$$\begin{aligned} sY + ar &= I^* + G \\ mY - hr &= M_s - M^* \end{aligned} \tag{23}$$

Choose numerical values for the parameters $s, a, m, h, I^*, G$, and $M^*$ in system (23). However, think of $M_s$, the money supply, as a variable policy parameter which a policymaker can set externally. For each choice of money supply, the economy reaches an equilibrium in $Y$ and $r$. Since we have two equations in two unknowns, Fact 7.10 tells us that system (23) will indeed determine a unique $(Y, r)$ pair for each choice of $M_s$ provided the coefficient matrix

$$\begin{pmatrix} s & a \\ m & -h \end{pmatrix}$$

has rank two.

In this IS-LM model the variables $Y$ and $r$ are called **endogenous variables** because their values are determined by the system of equations under consideration. On the other hand, $M_s$ is called an **exogenous variable** because its value is determined outside of system (23). If we were to treat $s, a, m, h, I^*, G$, and $M^*$ as parameters also, then they too would be exogenous variables. Mathematicians would call exogenous variables **independent variables** and endogenous variables **dependent variables.**

A general linear model will have $m$ equations in $n$ unknowns:

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 + & \cdots & + a_{1n}x_n & = & b_1 \\ \vdots & & & \vdots & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 + & \cdots & + a_{mn}x_n & = & b_m. \end{array} \tag{24}$$

Usually there will be a natural division of the $x_i$'s into exogenous and endogenous variables given by the model. This division will be successful only if, after choosing values for the exogenous variables and plugging them into system (24), one can then unambiguously solve the system for the rest of the variables, the endogenous ones. Fact 7.10 in the last section tells us the two conditions that must hold in order for this breakdown into exogenous variables and endogenous variables to be successful. There must be exactly as many endogenous variables as there are equations in (24), and the square matrix corresponding to the endogenous variables must have maximal rank $m$. This statement is a version of the Implicit Function Theorem for linear equations, and is summarized in the following theorem.

**Theorem 7.1** Let $x_1, \ldots, x_k$ and $x_{k+1}, \ldots, x_n$ be a partition of the $n$ variables in (24) into endogenous and exogenous variables, respectively. There is, for each choice of values $x_{k+1}^0, \ldots, x_n^0$ for the exogenous variables, a unique set of values $x_1^0, \ldots, x_k^0$ which solves (24) if and only if:

($a$) $k = m$ (number of endogenous variables = number of equations) and
($b$) the rank of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix},$$

corresponding to the endogenous variables, is $k$.

Under the conditions of Theorem 7.1, we can think of system (24) as implicitly presenting each of the endogenous variables as functions of all the exogenous variables. Later, we will strengthen this result and use it as motivation for the Implicit Function Theorem for nonlinear systems of equations — a result which will be the cornerstone of our treatment of nonlinear equations, especially applied to comparative statics in economic models.

## EXERCISES

**7.25** For each of the following two systems, we want to separate the variables into exogenous and endogenous ones so that each choice of values for the exogenous variables determines unique values for the endogenous variables. For each system, $a$) determine how many variables can be endogenous at any one time, $b$) determine a successful separation into exogenous and endogenous variables, and $c$) find an explicit formula for the endogenous variables in terms of the exogenous ones:

$$i)\quad \begin{aligned} x + 2y + z - w &= 1 \\ 3x + 6y - z - 3w &= 2; \end{aligned} \qquad ii)\quad \begin{aligned} x + 2y + z - w &= 1 \\ 3x - y - 4z + 2w &= 3 \\ y + z + w &= 0. \end{aligned}$$

**7.26** For Example 1 in Chapter 6, write out the linear system which corresponds to equation (1) in Chapter 6 but with the \$100,000 before-tax profit replaced by a general before-tax profit $P$. Solve the resulting system for $C$, $S$, and $F$ in terms of $P$.

**7.27** For the values of the constants in Exercise 6.7, show that each choice of $M_s$ uniquely determines an equilibrium $(Y, r)$.

**7.28** *a*) In IS-LM model (23), use Gaussian elimination to find a general formula involving $s, a, m$, and $h$ which, when satisfied, will guarantee that system (23) determines a unique value of $Y$ and $r$ for each choice of $I^*$, $M^*$, $G$, and $M_s$.

*b*) In this case, find an explicit formula for $Y$ and $r$ in terms of all the other variables.

*c*) Note how changes in each of the exogenous variables affect the values of $Y$ and $r$.

**7.29** Consider the system

$$\begin{aligned} w - x + 3y - z &= 0 \\ w + 4x - y + 2z &= 3 \\ 3w + 7x + y + z &= 6. \end{aligned}$$

*a*) Separate the variables into endogenous and exogenous ones so that each choice of the exogenous variables uniquely determines values for the endogenous ones.

*b*) For your answer to *a*, what are the values of the endogenous variables when all the exogenous variables are set equal to 0?

*c*) Find a separation into endogenous and exogenous variables (same number of each as in part *a*) that will not work in the sense of *a*. Find a value of the new exogenous variables for which there are infinitely many corresponding values of the endogenous variables.

**7.30** Consider the system

$$\begin{aligned} w - x + 3y - z &= 0 \\ w + 4x - y + z &= 3 \\ 3w + 7x + y + z &= 6. \end{aligned}$$

Is there any successful decomposition into endogenous and exogenous variables? Explain.

# Matrix Algebra

Matrices were introduced in the previous chapter to organize our calculations for solving systems of linear equations. Matrices play an important role in many other areas of economics and applied mathematics. The *input-output matrix* of Example 2 in Chapter 6, and the *Markov matrix* of Example 3 are but two examples. Other examples include *payoff matrices* from the theory of games, *coefficient matrices* and *correlation matrices* from econometrics, *Slutsky* and *Antonelli matrices* from consumer theory, and the *Hessian* and *bordered Hessian* matrices that embody the second order conditions in multivariable optimization theory.

A **matrix** is simply a rectangular array of numbers. So, any table of data is a matrix. The size of a matrix is indicated by the number of its rows and the number of its columns. A matrix with $k$ rows and $n$ columns is called a $k \times n$ ("k by n") matrix. The number in row $i$ and column $j$ is called the $(i, j)$th entry, and is often written $a_{ij}$, as we did in Chapter 7. Two matrices are *equal* if they both have the same size and if the corresponding entries in the two matrices are equal.

Matrices are in a sense generalized numbers. When the sizes are right, two matrices can be added, subtracted, multiplied and even divided. Whenever an economic model uses matrices, we can learn a lot about the underlying model via these algebraic operations. In this chapter, we describe the algebra of matrices. This chapter is a bit more abstract than previous chapters since it focuses on algebraic operations and their properties. But we will use these operations throughout this book. We illustrate this use in Section 8.5 where we derive the basic property of Leontief input-output models.

## 8.1 MATRIX ALGEBRA

### Addition

We begin with addition of matrices. One can add two matrices of the same size, which is to say, with the same number of rows and columns. Their sum is a new matrix of the same size as the two matrices being added. The $(i, j)$th entry of the sum matrix is simply the sum of the $(i, j)$th entries of the two matrices being added.

In symbols

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & b_{ij} & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & a_{ij} + b_{ij} & \vdots \\ a_{k1} + b_{k1} & \cdots & a_{kn} + b_{kn} \end{pmatrix}$$

For example,

$$\begin{pmatrix} 3 & 4 & 1 \\ 6 & 7 & 0 \\ -1 & 3 & 8 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 7 \\ 6 & 5 & 1 \\ -1 & 7 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 8 \\ 12 & 12 & 1 \\ -2 & 10 & 8 \end{pmatrix}$$

but

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$$

is not defined.

The matrix **0** whose entries are all zero is an *additive identity* since

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} + \begin{pmatrix} 0_{11} & \cdots & 0_{1n} \\ \vdots & 0_{ij} & \vdots \\ 0_{k1} & \cdots & 0_{kn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix},$$

that is, $A + \mathbf{0} = A$ for all matrices $A$.

## Subtraction

Since $-A$ is what one adds to $A$ to obtain **0**,

$$-\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} = \begin{pmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & -a_{ij} & \vdots \\ -a_{k1} & \cdots & -a_{kn} \end{pmatrix}.$$

Since $A - B$ is just shorthand for $A + (-B)$, we *subtract* matrices of the same size simply by subtracting their corresponding entries:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} - \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & b_{ij} & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - b_{11} & \cdots & a_{1n} - b_{1n} \\ \vdots & a_{ij} - b_{ij} & \vdots \\ a_{k1} - b_{kn} & \cdots & a_{kn} - b_{kn} \end{pmatrix}$$

## Scalar Multiplication

Matrices can be multiplied by ordinary numbers, which we also call **scalars**. This operation is called **scalar multiplication**. Implicitly we have already used this operation in defining $-A$, which is $(-1)A$. More generally, the product of the matrix $A$ and the number $r$, denoted $rA$, is the matrix created by multiplying each entry of $A$ by $r$.

$$r\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} = \begin{pmatrix} ra_{11} & \cdots & ra_{1n} \\ \vdots & ra_{ij} & \vdots \\ ra_{k1} & \cdots & ra_{kn} \end{pmatrix}.$$

In summary, within the class of $k \times n$ matrices, addition, subtraction, and scalar multiplication are all defined in the obvious way and act just as one would expect.

## Matrix Multiplication

Just as two numbers can be multiplied together, so can two matrices. But at this point matrix algebra becomes a little bit more complicated than the algebra for real numbers. There are two differences: Not all pairs of matrices can be multiplied together, and the order in which matrices are multiplied can matter.

We can define the matrix product $AB$ if and only if

*number of columns of A = number of rows of B.*

For the matrix product to exist, $A$ must be $k \times m$ and $B$ must be $m \times n$. To obtain the $(i, j)$th entry of $AB$, multiply the $i$th row of $A$ and the $j$th column of $B$ as follows:

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{im} \end{pmatrix} \cdot \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}.$$

In other words, the $(i, j)$th entry of the product $AB$ is defined to be

$$\sum_{h=1}^{m} a_{ih}b_{hj}.$$

For example,

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \\ eA + fC & eB + fD \end{pmatrix}$$

Note that in this case, the product taken in reverse order,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix},$$

is not defined. See Exercise 8.2.

If $A$ is $k \times m$ and $B$ is $m \times n$, then the product $AB$ will be $k \times n$. The product matrix $AB$ inherits the number of its rows from $A$ and the number of its columns from $B$:

$$\begin{aligned} \text{number of rows of } AB &= \text{number of rows of } A; \\ \text{number of columns of } AB &= \text{number of columns of } B; \\ (k \times m) \cdot (m \times n) &= (k \times n). \end{aligned}$$

The $n \times n$ matrix

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

with $a_{ii} = 1$ for all $i$ and $a_{ij} = 0$ for all $i \neq j$, has the property that for any $m \times n$ matrix $A$,

$$AI = A,$$

and for any $n \times l$ matrix $B$,

$$IB = B.$$

The matrix $I$ is called the $n \times n$ **identity matrix** because it is a multiplicative identity for matrices just as the number 1 is for real numbers.

### Laws of Matrix Algebra

We can think of matrices as generalized numbers because matrix addition, subtraction and multiplication obey most of the same laws that numbers do.

*Associative Laws:* $(A + B) + C = A + (B + C)$,

$$(AB)C = A(BC),$$

*Commutative Law for Addition:* $A + B = B + A$, *Distributive Laws:*

$$A(B + C) = AB + AC,$$
$$(A + B)C = AC + BC.$$

The one important law which numbers satisfy but matrices do not, is the *commutative law for multiplication.* Although $ab = ba$ for all numbers $a$ and $b$, it is not true that $AB = BA$ for matrices, *even when both products are defined.* We have already seen examples where only one product is defined. But notice that even if both products exist, they need not be the same size. For example, if $A$ is $2 \times 3$ and $B$ is $3 \times 2$, then $AB$ is $2 \times 2$ while $BA$ is $3 \times 3$. Even if $AB$ and $BA$ have the same size, $AB$ need not equal $BA$. For example,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix},$$

while

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}.$$

## Transpose

Finally, there is one other operation on matrices which we shall frequently use. The **transpose** of a $k \times n$ matrix $A$ is the $n \times k$ matrix obtained by interchanging the rows and columns of $A$. This matrix is often written as $A^T$. The first row of $A$ becomes the first column of $A^T$. The second row of $A$ becomes the second column of $A^T$, and so on. Thus, the $(i, j)$th entry of $A$ becomes the $(j, i)$th entry of $A^T$. For example,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix},$$

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \end{pmatrix}.$$

The following rules are fairly straightforward to verify:

$$(A + B)^T = A^T + B^T,$$
$$(A - B)^T = A^T - B^T,$$
$$(A^T)^T = A,$$
$$(rA)^T = rA^T,$$

where $A$ and $B$ are $k \times n$ and $r$ is a scalar. The following rule is not so obvious and takes a little work to prove:

$$(AB)^T = B^T A^T.$$

Note the change in the order of the matrix multiplication.

**Theorem 8.1** Let $A$ be a $k \times m$ matrix and $B$ be an $m \times n$ matrix. Then, $(AB)^T = B^T A^T$.

*Proof* We will be working with six different matrices: $A, B, A^T, B^T, (AB)^T$, and $B^T A^T$. For notation's sake, if $C$ is any of these matrices, we will write $C_{ij}$ for the $(i, j)$th element of $C$. For example, $((AB)^T)_{ij}$ will denote the $(i, j)$th element of the matrix $(AB)^T$. Now,

$$\begin{aligned}
((AB)^T)_{ij} &= (AB)_{ji} && \text{(definition of transpose)} \\
&= \sum_h A_{jh} \cdot B_{hi} && \text{(definition of matrix multiplication)} \\
&= \sum_h (A^T)_{hj} \cdot (B^T)_{ih} && \text{(definition of transpose, twice)} \\
&= \sum_h (B^T)_{ih} \cdot (A^T)_{hj} && (a \cdot b = b \cdot a \text{ for scalars}) \\
&= (B^T A^T)_{ij} && \text{(definition of matrix multiplication.)}
\end{aligned}$$

Therefore, $(AB)^T = B^T A^T$ ■

## Systems of Equations in Matrix Form

The algebra that we have developed so far is already very powerful. Consider the systems of linear equations from the previous chapter. The typical system looked like

$$\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad & \quad \vdots \\
a_{k1}x_1 + \cdots + a_{kn}x_n &= b_k.
\end{aligned}$$

This system can be expressed much more compactly using the notation suggested by matrix algebra. As before, let $A$ denote the coefficient matrix of the system:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}$$

Also, let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}.$$

Both $\mathbf{x}$ and $\mathbf{b}$ are matrices, called column matrices. The $n \times 1$ matrix $\mathbf{x}$ contains variables, and the $k \times 1$ matrix $\mathbf{b}$ contains the parameters from the right-hand side of the system. Then, the system of equations can be written as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix},$$

or simply as

$$A\mathbf{x} = \mathbf{b},$$

where $A\mathbf{x}$ refers to the matrix product of the $k \times n$ matrix $A$ with the $n \times 1$ matrix $\mathbf{x}$. This product is a $k \times 1$ matrix, which must be made equal to the $k \times 1$ matrix $\mathbf{b}$. Check that carrying out the matrix multiplication in $A\mathbf{x} = \mathbf{b}$ and applying the definition of equality of matrices gives back exactly the original system of linear equations. The matrix notation is much more compact than writing out arrays of coefficients, and, as we shall see, it suggests how to find the solution to the system by analogy with the one-variable case.

## EXERCISES

**8.1** Let

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix},$$

$$D = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad E = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

*a*) Compute each of the following matrices if it is defined:

| | | | | | |
|---|---|---|---|---|---|
| $A + B$, | $A - D$, | $3B$, | $DC$, | $B^T$, | $A^TC^T$, |
| $C + D$, | $B - A$ | $AB$, | $CE$, | $-D$, | $(CE)^T$, |
| $B + C$, | $D - C$, | $CA$, | $EC$, | $(CA)^T$, | $E^TC^T$. |

*b*) Verify that $(DA)^T = A^TD^T$.

*c*) Verify that $CD \neq DC$.

**8.2** Check that

$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 5 & 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 2 & 3 \\ 10 & 21 \end{pmatrix}.$$

Note that the reverse product is not defined.

**8.3** Show that if $AB$ is defined, then $B^T A^T$ is defined but $A^T B^T$ need not be defined.

**8.4** If you choose four numbers at random for the entries of a $2 \times 2$ matrix $A$, and four others for another $2 \times 2$ matrix $B$, $AB$ will probably not equal $BA$. Carry out this procedure a few times.

**8.5** It sometimes happens that $AB = BA$.

*a*) Check this for $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$.

*b*) Show that if $B$ is a scalar multiple of the $2 \times 2$ identity matrix, then $AB = BA$ for *all* $2 \times 2$ matrices $A$.

## 8.2 SPECIAL KINDS OF MATRICES

Special problems use special kinds of matrices. In this section we describe some of the important classes of $k \times n$ matrices which arise in economic analysis.

**Square Matrix.** $k = n$, that is, equal number of rows and columns.

**Column Matrix.** $n = 1$, that is, one column. For example,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

**Row Matrix.** $k = 1$, that is, one row. For example,

$$(2 \quad 1 \quad 0) \quad \text{and} \quad (2 \quad 3).$$

**Diagonal Matrix.** $k = n$ and $a_{ij} = 0$ for $i \neq j$, that is, a square matrix in which all nondiagonal entries are 0. For example,

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

**Upper-Triangular Matrix.** $a_{ij} = 0$ if $i > j$, that is, a matrix (usually square) in which all entries below the diagonal are 0. For example,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

**Lower-Triangular Matrix.** $a_{ij} = 0$ if $i < j$, that is, a matrix (usually square) in which all entries above the diagonal are 0. For example,

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$$

**Symmetric Matrix.** $A^T = A$, that is, $a_{ij} = a_{ji}$ for all $i, j$. These matrices are necessarily square. For example,

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}.$$

**Idempotent Matrix.** A square matrix $B$ for which $B \cdot B = B$, such as $B = I$ or

$$\begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}.$$

**Permutation Matrix.** A square matrix of 0s and 1s in which each row and each column contains exactly one 1. For example,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Nonsingular Matrix.** A square matrix whose rank equals the number of its rows (or columns). When such a matrix arises as a coefficient matrix in a system of linear equations, the system has one and only one solution.

## EXERCISES

**8.6** Give an example with more than two rows or more than two columns of each of the above types of matrices.

**8.7** Show that $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$ are idempotent.

**8.8** Let $D$, $U$, $L$, and $S$ denote, respectively, the sets of all diagonal, upper-triangular, lower-triangular, and symmetric matrices.

*a*) Show that $D$, $U$ and $L$ are each closed under matrix addition and multiplication, that is, that the sum or product of two matrices in one of the above sets is also a matrix in that set.

*b*) Show that $D \cap U = D$, $S \cap U = D$, and $D \subset S$.

*c*) Show that all matrices in $D$ commute with each other. Is this true for matrices in $U$ or $S$, too?

*d*) Show that $S$ is closed under addition but not under multiplication.

**8.9** How many $n \times n$ permutation matrices are there?

**8.10** Is the set of $n \times n$ permutation matrices closed under addition or under matrix multiplication?

## 8.3 ELEMENTARY MATRICES

Another important class of matrices is the class of **elementary matrices**. Recall that the three elementary row operations that are used to bring a matrix to row echelon form are:

(1) interchanging rows,
(2) adding a multiple of one row to another, and
(3) multiplying a row by a nonzero scalar.

These operations can be performed on a matrix $A$ by premultiplying $A$ by certain special matrices called *elementary matrices*. For example, the following theorem illustrates how to interchange rows $i$ and $j$ of a given matrix $A$.

**Theorem 8.2** Form the permutation matrix $E_{ij}$ by interchanging the $i$th and $j$th rows of the identity matrix $I$. Left-multiplication of a given matrix $A$ by $E_{ij}$ has the effect of interchanging the $i$th and $j$th rows of $A$.

*Proof* To see this, let $e_{hk}$ denote a generic element of $E_{ij}$:

$$\begin{aligned} e_{ij} &= e_{ji} = 1, \\ e_{ii} &= e_{jj} = 0, \\ e_{hh} &= 1 \qquad \text{if } h \neq i, j, \\ e_{hk} &= 0 \qquad \text{otherwise.} \end{aligned} \tag{1}$$

The element in row $k$ and column $n$ of $E_{ij}A$ is

$$\sum_m e_{km}a_{mn} = \begin{cases} a_{jn} & k = i, \\ a_{in} & k = j, \\ a_{kn} & k \neq i, j, \end{cases}$$

using (1). Therefore, $E_{ij}A$ is simply $A$ with rows $i$ and $j$ interchanged. ∎

To carry out Row Operation 3, the multiplication of row $i$ by the scalar $r \neq 0$, construct the matrix $E_i(r)$ by multiplying the $i$th row of the identity matrix $I$ by the scalar $r$. The effect of premultiplication of $A$ by $E_i(r)$ is to multiply each entry of the $i$th row of $A$ by $r$. For example, in the case of the general $3 \times 3$ matrix $A$,

$$E_2(5) \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 5a_{21} & 5a_{22} & 5a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Finally, to perform Row Operation 2, the addition of $r$ times the $i$th row of $A$ to the $j$th row of $A$, form the matrix $E_{ij}(r)$ by adding $r$ times row $i$ to row $j$ in the identity matrix $I$. In other words, replace the zero in column $i$ and row $j$ of $I$ with $r$. Premultiplication of $A$ by $E_{ij}(r)$ will add $r$ times row $i$ to row $j$ in matrix $A$ while leaving the entries in all other rows of $A$ unchanged. For example, in the $3 \times 3$ case

$$E_{23}(5) \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 5a_{21} + a_{31} & 5a_{22} + a_{32} & 5a_{23} + a_{33} \end{pmatrix}.$$

**Definition** The matrices $E_{ij}$, $E_{ij}(r)$ and $E_i(r)$, which are obtained by performing an elementary row operation on the identity matrix, are called **elementary matrices.**

We summarize this discussion in the following theorem, whose proof is left as an exercise.

**Theorem 8.3** Let $E$ be an elementary $n \times n$ matrix obtained by performing a particular row operation on the $n \times n$ identity matrix. For any $n \times m$ matrix $A$, $EA$ is the matrix obtained by performing that same row operation on $A$.

In Chapter 7, we showed that elementary row operations can be used to reduce any matrix to row echelon form. The matrix version of that fact is stated in the next theorem, whose proof is also left as an exercise.

**Theorem 8.4** For any $k \times n$ matrix $A$ there exist elementary matrices $E_1, E_2, \ldots, E_m$ such that the matrix product $E_m \cdot E_{m-1} \cdots E_1 \cdot A = U$ where $U$ is in (reduced) row echelon form.

One can represent an elementary equation operation on the linear system $A\mathbf{x} = \mathbf{b}$ by multiplying both sides of the equation by the corresponding elementary matrix $E$ to obtain the new system $EA\mathbf{x} = E\mathbf{b}$. This fact illustrates the convenience of matrix notation for representing systems of equations.

*Example 8.1* Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}.$$

To bring $\mathbf{A}$ to row echelon form, we first add $-12$ times row 1 to row 2. This operation corresponds to the elementary matrix

$$E_{12}(-12) = \begin{pmatrix} 1 & 0 & 0 \\ -12 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then add $-3$ times row 1 to row 3 and finally $1/10$ times row 2 to row 3. These operations correspond to the elementary matrices

$$E_{13}(-3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

and

$$E_{23}(0.1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & .1 & 1 \end{pmatrix},$$

respectively. Check that the row echelon form of $\mathbf{A}$ is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -10 & -15 \\ 0 & 0 & -3.5 \end{pmatrix} = E_{23}(0.1) \cdot E_{13}(-3) \cdot E_{12}(-12) \cdot \mathbf{A}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -12 & 1 & 0 \\ -4.2 & .1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}.$$

**EXERCISES**

**8.11** Carry out the matrix multiplication in Example 8.1.

**8.12** Prove Theorem 8.3.

**8.13** Prove Theorem 8.4.

**8.14** *a*) Prove the following statement. If $P$ is an $m \times m$ permutation matrix and $A$ is $m \times n$, then $PA$ is the matrix $A$ with its rows permuted according to $P$. If $p_{ij} = 1$, then the $i$th row of $PA$ will be the $j$th row of $A$.

*b*) State and prove a similar statement about the permutation of columns by the multiplication $AP$.

## 8.4 ALGEBRA OF SQUARE MATRICES

Within the class $\mathbf{M_n}$ of $n \times n$ (square) matrices, all the arithmetic operations defined so far can be used. The sum, difference and product of two $n \times n$ matrices is $n \times n$. Even transposes of matrices in $\mathbf{M_n}$ are $n \times n$. The $n \times n$ identity matrix $I$ is a true multiplicative identity in $\mathbf{M_n}$ in that $AI = IA = A$ for all $A$ in $\mathbf{M_n}$. The matrix $I$ plays the role in $\mathbf{M_n}$ that the number 1 plays among the real numbers ($\mathbf{M_1}$). Recall, however, that if $A$ and $B$ are in $\mathbf{M_n}$, $AB$ usually will not equal $BA$.

Since we can add, subtract, and multiply square matrices, it is reasonable to ask if we can divide square matrices too. For numbers, dividing by $a$ is the same as multiplying by $1/a = a^{-1}$, and $a^{-1}$ makes sense as long as $a \neq 0$. To carry out this program for matrices (if we can), we need to make sense of $A^{-1}$ for matrices in $\mathbf{M_n}$. The number $a^{-1}$ is defined to be that number $b$ such that $ab = ba = 1$. The number $b$ is called the inverse of the number $a$. We do the same for matrices in $\mathbf{M_n}$.

**Definition** Let $A$ be a matrix in $\mathbf{M_n}$. The matrix $B$ in $\mathbf{M_n}$ is an **inverse** for $A$ if $AB = BA = I$.

If the matrix $B$ exists, we say that $A$ is **invertible**. Our definition has left open the possibility that a matrix $A$ can have several inverses. This is not true for numbers, and neither is it true for matrices.

**Theorem 8.5** An $n \times n$ matrix $A$ can have at most one inverse.

*Proof* Suppose that $B$ and $C$ are both inverses of $A$. Then

$$C = CI = C(AB) = (CA)B = IB = B. \quad \blacksquare$$

If an $n \times n$ matrix $A$ is invertible, we write $A^{-1}$ for its unique inverse matrix. Note that if $A$ is $1 \times 1$, then $A^{-1} = 1/A$. So, multiplying by $A^{-1}$ is the analog of dividing by the matrix $A$.

The only $1 \times 1$ matrix which is not invertible is 0. A main goal of this section is to identify exactly which $n \times n$ matrices are not invertible. We will see that a matrix is invertible if and only if it is nonsingular. In fact, the two properties reinforce each other. Recall that a square matrix $\mathbf{A}$ is called nonsingular if and only if the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for every right-hand side $\mathbf{b}$. Theorem 8.6 below states that if a square matrix has an inverse, then it is nonsingular. The proof of this theorem shows how to use the inverse of $\mathbf{A}$ to solve a general system $\mathbf{Ax} = \mathbf{b}$. Theorem 8.7 below is the converse statement: if a matrix is nonsingular, then it is invertible. Its proof shows how to use the fact that $\mathbf{A}$ is nonsingular to compute the inverse of $\mathbf{A}$. Before proving these theorems, we need two more definitions and a lemma.

**Definition** Let $A$ be an $k \times n$ matrix. The $n \times k$ matrix $B$ is a **right inverse** for $A$ if $AB = I$. The $n \times k$ matrix $C$ is a **left inverse** for $A$ if $CA = I$.

*Example 8.2* The matrix $\begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}$ is a right inverse for the matrix $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}$, but not a left inverse. On the other hand, the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ is a left inverse for $\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}$, but not a right inverse.

**Lemma 8.1** If $A$ has a right inverse $B$ and a left inverse $C$, then $A$ is invertible, and $B = C = A^{-1}$.

*Proof* Exactly the same as the proof of Theorem 8.5. ■

**Theorem 8.6** If an $n \times n$ matrix $A$ is invertible, then it is nonsingular, and the unique solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$.

*Proof* We want to show that if $A$ is invertible, we can solve any system of equations $A\mathbf{x} = \mathbf{b}$. Multiply each side of this system by $A^{-1}$ to solve for $\mathbf{x}$, as follows:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b}, \\ (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b}, \\ I\mathbf{x} &= A^{-1}\mathbf{b}, \\ \mathbf{x} &= A^{-1}\mathbf{b}. \end{aligned}$$

Make sure you can justify all the steps in this calculation. ■

**Theorem 8.7** If an $n \times n$ matrix $A$ is nonsingular, then it is invertible.

*Proof* Suppose that $A$ is nonsingular. We shall prove that it has an inverse by showing how to compute this inverse. Let $\mathbf{e}_i$ denote the $i$th column of $I$. For example, when $n = 3$,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Write $I$ with a focus on its columns as $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$. Since $A$ is nonsingular, the equation $A\mathbf{x} = \mathbf{e}_i$ has a unique solution $\mathbf{x} = \mathbf{c}_i$. (Of course, $\mathbf{c}_i$ is an $n \times 1$ matrix.) Let $C$ be the matrix whose $n$ columns are the respective solutions $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$. Since one multiplies each row of $A$ by the $j$th column of $C$ to obtain the $j$th column of $AC$, we can write

$$\begin{aligned} AC &= A[\mathbf{c}_1, \ldots, \mathbf{c}_n] \\ &= [A\mathbf{c}_1, \ldots, A\mathbf{c}_n] \\ &= [\mathbf{e}_1, \ldots, \mathbf{e}_n] \\ &= I. \end{aligned} \tag{2}$$

So $C$ is a right inverse of $A$.

To see that $A$ has a left inverse too, use Theorem 8.4 to write $EA = U$ where $E$ is a product of elementary matrices and $U$ is the *reduced* row echelon form of $A$. Since $A$ is nonsingular, $U$ has no zero rows and each column contains exactly one 1. In other words, $U$ is the identity matrix. Therefore, $E$ is a left inverse of $A$. Since $A$ has a right inverse and a left inverse, it is invertible. ■

Take time to study the calculation labeled (2) in the proof of Theorem 8.7, since we shall use it often. Once again, it follows from the fact that, to obtain the $j$th column of $AC$, one multiplies the rows of $A$ by the $j$th column of $C$. No other column of $C$ enters this calculation. In other words, if $\mathbf{c}_j$ is the $j$th column of $C$, then $A\mathbf{c}_j$ is the $j$th column of $AC$.

The proof of Theorem 8.7 actually shows how to compute the inverse of a nonsingular matrix. To find the $i$th column $\mathbf{c}_i$ of $A^{-1}$, we solve the system

$$A\mathbf{x} = \mathbf{e}_i$$

to find the solution $\mathbf{x} = \mathbf{c}_i$. Gauss-Jordan elimination can be used to solve this system for each $i$. In this case the augmented matrix is $[A \mid \mathbf{e}_i]$. The row operations which will reduce this depend only on the first $n$ columns of the augmented matrix, in other words, only on the matrix $A$. One never uses the last column of an augmented matrix to determine which row operation to use on a system.

Therefore, the same row operations that reduce $[A \mid \mathbf{e}_i]$ to $[I \mid \mathbf{c}_i]$ will also reduce $[A \mid \mathbf{e}_j]$ to $[I \mid \mathbf{c}_j]$. We can be more efficient and pool all these data into a gigantic augmented matrix $[A \mid \mathbf{e}_1 \cdots \mathbf{e}_n] = [A \mid I]$ and perform Gauss-Jordan elimination only once rather than $n$ times. In this process, the augmented matrix $[A \mid I]$ reduces to $[I \mid A^{-1}]$.

*Example 8.3* We can apply this method to find the inverse of matrix $A$ in Example 8.1:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}. \tag{3}$$

First, augment $A$ with the identity matrix:

$$[A \mid I] = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 12 & 2 & -3 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right).$$

Then, perform the row operations on $[A \mid I]$ which reduce $A$ to row echelon form. The first three such operations are described in Example 8.1 and result in the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -10 & -15 & -12 & 1 & 0 \\ 0 & 0 & -3.5 & -4.2 & 0.1 & 1 \end{array}\right).$$

Next reduce this matrix to *reduced row echelon form* using the operations described in Section 7.2:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.4 & \frac{3}{35} & -\frac{1}{7} \\ 0 & 1 & 0 & -0.6 & -\frac{2}{35} & \frac{3}{7} \\ 0 & 0 & 1 & 1.2 & -\frac{1}{35} & -\frac{2}{7} \end{array}\right).$$

As implied by the proof of Theorem 8.7, the right half of this augmented matrix,

$$\begin{pmatrix} 0.4 & \frac{3}{35} & -\frac{1}{7} \\ -0.6 & -\frac{2}{35} & \frac{3}{7} \\ 1.2 & -\frac{1}{35} & -\frac{2}{7} \end{pmatrix}, \tag{4}$$

is the inverse of $A$.

*Example 8.4* We next apply this method to compute the inverse of an arbitrary $2 \times 2$ matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5}$$

Begin by writing the augmented matrix

$$[A \mid I] = \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right).$$

If $a$ and $c$ are both $0, A$ will clearly be singular. Let us assume, then, that $a \neq 0$. First, add $-c/a$ times row 1 to row 2, to obtain the row echelon form

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \dfrac{ad - bc}{a} & -\dfrac{c}{a} & 1 \end{array}\right). \tag{6}$$

This short calculation tells us that when $a \neq 0, A$ is nonsingular (and therefore invertible) if and only if $ad - bc \neq 0$. Now we continue with Gauss-Jordan elimination to transform (6) to reduced row echelon form. Multiply the first row of (6) by $1/a$ and the second row of (6) by $a/(ad - bc)$ to obtain the matrix

$$\left(\begin{array}{cc|cc} 1 & \dfrac{b}{a} & \dfrac{1}{a} & 0 \\ 0 & 1 & -\dfrac{c}{ad - bc} & \dfrac{a}{ad - bc} \end{array}\right)$$

whose leading entries are both 1s. To complete the reduction, add $-b/a$ times row 2 to row 1. The final product is

$$\left(\begin{array}{cc|cc} 1 & 0 & \dfrac{d}{ad - bc} & -\dfrac{b}{ad - bc} \\ 0 & 1 & -\dfrac{c}{ad - bc} & \dfrac{a}{ad - bc} \end{array}\right).$$

Reading off the last half of the augmented matrix, we see that

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{7}$$

Note that if $ad - bc \neq 0$, $a$ and $c$ cannot both be 0. Thus, by Example 8.3 and Exercise 8.17, we have proven the following theorem on $2 \times 2$ matrices.

**Theorem 8.8** The general $2 \times 2$ matrix given by (5) is nonsingular (and therefore invertible) if and only if $ad - bc \neq 0$. Its inverse is matrix (7).

The goal of the next chapter will be to generalize this convenient criterion to the case of arbitrary $n \times n$ matrices.

Putting together the facts about nonsingularity from Chapter 7 with what we have done here, we arrive at the following equivalencies.

**Theorem 8.9** For any square matrix $A$, the following statements are equivalent:

(*a*) $A$ is invertible.
(*b*) $A$ has a right inverse.
(*c*) $A$ has a left inverse.
(*d*) Every system $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$.
(*e*) Every system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$.
(*f*) $A$ is nonsingular.
(*g*) $A$ has maximal rank $n$.

*Proof* We saw the equivalence of statements *d*) through *g*) in Section 7.4. The statements and proofs of Theorems 8.6 and 8.7 indicate that statements *a*) through *d*) are equivalent. ■

The following facts about the behavior of the inverse are easy to prove, and are left as an exercise.

**Theorem 8.10** Let $A$ and $B$ be square invertible matrices. Then,

(*a*) $(A^{-1})^{-1} = A$,
(*b*) $(A^T)^{-1} = (A^{-1})^T$,
(*c*) $AB$ is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

The inverse for matrices works very much like the inverse for numbers. If $A$ and $B$ are invertible, $A + B$ need not be invertible, and even when it is, $(A + B)^{-1}$ is generally not $A^{-1} + B^{-1}$. Even for $1 \times 1$ matrices or scalars,

$$(3 + 2)^{-1} = \frac{1}{5}, \quad \text{but} \quad 3^{-1} + 2^{-1} = \frac{5}{6}.$$

If $A$ is a square matrix, we can take integral powers of $A$. The matrix $A^m$ is defined as the product $A \cdot A \cdots A$ ($m$ times). For example, if

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

then
$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

If $A$ is invertible, we can define negative powers of $A$ as well:

$$A^{-m} = (A^{-1})^m = A^{-1} \cdot A^{-1} \cdots A^{-1} \qquad (m \text{ times}).$$

Taking powers of matrices follows most of the same basic rules as taking powers of scalars. This is summarized in the following theorem.

**Theorem 8.11** If $A$ is invertible:

(*a*) $A^m$ is invertible for any integer $m$ and $(A^m)^{-1} = (A^{-1})^m = A^{-m}$,
(*b*) for any integers $r$ and $s$, $A^r A^s = A^{r+s}$, and
(*c*) for any scalar $r \neq 0$, $rA$ is invertible and $(rA)^{-1} = (1/r)A^{-1}$.

*Proof* These easy computations are left as an exercise. ■

There are some differences between exponentiation of matrices and exponentiation of numbers, all due to the fact that matrix multiplication need not be commutative — that $AB$ need not equal $BA$. These differences are explored in Exercise 8.27.

*Example 8.5* Since each of the elementary row operations is reversible, each of the elementary matrices is invertible and has an elementary matrix for its inverse. For example, the inverse of the permutation matrix $E_{ij}$ is $E_{ji}$ $(= E_{ij})$, the inverse of $E_i(r)$ is $E_i(1/r)$, and the inverse of $E_{ij}(r)$ is $E_{ij}(-r)$.

Since each elementary matrix is invertible, any product of elementary matrices is also invertible by Theorem 8.10*c*. By inverting the elementary matrices in the statement of Theorem 8.4, we can write any matrix $A$ as a product of elementary matrices times a reduced row echelon matrix $U$:

$$A = E_1^{-1} \cdot E_2^{-1} \cdots E_m^{-1} \cdot U.$$

Furthermore, if $A$ is nonsingular, its reduced row echelon form is the identity matrix, as we saw in the proof of Theorem 8.7.

The foregoing discussion gives us a decomposition theorem for matrices which we will use in Chapter 26.

**Theorem 8.12** Any matrix $A$ can be written as a product

$$A = F_1 \cdots F_m \cdot U$$

where the $F_i$'s are elementary matrices and $U$ is in reduced row echelon form. When $A$ is nonsingular, $U = I$ and $A = F_1 \cdots F_m$.

## EXERCISES

**8.15** Check that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} .5 & 0 & -.5 \\ .5 & 0 & .5 \\ -.5 & 1 & -.5 \end{pmatrix}.$$

**8.16** Verify that matrix (4) is the inverse of matrix (3) by direct matrix multiplication.

**8.17** Suppose that $a = 0$ but $c \neq 0$ in (5). Show that one obtains the same inverse (7) for $A$.

**8.18** Show by simple matrix multiplication that, if $ad - bc \neq 0$,

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is both a left and a right inverse of $A$.

**8.19** Use the technique of Example 8.3 to either invert each of the following matrices or prove that it is singular:

$$a)\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad b)\ \begin{pmatrix} 4 & 5 \\ 2 & 4 \end{pmatrix}, \quad c)\ \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix},$$

$$d)\ \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \quad e)\ \begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix},$$

$$f)\ \begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 21 & 8 & 17 \\ 4 & 12 & -4 & 13 \\ 0 & -3 & -12 & 2 \end{pmatrix}.$$

**8.20** Invert the coefficient matrix to solve the following systems of equations:

$$a)\ \begin{aligned} 2x_1 + x_2 &= 5 \\ x_1 + x_2 &= 3; \end{aligned} \qquad b)\ \begin{aligned} 2x_1 + x_2 \phantom{+ 6x_3} &= 4 \\ 6x_1 + 2x_2 + 6x_3 &= 20 \\ -4x_1 - 3x_2 + 9x_3 &= 3; \end{aligned}$$

$$c)\quad \begin{aligned} 2x_1 + 4x_2 \quad &= 2 \\ 4x_1 + 6x_2 + 3x_3 &= 1 \\ -6x_1 - 10x_2 \quad &= -6. \end{aligned}$$

**8.21** Show that if $A$ is $n \times n$ and $AB = BA$, then $B$ is also $n \times n$.

**8.22** For $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, compute $A^3$, $A^4$, and $A^{-2}$.

**8.23** Verify the statements about the inverses of elementary matrices in the last sentence of Example 8.5.

**8.24** *a*) Use Theorem 8.8 to prove that a $2 \times 2$ lower- or upper-triangular matrix is invertible if and only if each diagonal entry is nonzero.
*b*) Show that the inverse of a $2 \times 2$ lower triangular matrix is lower triangular.
*c*) Show that the inverse of a $2 \times 2$ upper triangular matrix is upper triangular.

**8.25** *a*) Prove Theorem 8.10.
*b*) Generalize part *c* to the case of the product of $k$ nonsingular matrices.
*c*) Show by example that if $A$ and $B$ are invertible, $A + B$ need not be invertible.
*d*) Show that, when it exists, $(A + B)^{-1}$ is generally not $A^{-1} + B^{-1}$.

**8.26** Prove Theorem 8.11.

**8.27** *a*) Prove that $(AB)^k = A^k B^k$ if $AB = BA$.
*b*) Show that $(AB)^k \neq A^k B^k$ in general.
*c*) Conclude that $(A + B)^2$ does not equal $A^2 + 2AB + B^2$ unless $AB = BA$.

**8.28** What is the inverse of the $n \times n$ diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}?$$

**8.29** Show that the inverse of a $2 \times 2$ symmetric matrix $S$ is symmetric.

**8.30** Show that the inverse of an $n \times n$ upper-triangular matrix $U$ is upper-triangular. Can you find an easy argument to extend this result to lower-triangular matrices?
[Hint: There are a number of ways to do the first part. You can use the inversion method described in the proof of Theorem 8.7, keeping track of the status of the 0s below the diagonal. Or, you can show by direct calculation that $BU = I$ implies that $B$ has only 0s below the diagonal.]

**8.31** Show that for any permutation matrix $P$, $P^{-1} = P^T$.

**8.32** Use Gauss-Jordan elimination to derive a criterion for the invertibility of $3 \times 3$ matrices similar to the $ad - bc$ criterion for the $2 \times 2$ case. For simplicity, assume that no row interchanges are needed in the elimination process.

**8.33** The definitions of left inverse and right inverse apply to nonsquare matrices. Use the ideas in the proof of Theorem 8.7 to prove the following statements for an $m \times n$ matrix $A$, where $m \neq n$.
*a*) A nonsquare matrix cannot have both a left and a right inverse.
*b*) If $A$ has one left (right) inverse, it has infinitely many.
*c*) If $m < n$, $A$ has a right inverse if and only if $\operatorname{rank} A = m$.
*d*) If $m > n$, $A$ has a left inverse if and only if $\operatorname{rank} A = n$.

## 8.5 INPUT-OUTPUT MATRICES

The last section showed that solving a system $A\mathbf{x} = \mathbf{b}$ of $n$ equations in $n$ unknowns is closely related to inverting the matrix $A$ since

$$\mathbf{x} = A^{-1}\mathbf{b}. \tag{8}$$

For a single fixed $\mathbf{b}$, it is usually quicker to solve $A\mathbf{x} = \mathbf{b}$ by Gaussian elimination (and back substitution). However, if one is going to work with many different right-hand sides $\mathbf{b}$ and the same $A$, it may be easier to invert $A$ and use (8).

For example, consider the input-output example of Chapter 6. This is a model of an economy with $n$ industries. Each industry produces a single output, using as inputs the products produced by the other industries. Write $x_i$ for the gross output of product $i$, and let $a_{ij}$ denote the amount of good $i$ needed to produce one unit of good $j$. Let $c_i$ denote consumer demand for product $i$. In Chapter 6, we saw that the market equilibrium condition that supply equal demand is given by the $n$ equations

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + c_i,$$

for $i = 1, \ldots, n$. In matrix notation this system of equations becomes

$$\mathbf{x} = A\mathbf{x} + \mathbf{c},$$

which is more conveniently written as

$$(I - A)\mathbf{x} = \mathbf{c}. \tag{9}$$

(To keep all $n$-tuples nonnegative, we will ignore the labor sector described in Chapter 6.)

The matrix $A$ of intermediate factor demands is sometimes called the **technology matrix**. We might expect this to remain relatively constant over long periods of time. The right-hand side of (9), $\mathbf{c}$, can be expected to vary more frequently. Thus it is convenient to study solutions to (9) by working with the inverse:

$$\mathbf{x} = (I - A)^{-1}\mathbf{c}.$$

Notice that in addition to requiring that $I - A$ be invertible, we also require that the solution to (9) be nonnegative whenever $\mathbf{c}$ is nonnegative. This corresponds to the requirement that any solution to our economic system produces nonnegative amounts of each commodity. For this to happen, all entries of the matrix $(I - A)^{-1}$ must be nonnegative. Furthermore, the study of this system is complicated by the fact that all the economic data in the model are contained in the matrix $A$. It is not enough simply to assume that $I - A$ has a nonnegative inverse. We must find assumptions on $A$ which will imply the desired behavior of $I - A$.

Since the factors of production have different natural units, it is convenient to express them all in monetary terms, say in millions of dollars, in an input-output analysis. In this case, the $(i, j)$th entry $a_{ij}$ of technology matrix $A$ indicates how many millions of dollars of good $i$ are needed to produce 1 million dollars of good $j$. The sum of the entries in each column of $A$ gives the total cost of producing 1 million dollars of the product that column represents. Since we expect each industry to make a positive accounting profit, the sum of the entries in each column should be less than 1. This turns out to be one of the conditions on a technology matrix $A$ which will guarantee that $I - A$ has a nonnegative inverse.

**Theorem 8.13** Let $A$ be an $n \times n$ matrix with the properties that each entry is nonnegative and the sum of the entries in each column is less than 1. Then, $(I - A)^{-1}$ exists and contains only nonnegative entries.

We will prove Theorem 8.13 at the end of this section. First, to make the preceding discussion concrete, consider a simple three-industry economy, with input-output matrix

$$A = \begin{pmatrix} 0.15 & 0.5 & 0.25 \\ 0.3 & 0.1 & 0.4 \\ 0.15 & 0.3 & 0.2 \end{pmatrix}.$$

Suppose that consumer demand fluctuates between

$$\mathbf{c} = \begin{pmatrix} 20 \\ 20 \\ 10 \end{pmatrix} \quad \text{and} \quad \mathbf{c}' = \begin{pmatrix} 10 \\ 20 \\ 20 \end{pmatrix}.$$

What will be the corresponding industry outputs?

First, compute $I - A$:

$$I - A = \begin{pmatrix} 0.85 & -0.5 & -0.25 \\ -0.3 & 0.9 & -0.4 \\ -0.15 & -0.3 & 0.8 \end{pmatrix}.$$

To invert $I - A$, write the augmented matrix

$$\left(\begin{array}{ccc|ccc} 0.85 & -0.5 & -0.25 & 1 & 0 & 0 \\ -0.3 & 0.9 & -0.4 & 0 & 1 & 0 \\ -0.15 & -0.3 & 0.8 & 0 & 0 & 1 \end{array}\right)$$

and use Gauss-Jordan elimination to reduce the first three columns to the identity matrix. The result, rounded to three decimal places, is

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1.975 & 1.564 & 1.399 \\ 0 & 1 & 0 & 0.988 & 2.115 & 1.366 \\ 0 & 0 & 1 & 0.741 & 1.086 & 2.025 \end{array}\right).$$

The last three columns are $(I - A)^{-1}$. Note that, as Theorem 8.13 predicts, all entries are positive.

When consumer demand is $\mathbf{c} = \begin{pmatrix} 20 \\ 20 \\ 10 \end{pmatrix}$, the total output should be

$$\mathbf{x} = (I - A)^{-1}\mathbf{c} = \begin{pmatrix} 1.975 & 1.564 & 1.399 \\ 0.988 & 2.115 & 1.366 \\ 0.741 & 1.086 & 2.025 \end{pmatrix} \begin{pmatrix} 20 \\ 20 \\ 10 \end{pmatrix} = \begin{pmatrix} 84.77 \\ 75.72 \\ 56.79 \end{pmatrix}.$$

When consumer demand is $\mathbf{c} = \begin{pmatrix} 10 \\ 20 \\ 20 \end{pmatrix}$, the total output should be

$$\mathbf{x} = (I - A)^{-1}\mathbf{c} = \begin{pmatrix} 1.975 & 1.564 & 1.399 \\ 0.988 & 2.115 & 1.366 \\ 0.741 & 1.086 & 2.025 \end{pmatrix} \begin{pmatrix} 10 \\ 20 \\ 20 \end{pmatrix} = \begin{pmatrix} 79.01 \\ 79.51 \\ 69.63 \end{pmatrix}.$$

Leontief used input-output analysis to study the 1958 U.S. economy. He divided the economy into 81 sectors and aggregated these sectors into six groups of related sectors. We will treat each of the six families as a separate industry in order to simplify our presentation. These six industries are listed in Table 8.1, and their intermediate factor demands are listed in Table 8.2. The units are millions of dollars. So the .173 in row 3 column 2 means that the production of $1 million worth of final metal products requiresthe expenditure of $173,000 on basic metal

| | **Sector** | **Examples** |
|---|---|---|
| FN, | Final nonmetal | Leather goods, furniture, foods |
| FM, | Final metal | Construction mach'ry, household appliances |
| BM, | Basic metal | Mining, machine shop products |
| BN, | Basic nonmetal | Glass, wood, textile, and livestock products |
| E, | Energy | Coal, petroleum, electricity, gas |
| S, | Services | Govt. services, transportation, real estate |

**Table 8.1** *The Six Sectors*

| | FN | FM | BM | BN | E | S |
|---|---|---|---|---|---|---|
| FN | 0.170 | 0.004 | 0.000 | 0.029 | 0.000 | 0.008 |
| FM | 0.003 | 0.295 | 0.018 | 0.002 | 0.004 | 0.016 |
| BM | 0.025 | 0.173 | 0.460 | 0.007 | 0.011 | 0.007 |
| BN | 0.348 | 0.037 | 0.021 | 0.403 | 0.011 | 0.048 |
| E | 0.007 | 0.001 | 0.039 | 0.025 | 0.358 | 0.025 |
| S | 0.120 | 0.074 | 0.104 | 0.123 | 0.173 | 0.234 |

**Table 8.2** *Internal demands for 1958 U.S. Economy*

**Table 8.3**

| | |
|---|---|
| FN | \$ 99,640 |
| FM | 75,548 |
| BM | 14,444 |
| BN | 33,501 |
| E | 23,527 |
| S | 263,985 |

*External Demands for 1958 U.S. Economy (in millions of dollars)*

goods. Table 8.3 lists Leontief's estimates of final demands in the 1958 U.S. economy. The problem is to determine how many units had to be produced in each of the six sectors in order to run the U.S. economy in 1958.

To solve the problem, we turn Table 8.2 into the technology matrix $A$ and Table 8.3 into the final demand column matrix $\mathbf{c}$. As before, the goal is to solve $(I - A)\mathbf{x} = \mathbf{c}$ for the output column matrix $\mathbf{x}$:

$$\mathbf{x} = (I - A)^{-1}\mathbf{c}.$$

First, we need to compute the net input-output matrix $I - A$.

$$I - A$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.170 & 0.004 & 0 & 0.029 & 0 & 0.008 \\ 0.003 & 0.295 & 0.018 & 0.002 & 0.004 & 0.016 \\ 0.025 & 0.173 & 0.460 & 0.007 & 0.011 & 0.007 \\ 0.348 & 0.037 & 0.021 & 0.403 & 0.011 & 0.048 \\ 0.007 & 0.001 & 0.039 & 0.025 & 0.358 & 0.025 \\ 0.120 & 0.074 & 0.104 & 0.123 & 0.173 & 0.234 \end{pmatrix}$$

$$= \begin{pmatrix} 0.830 & -0.004 & 0 & -0.029 & 0 & -0.008 \\ -0.003 & 0.705 & -0.018 & -0.002 & -0.004 & -0.016 \\ -0.025 & -0.173 & 0.540 & -0.007 & -0.011 & -0.007 \\ -0.348 & -0.037 & -0.021 & 0.597 & -0.011 & -0.048 \\ -0.007 & -0.001 & -0.039 & -0.025 & 0.642 & -0.025 \\ -0.120 & -0.074 & -0.104 & -0.123 & -0.173 & 0.766 \end{pmatrix}.$$

The inverse of this net input-output matrix can be computed by the methods of Section 8.4 and then used to compute the gross output column matrix.

$$\mathbf{x} = (I - A)^{-1}$$

$$= \begin{pmatrix} 1.234 & 0.014 & 0.006 & 0.064 & 0.007 & 0.018 \\ 0.017 & 1.436 & 0.057 & 0.012 & 0.020 & 0.032 \\ 0.071 & 0.465 & 1.877 & 0.019 & 0.045 & 0.031 \\ 0.751 & 0.134 & 0.100 & 1.740 & 0.066 & 0.124 \\ 0.060 & 0.045 & 0.130 & 0.082 & 1.578 & 0.059 \\ 0.339 & 0.236 & 0.307 & 0.312 & 0.376 & 1.349 \end{pmatrix} \begin{pmatrix} 99{,}640 \\ 75{,}548 \\ 14{,}444 \\ 33{,}501 \\ 23{,}527 \\ 263{,}985 \end{pmatrix}$$

$$= \begin{pmatrix} 131{,}161 \\ 120{,}324 \\ 79{,}194 \\ 178{,}936 \\ 66{,}703 \\ 426{,}542 \end{pmatrix}$$

We conclude, for example, that it requires \$131,161 million worth of final nonmetal products to meet both intermediate and final demands in the 1958 U.S. economy.

### Proof of Theorem 8.13

We conclude this section by proving Theorem 8.13. Let $A$ be a technology matrix that satisfies the hypotheses of Theorem 8.13: nonnegative entries and column sums less than 1. Then, $-A$ has all its entries and its column sums between 0 and $-1$ and $I - A$ satisfies the following three properties:

(*a*) each off-diagonal entry is $\leq 0$,
(*b*) each diagonal entry is positive, and
(*c*) the sum of the entries in each column is positive.

Matrices which satisfy these three conditions are a special case of the class of **dominant diagonal matrices**. A more general definition of a dominant diagonal matrix requires that in each column the *absolute value* of the diagonal entry is at least as large as the sum of the absolute values of the other entries in that column. To prove Theorem 8.13, we need only prove the following result.

**Theorem 8.14** Let $B$ be a square matrix which satisfies conditions $a$, $b$, and $c$ above. Then, all entries of $B^{-1}$ are nonnegative.

*Proof* To keep better track of the signs and sizes of the entries of the matrix $B$, we write it as

$$B = \begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & b_{22} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & b_{nn} \end{pmatrix},$$

where $$\text{each } b_{ij} \geq 0 \quad \text{and} \quad 0 \leq \sum_{h \neq j} b_{hj} < b_{jj} \qquad (10)$$

for all $j$. Let $\mathbf{c}$ be a vector with all positive entries and consider the system $B\mathbf{x} = \mathbf{c}$. To solve this system, we perform Gaussian elimination on the augmented matrix $[B \mid \mathbf{c}]$. Add $b_{j1}/b_{11}$ times row 1 to row $j$ for all $j > 1$. The result is the new augmented matrix

$$\begin{pmatrix} b_{11} & -b_{12} & \cdots & -b_{1n} & | & c_1 \\ 0 & b_{22} - \dfrac{b_{21}}{b_{11}} b_{12} & \cdots & -b_{2n} - \dfrac{b_{21}}{b_{11}} b_{1n} & | & c_2 + \dfrac{b_{21}}{b_{11}} c_1 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & -b_{n2} - \dfrac{b_{n1}}{b_{11}} b_{12} & \cdots & b_{nn} - \dfrac{b_{n1}}{b_{11}} b_{1n} & | & c_n + \dfrac{b_{n1}}{b_{11}} c_1 \end{pmatrix}$$

$$\equiv \begin{pmatrix} b_{11} & * & | & c_1 \\ \mathbf{0} & \hat{B} & | & \hat{\mathbf{c}} \end{pmatrix}.$$

The $(n-1) \times (n-1)$ matrix $\hat{B}$ is still dominant diagonal, since its off-diagonal entries are still nonpositive and the sum of the entries in its $(j-1)$th column is

$$\begin{aligned} \left(b_{jj} - \frac{b_{j1}}{b_{11}} b_{1j}\right) &+ \sum_{h \neq 1,j} \left(-b_{hj} - \frac{b_{h1}}{b_{11}} b_{1j}\right) \\ &= b_{jj} - \left(\sum_{h \neq 1,j} b_{hj}\right) - b_{1j} \frac{b_{21} + \cdots + b_{n1}}{b_{11}} \\ &> b_{jj} - \sum_{h \neq 1,j} b_{hj} - b_{1j} \\ &> 0, \qquad \text{(by (10) twice).} \end{aligned}$$

The new RHS $\hat{\mathbf{c}}$ has all entries positive. Continue applying Gaussian elimination; at each stage, the resulting submatrix still satisfies $a$, $b$, and $c$. We conclude that the row echelon form of $[B \mid \mathbf{c}]$ has the sign pattern

$$\begin{pmatrix} + & - & - & \cdots & - & | & + \\ 0 & + & - & \cdots & - & | & + \\ 0 & 0 & + & \cdots & - & | & + \\ \vdots & \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & 0 & 0 & \cdots & - & | & + \\ 0 & 0 & 0 & \cdots & + & | & + \end{pmatrix}$$

Back substitution from such a matrix yields a *positive* solution $\mathbf{x}$ to the system $B\mathbf{x} = \mathbf{c}$. If the nonzero right-hand side $\mathbf{c}$ had some zero entries and if $A$ had some zero off-diagonal terms, the same argument yields a nonnegative solution of $B\mathbf{x} = \mathbf{c}$. Since the columns of $B^{-1}$ are the solution vectors of $B\mathbf{x} = \mathbf{e}_i$ (Theorem 8.7), the entries of $B^{-1}$ are all nonnegative numbers. ■

## EXERCISES

**8.34** Let the technology matrix be given by $A = \begin{pmatrix} .7 & .2 & .2 \\ .1 & .6 & .1 \\ .1 & .1 & .6 \end{pmatrix}$. Find the gross output vectors when final demand is:

$$a)\ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad c)\ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

**8.35** Let the general $2 \times 2$ technology matrix be given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove Theorem 8.13 directly for such a matrix using Theorem 8.8.

## 8.6 PARTITIONED MATRICES (optional)

Let $A$ be an $m \times n$ matrix. A **submatrix** of $A$ is a matrix formed by discarding some entire rows and/or columns of $A$. A **partitioned matrix** is a matrix which has been partitioned into submatrices by horizontal and/or vertical lines which extend along *entire* rows or columns of $A$. For example,

$$A = \left(\begin{array}{cc|c|ccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{array}\right), \tag{11}$$

which we can write as

$$A = \left(\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right).$$

Each submatrix $A_{ij}$ is called a **block** of $A$. Augmented matrices are an example of partitioned matrices. They have been partitioned vertically into two blocks.

If $A$ is a square matrix which has been partitioned as

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix} \tag{12}$$

where each $A_{ii}$ is square and $A_{ij} = 0$ for $i \neq j$, then $A$ is called a **block diagonal** matrix.

Suppose that $A$ and $B$ are two $m \times n$ matrices which are partitioned the same way; that is,

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}$$

where $A_{11}$ and $B_{11}$ have the same dimensions, $A_{12}$ and $B_{12}$ have the same dimensions, and so on. Then $A$ and $B$ can be added as if the blocks are scalar entries:

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} & A_{13} + B_{13} \\ A_{21} + B_{21} & A_{22} + B_{22} & A_{23} + B_{23} \end{pmatrix}.$$

Similarly, two partitioned matrices $A$ and $C$ can be multiplied, treating the blocks as scalars, if the blocks are all of a size such that the matrix multiplication of blocks can be done. For example, if

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{pmatrix}, \tag{13}$$

then

$$AC = \begin{pmatrix} A_{11}C_{11} + A_{12}C_{21} & A_{11}C_{12} + A_{12}C_{22} & A_{11}C_{13} + A_{12}C_{23} \\ A_{21}C_{11} + A_{22}C_{21} & A_{21}C_{12} + A_{22}C_{22} & A_{21}C_{13} + A_{22}C_{23} \end{pmatrix}$$

so long as the various matrix products $A_{ij}C_{jk}$ can be formed. For example, $A_{11}$ must have as many columns as $C_{11}$ has rows, and so on.

We used the block multiplication of partitioned matrices in Section 8.4 when we wrote the matrix product $AA^{-1} = I$ as

$$A(\mathbf{c}_1 \quad \cdots \quad \mathbf{c}_n) = (\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n),$$

where $\mathbf{c}_i$ is the $i$th column of $A^{-1}$ and $\mathbf{e}_j$ is the $j$th column of the identity matrix. In this case, the $j$th block product yielded the equation $A\mathbf{c}_j = \mathbf{e}_j$ in (2).

One reason for partitioning matrices is that frequently inverses can be computed (or found not to exist) much more easily using the blocks than they can by direct computation. For example, the following result on partitions is useful for deriving propositions about how demand functions depend on the price level.

**Theorem 8.15** Let $A$ be a square matrix partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11}$ and $A_{22}$ are square submatrices. If both $A_{22}$ and the matrix

$$D = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

are nonsingular, then $A$ is nonsingular and

$$A^{-1} = \begin{pmatrix} D^{-1} & -D^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}D^{-1} & A_{22}^{-1}(I + A_{21}D^{-1}A_{12}A_{22}^{-1}) \end{pmatrix} \tag{14}$$

The proof of this theorem is left as an exercise.

## EXERCISES

**8.36** What must be true about the sizes of the various blocks $A_{11}, A_{12}, C_{11}$, and so on, in (13) in order for the block multiplications to make sense?

**8.37** Suppose that $A$ is given by (11) and the matrix $C$ is given by

$$C = \left(\begin{array}{c|cc|c} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ \hline c_{31} & c_{32} & c_{33} & c_{34} \\ \hline c_{41} & c_{42} & c_{43} & c_{44} \\ c_{51} & c_{52} & c_{53} & c_{54} \\ c_{61} & c_{62} & c_{63} & c_{64} \end{array}\right) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}.$$

*a*) Check that block multiplication can be carried out for the matrix product $AC$.
*b*) Compute the six block products $\Sigma A_{ij}C_{jk}$ for $i = 1, 2$ and $k = 1, 2, 3$.
*c*) Check that you reach the same answer for the matrix product whether you compute it with the block products or directly.

**8.38** Show that the block diagonal matrix $A$ in (12) is invertible if and only if each $A_{ii}$ is invertible. Find $A^{-1}$.

**8.39** Prove Theorem 8.15. First show that the matrix $D$ exists. Then verify by block multiplication that matrix (14) is the inverse of $A$.

**8.40** Replace the hypotheses on the matrix $A$ of Theorem 8.15 by the hypothesis that both $A_{11}$ and $A_{22} - A_{21}A_{11}^{-1}A_{12}$ are invertible. Prove that $A$ is invertible and find its inverse.

**8.41** Rewrite the invertibility conditions of Theorem 8.15 for the following cases.
*a*) $A_{21} = 0$;
*b*) $A_{22}$ is $1 \times 1$ (a scalar);
*c*) $A_{11}$ is the scalar 0, and $A_{21} = A_{12}^T = \mathbf{p}$ where $\mathbf{p}$ is a column vector.

## 8.7 DECOMPOSING MATRICES (optional)

This section demonstrates how most matrices can be written as a product of a lower-triangular matrix $L$ and an upper-triangular matrix $U$. This **LU decomposition** leads to an efficient approach to solving systems of equations (Exercise 8.51 below). It is also the central technique in proving some important theorems about matrices (especially in Chapter 26). This decomposition is a direct consequence of Theorem 8.12 and the following lemma about the product of elementary matrices.

**Lemma 8.2** Let $L$ and $M$ be two $n \times n$ lower-triangular matrices. Then, the matrix product $LM$ is lower triangular. If $L$ and $M$ have only 1s on their diagonals, so does $LM$.

*Proof* The $(i, j)$th entry of the product $LM$ is the product of the $i$th row of $L$ and the $j$th column of $M$. Using the hypothesis that $l_{ik} = 0$ for $k > i$ and $m_{hj} = 0$ for $h < j$, we write this product as:

$$[LM]_{ij} = (l_{i1} \quad \cdots \quad l_{i,i-1} \quad l_{ii} \quad 0 \quad \cdots \quad 0) \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{jj} \\ m_{j+1,j} \\ \vdots \\ m_{nj} \end{pmatrix} \tag{15}$$

If $i < j$, each of the $i$ nonzero entries at the beginning of the $i$th row of $L$ will be multiplied by the $i$ zero entries beginning the $j$th column of $M$. The result is a zero entry in $LM$. Therefore, $LM$ is lower triangular.

It follows from (15) that the $(i, i)$th diagonal entry of $LM$ is $l_{ii}m_{ii}$. If $l_{ii} = m_{ii} = 1$, then $l_{ii}m_{ii} = 1$. ■

Now we can use our knowledge of elementary matrices to decompose matrices.

**Theorem 8.16** Let $A$ be a general $k \times n$ matrix, and suppose that no row interchanges are needed to reduce $A$ to its row echelon form. Then $A$ can be written as a product $LU$ where $L$ is an $k \times k$ lower-triangular matrix with only 1's on the diagonal, and $U$ is an upper-triangular $k \times n$ matrix.

The $U$ in Theorem 8.16 is the row echelon form of $A$. Although it is not necessarily a square matrix, we will call it upper triangular because its $(i, j)$th entries are all zero whenever $i > j$.

*Proof* Theorem 8.16 is a consequence of Theorems 8.4 and 8.12, which summarize the elementary matrix approach to Gaussian elimination. If no row interchanges are needed to reduce $A$ to its row echelon form $U$, the only row operation required is the addition of a multiple of one row to a row which is farther down in the matrix. This operation is described by the elementary matrix $E_{ij}(r)$ where $i < j$. These elementary matrices are lower-triangular with 1s on the diagonal. Theorems 8.4 and 8.12 tell us that

$$A = E_1 \cdots E_m \cdot U \tag{16}$$

where $E_1$ is the inverse of the first elementary matrix used in the row reduction of $A$, $E_2$ is the inverse of the second elementary matrix used in the row reduction of $A$, and so on. In Example 8.5, we noted that the inverse of $E_{ij}(r)$ is $E_{ij}(-r)$. So the matrices $E_1, \ldots, E_m$ are all lower triangular with only 1's on their diagonals. Applying Lemma 8.2, we see that the product $E_1 \cdot E_2$ is lower triangular and has only 1's on the diagonal. Since the matrices $E_3$ and $E_1 \cdot E_2$ satisfy the hypotheses of the Lemma, $E_1 \cdot E_2 \cdot E_3$ is lower triangular and has 1s on the diagonal. Repeating this argument as many times as is necessary, we can see that the product $L = E_1 \cdots E_m$ is lower triangular and has only 1s on the diagonal. Consequently (16) can be rewritten as $A = LU$ where $L$ is lower triangular with only 1s on the diagonal. ■

*Example 8.6* To illustrate Theorem 8.16, let us return to Example 8.1, where we wrote the row echelon form $U$ of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}$$

as

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -10 & -15 \\ 0 & 0 & -3.5 \end{pmatrix} = E_{23}(.1) \cdot E_{13}(-3) \cdot E_{12}(-12) \cdot A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -12 & 1 & 0 \\ -4.2 & .1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}.$$

Multiply the right-hand side by the inverses of the elementary matrices:

$$\begin{aligned} A &= E_{12}(-12)^{-1} \cdot E_{13}(-3)^{-1} \cdot E_{23}(.1)^{-1} \cdot U \\ &= E_{12}(12) \cdot E_{13}(3) \cdot E_{23}(-.1) \cdot U \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 3 & -.1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -10 & -15 \\ 0 & 0 & -3.5 \end{pmatrix} \\ &= LU. \end{aligned}$$

Notice that the negatives of the below-diagonal entries of $L$ reflect the elementary row operations used to reduce $A$ to $U$.

## Mathematical Induction

The proof of Theorem 8.16 is not completely rigorous, because the statement "repeating this argument as many times as is necessary" is a bit vague. How do we know that we can really do this? How many times are necessary? There is a formal technique for making this argument, which is called the *principle of mathematical induction*. The principle of mathematical induction is described fully in the first Appendix of this book. Here we will only show how we would apply it in the proof of Theorem 8.16.

In the proof of Theorem 8.16 we want to show that, for all $k$, the matrix product of $k$ lower-triangular matrices $L_1 \cdot L_2 \cdots L_k$ is lower triangular. This statement is clearly true when $k = 1$. Lemma 8.2 tells us that the statement is true for $k = 2$. For $k = 3$, we write $L_1 \cdot L_2 \cdot L_3$ as $(L_1 \cdot L_2) \cdot L_3$. Since the statement is true for $k = 2$, $(L_1 \cdot L_2)$ is lower triangular. Lemma 8.2 then assures us that the product $L_1 \cdot L_2 \cdot L_3$ is lower triangular, and so on.

To formalize this argument, we divide it into two steps:

(1) the product of two lower-triangular matrices is lower triangular, and
(2) *if* the product of $k$ lower-triangular matrices is lower triangular, *then* the product of $k + 1$ lower-triangular matrices is lower triangular.

Statements 1 and 2 are true by Lemma 8.2. Taken together, statements 1 and 2 allow us to conclude that the product of an arbitrary number $k$ of lower triangular matrices is lower triangular. First, let $k = 2$ in 2, then 1 and 2 imply that the statement is true for $k = 3$. Next, let $k = 3$ in 2 to conclude that the statement is true for $k = 4$, and so on. Statement 2 is called the **inductive hypothesis**. This proof by induction is a bootstrap method that is often used to prove propositions of the form: statement P($k$) is true for every positive integer $k$.

## Including Row Interchanges

In the hypothesis of Theorem 8.16 we assumed that no row interchanges were needed to reduce $A$ to its row echelon form. Of course this is not always the case, and so we would like to know what happens to the conclusions of Theorem 8.16 when row interchanges are required. First consider the case of nonsingular $A$. The answer is very simple (although the proof is sufficiently tricky that we will only sketch it here). Row interchanges are required only because, at some stage in the reduction process, there arises a pivot whose value is 0. So, reduce $A$ to its row echelon form, keeping track of the row interchanges that are required. Suppose now that these row interchanges were to be made *before* we began the reduction. Then all the pivots would be in the right places, and no 0 pivots would be encountered in the row reduction process. How do we swap the rows of $A$?

The row interchanges can be accomplished by premultiplying $A$ by permutation matrices — $E_{ij}$ matrices. The product of permutation matrices is a permutation matrix (Exercise 8.10), and so, to eliminate the need for row interchanges during the reduction process, we can just premultiply $A$ by the appropriate permutation matrix $P$. Thus there exists a permutation matrix $P$, an upper-triangular matrix $U$, and a lower-triangular matrix $L$ such that $PA = LU$.

When $A$ is singular, the story is not much different. Here, when a 0 pivot is encountered, it may not be possible to replace it with a nonzero pivot using a row interchange. Everything below the pivot may also be 0. This presents no problem; just go on to the next column. Of course some 0 pivots may have nonzero elements below them, so row interchanges may still be required. Nonetheless, our conclusions are not altered. We summarize them in the following theorem:

**Theorem 8.17** Let $A$ be a general $k \times n$ matrix. Then one can write $PA = LU$ where $P$ is a $k \times k$ permutation matrix, $L$ is a $k \times k$ lower-triangular matrix with only 1s on the diagonal, and $U$ is a $k \times n$ upper-triangular matrix.

## EXERCISES

**8.42** For each of the following matrices $A$, write down the string of elementary matrices which are needed to transform $A$ to its row echelon form.

*a)* $\begin{pmatrix} 2 & 4 \\ -6 & -13 \end{pmatrix}$, *b)* $\begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix}$,

*c)* $\begin{pmatrix} 2 & 4 & 0 & 1 \\ 4 & 6 & 3 & 3 \\ -6 & -10 & 0 & 4 \end{pmatrix}$, *d)* $\begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 21 & 8 & 17 \\ 0 & -3 & -12 & 2 \\ 4 & 12 & -4 & 13 \end{pmatrix}$.

**8.43** Write down the $LU$ decomposition of each matrix in Exercise 8.42.

**8.44** Show that the $LU$ decomposition of $A$ is unique if $A$ is square, invertible, and satisfies the hypotheses of Theorem 8.16.
[Hint: Write $A = L_1U_1 = L_2U_2$, where the $L_i$ are invertible and lower triangular, with 1s on the diagonal. Show that the $U_i$ are invertible and write $L_2^{-1}L_1 = U_2U_1^{-1}$. Conclude that both sides are diagonal and that the left side is in fact the identity matrix.]

**8.45** Show that the $LU$ decomposition of the $k \times n$ matrix $A$ satisfying the hypothesis of Theorem 8.16 is unique if $A$ has maximal rank.
[Hint: As in the previous exercise, write $L_2^{-1}L_1U_1 = U_2$. Check that $U_1$ and $U_2$ have no 0 rows and then show that the equation $L_2^{-1}L_1U_1 = U_2$ implies that $L_2^{-1}L_1$ is the identity matrix.]

**8.46** Show by example that if $A$ does not have maximal rank, then the $LU$ decomposition of $A$ need not be unique.

**8.47** Prove the following proposition: If $A$ is a square, nonsingular matrix and if row reduction of $A$ requires no row interchanges, then $A$ can be written uniquely as $A = LDU$ where $L$ and $U$ are lower- and upper-triangular matrices, respectively, with only 1s on their diagonals and $D$ is a diagonal matrix. The diagonal entries of $D$ are precisely the pivots of $A$.
[Hint: Start with the $LU$ decomposition of $A$ and decompose $U$ into the product of two matrices, each with the desired properties.]

**8.48** Find the $LDU$ decomposition for the matrices in Exercise 8.42*a*, *b*, *d*.

**8.49** The following two matrices require row interchanges to achieve their row echelon forms. For each matrix $A$:

*a*) Compute the row-echelon form.
*b*) Construct the permutation matrix $P$ which corresponds to these row interchanges.
*c*) Compute the row echelon form of $PA$ and compare your answer to that of part *a*.
*d*) Find the $LU$ decomposition of $PA$.

$$i)\ \begin{pmatrix} 3 & 2 & 0 \\ 6 & 4 & 1 \\ -3 & 4 & 1 \end{pmatrix}, \qquad ii)\ \begin{pmatrix} 0 & 1 & 1 & 4 \\ 1 & 1 & 2 & 2 \\ -6 & -5 & -11 & -12 \\ 2 & 3 & -2 & 3 \end{pmatrix}$$

**8.50** *a*) What must be true about the entries of the general $2 \times 2$ matrix if row interchanges are required for reduction to row-echelon form?
*b*) What about the general $3 \times 3$ case?

**8.51** The $LU$ decomposition provides an efficient way to solve a system of linear equations $A\mathbf{x} = \mathbf{b}$ for different values of $\mathbf{b}$. It requires many fewer arithmetic steps than matrix inversion, and it works when $A$ is not square. Use the $LU$ decomposition to rewrite the system of equations as $LU\mathbf{x} = \mathbf{b}$. Now the system can be solved by first letting $U\mathbf{x} = \mathbf{z}$, solving the system of equations $L\mathbf{z} = \mathbf{b}$ for $\mathbf{z}$, and then solving $U\mathbf{x} = \mathbf{z}$ for $\mathbf{x}$. Since both of these systems are triangular, only back substitution is required to solve them.

*a*) Verify that the solutions obtained this way are precisely the solutions to $A\mathbf{x} = \mathbf{b}$.
*b*) Solve the following systems using this technique:

$$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}; \quad \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix};$$

$$\begin{pmatrix} 5 & 3 & 1 \\ -5 & -4 & 1 \\ -10 & -9 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -10 \\ -24 \end{pmatrix}; \quad \begin{pmatrix} 5 & 3 & 1 \\ -5 & -4 & 1 \\ -10 & -9 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -14 \end{pmatrix}$$

## NOTES

For an excellent summary of Leontief's study, see W. Leontief, "The structure of the U.S. economy," *Scientific American* 212 (April 1965). Our discussion of Leontief's 1958 model is adapted from the presentation in Stanley Grossman, *Applied Mathematics for the Management, Life, and Social Sciences* (Belmont, Calif.: Wadsworth, 1983). Our proof of Theorem 8.14 is adapted from Carl Simon, "Some Fine-Tuning for Dominant Diagonal Matrices," *Economic Letters* 30 (1989), 217–221.

CHAPTER 9

# Determinants: An Overview

The most important matrices in economic models are square matrices, in which the number of unknowns equals the number of equations. For example, all the matrices for economic analysis listed in the first paragraph of Chapter 8 are square matrices. The most important square matrices are the nonsingular ones. These are precisely the coefficient matrices $A$ such that the system of $n$ equations in $n$ unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \qquad \vdots \qquad \qquad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

or in matrix notation $A\mathbf{x} = \mathbf{b}$, has one and only one solution for each right-hand side $\mathbf{b}$. As we saw in the last chapter, these are also the matrices which are invertible. Since not all square matrices are nonsingular, we will describe in this chapter a straightforward test to *determine* whether or not a given matrix is nonsingular. In particular, for any square matrix we will define a number called the *determinant*, with the property that the square matrix is nonsingular if and only if its determinant is not zero. Later we will use the determinant for other tasks, for example, for developing an *explicit formula* for the solution of (1) in terms of the $a_{ij}$'s and $b_i$'s, for deriving a formula for the inverse of a matrix, and for classifying the behavior of quadratic functions.

Many mathematical models in economics center around constrained maximization or minimization problems. Determinants play a role here too, because the second order condition for such problems requires that one check the signs of determinants of certain matrices of second derivatives.

The determinant can be a fairly complex expression. For a general $n \times n$ matrix there are $n!$ terms, each the product of $n$ different entries of the matrix. Some of the proofs of its properties are also fairly complex. Consequently, this chapter presents a comprehensive *overview* of the determinant: how to compute it and how to use it, with relatively little motivation and no complex proofs. Chapter 26 contains a complete analysis of the determinant, including proofs of its important properties and major uses. Depending on the amount of detail with which one wants to cover determinants, one can: 1) read this chapter and skip Chapter 26, at least for the time being; 2) read Chapter 26 now and skip this chapter; or 3) read this chapter as an overview on determinants, follow it with a careful reading of Chapter 26, and then return to Chapter 10.

## 9.1 THE DETERMINANT OF A MATRIX

### Defining the Determinant

The determinant of a matrix is defined inductively. There is a natural definition for $1 \times 1$ matrices. Then, we use this definition to define the determinant of $2 \times 2$ matrices. Once we have defined the determinant for $2 \times 2$ matrices, we use this definition to define the determinant for $3 \times 3$ matrices, and so on.

A $1 \times 1$ matrix is just a scalar $(a)$. Since the inverse of $a$, $1/a$, exists if and only if $a$ is nonzero, it is natural to define the determinant of such a matrix to be just that scalar $a$:

$$\det(a) = a.$$

For a $2 \times 2$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

Theorem 8.8 states that $A$ is nonsingular if and only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Therefore, we define the determinant of a $2 \times 2$ matrix $A$:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{2}$$

Notice that (2) is just the product of the two diagonal entries minus the product of the two off-diagonal entries. In order to motivate the general definition of a determinant, we write (2) as follows:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \det(a_{22}) - a_{12} \det(a_{21}). \tag{3}$$

The first term on the right-hand side of (3) is the $(1, 1)$th entry of $A$ times the determinant of the submatrix obtained by deleting from $A$ the row and column which contain that entry; the second term is the $(1, 2)$th entry times the determinant of the submatrix obtained by deleting from $A$ the row and column which contain that entry. The terms alternate in sign; the term containing $a_{11}$ receives a plus sign and the term containing $a_{12}$ receives a minus sign.

The following definitions will simplify the task of defining the determinant of an $n \times n$ matrix.

**Definition** Let $A$ be an $n \times n$ matrix. Let $A_{ij}$ be the $(n-1) \times (n-1)$ submatrix obtained by deleting row $i$ and column $j$ from $A$. Then, the scalar

$$M_{ij} \equiv \det A_{ij}$$

is called the $(i, j)$th **minor** of $A$ and the scalar

$$C_{ij} \equiv (-1)^{i+j} M_{ij}$$

is called the $(i, j)$th **cofactor** of $A$. A cofactor is a signed minor. Note that $M_{ij} = C_{ij}$ if $(i + j)$ is even and $M_{ij} = -C_{ij}$ if $(i + j)$ is odd.

Formula (3) can be written as

$$\det A = a_{11}M_{11} - a_{12}M_{12} = a_{11}C_{11} + a_{12}C_{12}.$$

We use this expression as motivation for the definition of the determinant of a $3 \times 3$ matrix.

**Definition** The **determinant** of a $3 \times 3$ matrix is given by

$$\begin{aligned}
\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\
&= a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \\
&\quad + a_{13} \cdot \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.
\end{aligned}$$

The $j$th term on the right-hand side of the definition is $a_{1j}$ times the determinant of the submatrix obtained by deleting row 1 and column $j$ from $A$. The term is preceeded by a plus sign if $1 + j$ is even and by a minus sign if $1 + j$ is odd.

**Definition** The **determinant** of an $n \times n$ matrix $A$ is given by

$$\begin{aligned}
\det A &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\
&= a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{n+1}a_{1n}M_{1n}.
\end{aligned} \tag{4}$$

**Notation** In referring to the determinant of a $n \times n$ matrix $A$, one sometimes writes

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{for} \quad \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

and $|A|$ for $\det A$.

### Computing the Determinant

Our definition of the determinant of a matrix involves expanding along its first row. There is nothing special about the first row. It turns out that one can use any row or column to compute the determinant of a matrix. For example, if one uses, say, the *second column* to compute the determinant of a $3 \times 3$ matrix, one computes

$$\det A = -a_{12} \cdot \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{22} \cdot \det\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} - a_{32} \cdot \det\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix},$$

or equivalently,

$$\det A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}. \tag{5}$$

The $j$th term on the right-hand side of (5) is $a_{j2}$ times the determinant of the submatrix obtained by deleting the row and column of $A$ which contains $a_{j2}$; it is preceeded by a plus sign if $(j + 2)$ is even and by a minus sign if $(j + 2)$ is odd.

In general, the determinant of an $n \times n$ matrix involves $n!$ terms, each a product of $n$ entries. This can be a time-consuming computation. There are certain classes of matrices whose determinants are easy to compute, as the following theorem illustrates.

**Theorem 9.1** The determinant of a lower-triangular, upper-triangular, or diagonal matrix is simply the product of its diagonal entries.

*Example 9.1* For a lower- or upper-triangular $2 \times 2$ matrix $A$, $a_{12} = 0$ or $a_{21} = 0$. Therefore, by (2)

$$\det A = a_{11}a_{22} - 0 = a_{11}a_{22}.$$

For a lower-triangular $3 \times 3$ matrix, use the definition to compute

$$\det\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}C_{11} + 0 \cdot C_{12} + 0 \cdot C_{13}$$

$$= a_{11} \det\begin{pmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33}.$$

Theorem 9.1 along with the following theorem often leads to simpler calculations of $\det A$.

**Theorem 9.2** Let $A$ be an $n \times n$ matrix and let $R$ be its row echelon form. Then

$$\det A = \pm \det R.$$

If no row interchanges are used to compute $R$ from $A$, then $\det A = \det R$.

One can frequently combine the previous two theorems to compute $\det A$ more efficiently. First, convert $A$ to its row echelon form $R$. Since $R$ is an upper-triangular matrix, its determinant is simply the product of its diagonal entries.

**Remark** There is an easy-to-remember mnemonic device for computing the determinant of a $3 \times 3$ matrix $A$, that **works only for** $3 \times 3$ **matrices**. Form the partitioned matrix $\hat{A}$ by recopying the first and second rows of $A$ right below $A$, as in Figure 9.1. Starting from $a_{11}$ at the top left corner of $\hat{A}$, add together the three products along the three "diagonals" indicated by the solid lines in Figure 9.1:

$$a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}. \tag{6}$$

Then, starting from $a_{21}$ at the bottom left corner of $\hat{A}$, subtract from (6) the three products along the three "counterdiagonals" indicated by the dotted lines in Figure 9.1:

$$-a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23} - a_{31}a_{22}a_{13}. \tag{7}$$

The result (6) + (7) is the determinant of $A$.

*Example 9.2* Using this method, it is easy to see that

$$\begin{aligned} \det\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} &= 0 \cdot 4 \cdot 8 + 3 \cdot 7 \cdot 2 + 6 \cdot 1 \cdot 5 \\ &\quad - 3 \cdot 1 \cdot 8 - 0 \cdot 7 \cdot 5 - 6 \cdot 4 \cdot 2 \\ &= 0 + 42 + 30 - 24 - 48 \\ &= 0. \end{aligned}$$

## Main Property of the Determinant

Finally, we put the above facts about determinants together to derive the main property of the determinant — the determinant *determines* whether or not a square matrix is nonsingular.

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

*Computing the determinant of a* $3 \times 3$ *matrix.* **Figure 9.1**

> **Theorem 9.3** A square matrix is nonsingular if and only if its determinant is nonzero.

*Proof Sketch* Recall that a square matrix $A$ is nonsingular if and only if its row echelon form $R$ has no all-zero rows. Since each row of the square matrix $R$ has more leading zeros than the previous row, $R$ has no all-zero rows if and only if the $j$th row of $R$ has exactly $(j - 1)$ leading zeros. This occurs if and only if $R$ has no zeros on its diagonal. Since $\det R$ is the product of its diagonal entries, $A$ is nonsingular if and only if $\det R$ is nonzero. Since $\det R = \pm \det A$, $A$ is nonsingular if and only if $\det A$ is nonzero. ■

Theorem 9.3 is obvious for $1 \times 1$ matrices, because the equation $ax = b$ has a unique solution, $x = b/a$, for every $b$ if and only if $a \neq 0$. Theorem 8.8 demonstrates Theorem 9.3 for $2 \times 2$ matrices.

## EXERCISES

**9.1** Write out the complete expression for the determinant of a $3 \times 3$ matrix — six terms, each a product of three entries.

**9.2** Write out the definition of the determinant of a $4 \times 4$ matrix in terms of the determinants of certain of its $3 \times 3$ submatrices. How many terms are there in the complete expansion of the determinant of a $4 \times 4$ matrix?

**9.3** Compute out the expression on the right-hand side of (5). Show that it equals the expression calculated in Exercise 9.1.

**9.4** Show that one obtains the same formula for the determinant of a $2 \times 2$ matrix, no matter which row or column one uses for the expansion.

**9.5** Use a formula for the determinant to verify Theorem 9.1 for upper-triangular $3 \times 3$ matrices.

**9.6** Verify the conclusion of Theorem 9.2 for $2 \times 2$ matrices by showing that the determinant of a general $2 \times 2$ matrix is not changed if one adds $r$ times row 1 to row 2.

**9.7** For each of the following matrices, compute the row echelon form and verify the conclusion of Theorem 9.2:

$$a)\ \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \qquad c)\ \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 7 & 8 \end{pmatrix}.$$

**9.8** Use the observation following Theorem 9.2 to carry out a quick calculation of the determinant of each of the following matrices:

$$a)\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 1 & 9 & 6 \end{pmatrix}.$$

**9.9** Use Theorem 9.3 to determine which of the matrices in Exercises 9.7 and 9.8 are nonsingular.

## 9.2 USES OF THE DETERMINANT

Since the determinant tells whether or not $A^{-1}$ exists and whether or not $A\mathbf{x} = \mathbf{b}$ has a unique solution, it is not surprising that one can use the determinant to derive a formula for $A^{-1}$ and a formula for the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$. First, we define the adjoint matrix of $A$ as the transpose of the matrix of cofactors of $A$.

**Definition** For any $n \times n$ matrix $A$, let $C_{ij}$ denote the $(i, j)$th cofactor of $A$, that is, $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting row $i$ and column $j$ from $A$. The $n \times n$ matrix whose $(i, j)$th entry is $C_{ji}$, the $(j, i)$th cofactor of $A$ (note the switch in indices), is called the **adjoint** of $A$ and is written **adj** $A$.

**Theorem 9.4** Let $A$ be a nonsingular matrix. Then,

(*a*) $A^{-1} = \dfrac{1}{\det A} \cdot \operatorname{adj} A$, and

(*b*) (**Cramer's rule**) the unique solution $\mathbf{x} = (x_1, \cdots, x_n)$ of the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ is

$$x_i = \frac{\det B_i}{\det A}, \qquad \text{for } i = 1, \ldots, n,$$

where $B_i$ is the matrix $A$ with the right-hand side $\mathbf{b}$ replacing the $i$th column of $A$.

For $3 \times 3$ systems,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

Cramer's rule states that

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

*Example 9.3* Use Theorem 9.4 to invert the matrix

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{8}$$

$$C_{11} = +\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3, \quad C_{12} = -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0, \quad C_{13} = +\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -3,$$

$$C_{21} = -\begin{vmatrix} 4 & 5 \\ 0 & 1 \end{vmatrix} = -4, \quad C_{22} = +\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = -3, \quad C_{23} = -\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 4,$$

$$C_{31} = +\begin{vmatrix} 4 & 5 \\ 3 & 0 \end{vmatrix} = -15, \quad C_{32} = -\begin{vmatrix} 2 & 5 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{33} = +\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} = 6,$$

$$\det A = -9,$$

$$\operatorname{adj} A = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 3 & -4 & -15 \\ 0 & -3 & 0 \\ -3 & 4 & 6 \end{pmatrix}$$

So,

$$A^{-1} = -\frac{1}{9}\begin{pmatrix} 3 & -4 & -15 \\ 0 & -3 & 0 \\ -3 & 4 & 6 \end{pmatrix}. \tag{9}$$

*Example 9.4* We can use Cramer's rule to calculate $x_3$ for the system in Example 7.1, which we write in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -4 \end{pmatrix}.$$

The determinant of the coefficient matrix $A$ is 35. The determinant of

$$B_3 = \begin{pmatrix} 1 & 1 & 0 \\ 12 & 2 & 5 \\ 3 & 4 & -4 \end{pmatrix}$$

is also 35. Thus, $x_3 = |B_3|/|A| = 1$.

Finally, we note three algebraic properties of the determinant function which we will find important in our use of determinants.

**Theorem 9.5** Let $A$ be a square matrix. Then,

(*a*) $\det A^T = \det A$,
(*b*) $\det(A \cdot B) = (\det A)(\det B)$, and
(*c*) $\det(A + B) \neq \det A + \det B$, in general.

Gaussian elimination is a much more efficient method of solving a system of $n$ equations in $n$ unknowns than is Cramer's rule. Cramer's rule requires the evaluation of $(n + 1)$ determinants. Each determinant is a sum of $n!$ terms and each term is a product of $n$ entries. So, Cramer's rule requires $(n + 1)!$ operations. On the other hand, the number of arithmetic operations required by Gaussian elimination for such a system is on the order of $n^3$. If $n = 6$ as in the Leontief model in Section 8.5, then $(n + 1)!$ is 5040, while $n^3$ is only 216; the difference grows exponentially as $n$ increases.

Nevertheless, Cramer's rule is particularly useful for small linear systems in which the coefficients $a_{ij}$ are parameters and for which one wants to obtain a general formula for the endogenous variables (the $x_i$'s) in terms of the parameters and the exogenous variables (the $b_j$'s). One can then see more clearly how changes in the parameters affect the values of the endogenous variables.

## EXERCISES

**9.10** Verify directly that matrix (9) really is the inverse of matrix (8) in Example 9.3.

**9.11** Use Theorem 9.4 to invert the following matrices:

$$a) \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \qquad b) \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}, \qquad c) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

**9.12** Use Cramer's rule to compute $x_1$ and $x_2$ in Example 9.4.

**9.13** Use Cramer's rule to solve the following systems of equations:

$$a) \begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4; \end{aligned} \qquad b) \begin{aligned} 2x_1 - 3x_2 \phantom{{}+x_3} &= 2 \\ 4x_1 - 6x_2 + x_3 &= 7 \\ x_1 + 10x_2 \phantom{{}+x_3} &= 1. \end{aligned}$$

**9.14** Verify the conclusions of Theorem 9.5 for the following pairs of matrices:

$$a) \; A = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix};$$

b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$;

c) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

## 9.3 IS-LM ANALYSIS VIA CRAMER'S RULE

As an illustrative example, consider the linear IS-LM national income model described in Chapter 6:

$$\begin{aligned} sY + ar &= I^o + G \\ mY - hr &= M_s - M^o \end{aligned} \tag{10}$$

where $Y$ = net national product
$r$ = interest rate
$s$ = marginal propensity to save,
$a$ = marginal efficiency of capital,
$I$ = investment $(= I^o - ar)$,
$m$ = money balances needed per dollar of transactions,
$G$ = government spending,
$M_s$ = money supply.

All the parameters are positive. Because the coefficients in this system are parameters instead of numbers, it is easiest to solve (10) using Cramer's rule:

$$Y = \frac{\begin{vmatrix} I^o + G & a \\ M_s - M^o & -h \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}} = \frac{(I^o + G)h + a(M_s - M^o)}{sh + am}$$

$$r = \frac{\begin{vmatrix} s & I^o + G \\ m & M_s - M^o \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}} = \frac{(I^o + G)m - s(M_s - M^o)}{sh + am}.$$

One can now use these expressions to see that, in this model, an increase in $I^o$, $G$, or $M_s$ or a decrease in $M^o$ or $m$ will lead to an increase in the equilibrium net product $Y$. An increase in $I^o$ or $M^o$ or a decrease in $M_s$, $h$, or $m$ will lead to an increase in equilibrium interest rate $r$.

## EXERCISES

**9.15** Verify the assertions in the last two sentences before these exercises.

**9.16** If you are familiar with partial derivatives, compute

$$\frac{\partial Y}{\partial a} = \frac{-rh}{(sh + am)} \leq 0.$$

So an increase in the marginal efficiency of capital $a$ will bring down the equilibrium $Y$ and $r$. How will the equilibrium $Y$ change if $h$ increases? How will the equilibrium $r$ change if $m$ or $s$ increases?

**9.17** If we introduce tax rate $t$ and let the consumption function depend on after-tax income, $C = b(Y - tY)$, then system (10) becomes

$$\begin{aligned}(1 - t)sY + ar &= I^o + G \\ mY - hr &= M_s - M^o.\end{aligned}$$

Use Cramer's rule to see how the equilibrium $Y$ and $r$ are affected by the tax rate $t$.

**9.18** Consider the following more elaborate linear IS-LM.

$a)\ Y = C + I + G$ $\qquad$ $b)\ C = c_0 + c_1(Y - T) - c_2 r$

$c)\ T = t_0 + t_1 Y$ $\qquad$ $d)\ I = I^o + a_0 Y - ar$

$e)\ M_s = mY + M^o - hr.$

Substitute $c$ into $b$ to obtain $b'$; then substitute $b'$ and $d$ into $a$ to get the new IS-curve. Combine this with $e$ and use Cramer's rule to solve this system for $Y$ and $r$ in terms of the exogenous variables. Show that an increase in $G$ or a reduction of $t_0$ or $t_1$ will increase $Y$; in macroeconomic terms, Keynesian fiscal policy "works" in this model. Show that these changes also increase $r$. Regarding monetary policy, show that an increase in $M_s$ increases $Y$ and lowers $r$.

**9.19** What is the effect of an increase in $I^o$, $c_0$, or $m$?

**9.20** For Example 1 in Chapter 6, write out the linear system which corresponds to equation (1) in Chapter 6, but with a general before-tax profit $P$, a general contribution percentage $c$, and general state and federal tax rates $r$ and $f$. Use Cramer's rule to compute $C$, $S$, and $F$ in terms of $P$, $c$, $s$, and $f$.

C H A P T E R 1 0

# Euclidean Spaces

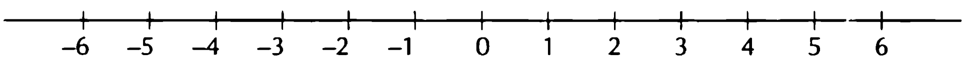
As we discussed at the end of Chapter 1, one of the main uses of mathematical analysis in economic theory is to help construct the appropriate geometric and analytic generalizations of the two-dimensional geometric models that are the mainstay of undergraduate economics courses. In this chapter, we begin these constructions by studying how to generalize notions of points, lines, planes, distances, and angles to $n$-dimensional Euclidean spaces. Later, our analyses of $n$-commodity economies will make heavy use of these concepts.

The first three sections of this chapter present the basic geometry of coordinates, points and displacements in $n$-space. If this material is familiar to most students, it can be left as a background reading assignment.

## 10.1 POINTS AND VECTORS IN EUCLIDEAN SPACE

### The Real Line

The simplest geometric object is the number line — the geometric realization of the set of all real numbers. The number line was defined carefully at the beginning of Chapter 2. Every real number is represented by exactly one point on the line, and each point on the line represents one and only one number. Figure 10.1 shows part of a number line.




*The Real Line.*

**Figure 10.1**

### The Plane

In some of our economic examples, we have used pairs of numbers to represent economic objects, for example, consumption bundles in Chapter 1. Pairs of numbers also have a geometric representation, called the **Cartesian plane** or **Euclidean 2-space**, and written as $\mathbf{R}^2$. To depict $\mathbf{R}^2$, first draw two perpendicular number lines: one horizontal to represent the first **component** $x_1$ of the pair $(x_1, x_2)$ and the other vertical to represent the second component $x_2$ of $(x_1, x_2)$. The unit length is usually the same along each line (although it need not be). These two

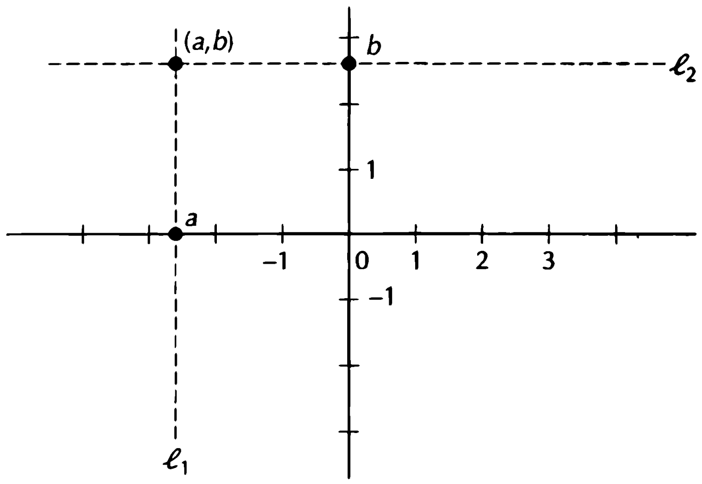



**Figure 10.2** *Identifying a point in the plane with an ordered pair.*

number lines are called **coordinate axes**. They intersect at their origins. Figure 10.2 shows how each point in the plane is identified with a unique pair of numbers. We have used the Cartesian plane in Chapter 2 to draw graphs of functions of one variable.

A point **p** in the Cartesian plane represents a pair of numbers $(a, b)$ as follows: draw a vertical line $\ell_1$ and a horizontal line $\ell_2$ through the point **p**. The vertical line crosses the $x_1$-axis at $a$, and the horizontal line crosses the $x_2$-axis at $b$. We associate the pair $(a, b)$ with the point **p**. To go the other way — to find the point **p** which represents the pair $(a, b)$ — find $a$ on the $x_1$-axis, and through it draw the vertical line $\ell_1$. Find $b$ on the $x_2$-axis, and through it draw the horizontal line $\ell_2$. The intersection of the two lines $\ell_1$ and $\ell_2$ is the point **p**, which we will sometimes write as $\mathbf{p}(a, b)$. The number $a$ is called the $x_1$**-coordinate** of **p**, and $b$ is called the $x_2$**-coordinate** of **p**. In Figure 10.3 we show a number of points and their coordinates.

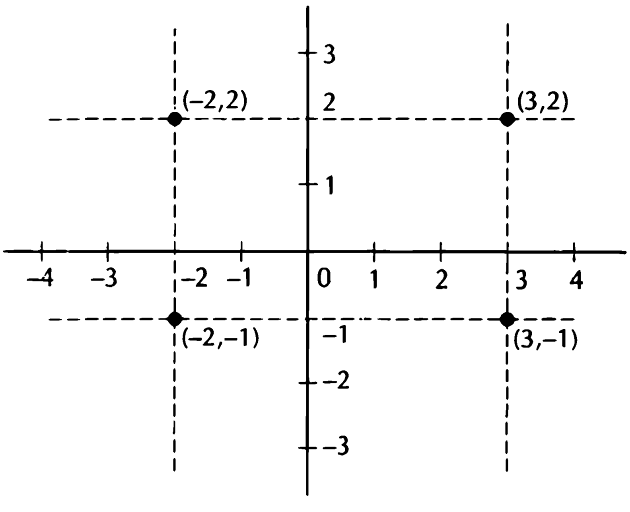


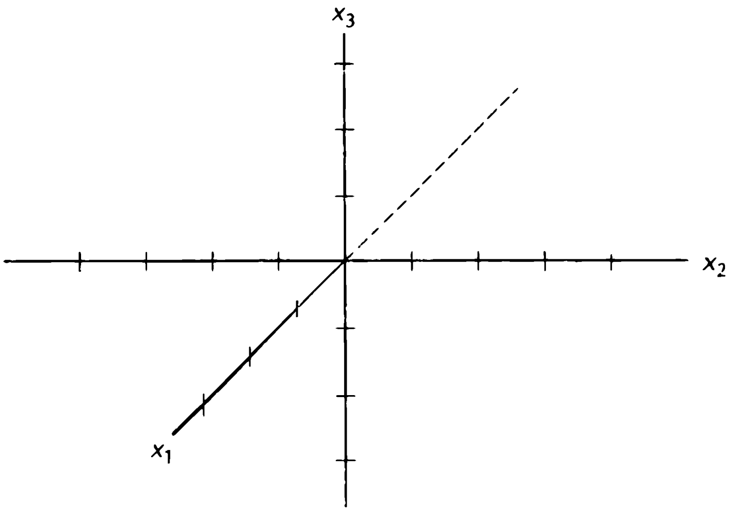

**Figure 10.3** *Coordinates of points in* $\mathbf{R}^2$.

The point of intersection of the horizontal and vertical number lines is our reference point for measuring the location of **p**. It is called the **origin**, and we denote it by the symbol **0**, since it is represented by the pair (0, 0).

## Three Dimensions and More

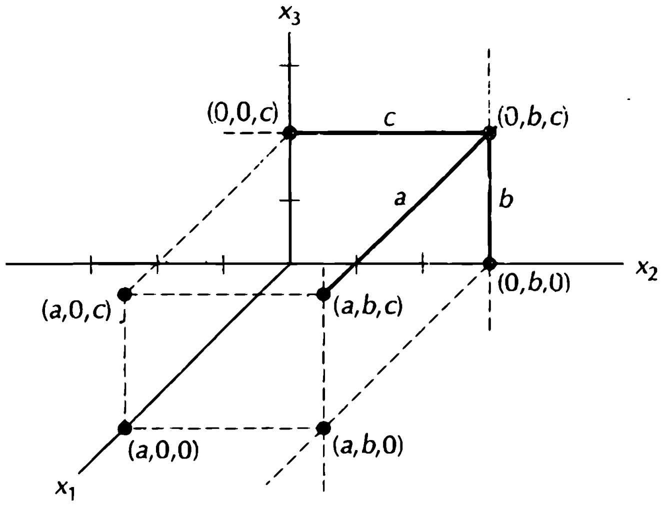
Similarly, one can visualize 3-dimensional Euclidean space $\mathbf{R}^3$ by drawing three mutually perpendicular number lines. As before, each one of these number lines is called a **coordinate axis**: the $x_1$-axis, the $x_2$-axis, and the $x_3$-axis, respectively. One usually draws the $x_2$-axis as the horizontal axis and the $x_3$-axis as the vertical axis on the plane of the page and then pictures the $x_1$-axis as coming out of the page toward one, as in Figure 10.4.

The process of identifying a point with a particular triple of numbers uses the techniques that we used in $\mathbf{R}^2$. The process is illustrated in Figure 10.5. To




*The coordinate axes in* $\mathbf{R}^3$.

**Figure 10.4**




*The point* **p** *with coordinates* $(a, b, c)$.

**Figure 10.5**

find the point represented by the triple $(a, b, c)$, forget about $a$ for a moment, and locate the point representing $(b, c)$ in the $x_2x_3$-plane — the plane of the page. This is a 2-space exercise that we already know how to do. From the point $(b, c)$ in the plane of the page, move $a$ units in the direction parallel to the $x_1$-axis. March out of the page if $a$ is positive, and march behind the page if $a$ is negative. If $a$ is 0, remain where you are. The point **p** at which you finish represents $(a, b, c)$ and is sometimes denoted by $\mathbf{p}(a, b, c)$. We could have just as easily started in the $x_1x_3$-plane and then moved $b$ units to the right (for positive $b$), or in the $x_1x_2$-plane and then moved $c$ units up (for positive $c$). Check to see that you end up at the same point no matter which method you use.

Finding the coordinates that describe a particular point **p** is just as easy. Starting from **p**, move parallel to the $x_1$-axis until you reach $x_2x_3$-plane. The distance moved is $a$; it is positive if the move was into the page and negative if the move was out toward you. The coordinates $b$ and $c$ are now found using the 2-space technique. Again, the answer is independent of which plane you head for first. This description and the accompanying diagram (Figure 10.5) is an example of a situation where a picture is worth a thousand words.

Of course, we cannot draw geometric pictures of higher-dimensional Euclidean spaces, but we can use our pictures of $\mathbf{R}^1$, $\mathbf{R}^2$, and $\mathbf{R}^3$ to guide our intuition. We will see that the formulas describing geometric objects and their properties in $\mathbf{R}^2$ and $\mathbf{R}^3$ generalize readily to higher dimensions. The real line $\mathbf{R}^1$ consists of single numbers. The plane $\mathbf{R}^2$ consists of **ordered pairs** of numbers. We say *ordered* pairs because the order of the numbers matters; (1, 0) is not the same as (0, 1). Euclidean $n$-space consists of ordered $n$-tuples of numbers — ordered lists of $n$ numbers. For example, Euclidean 3-space contains ordered triples $(a, b, c)$ of numbers. Euclidean 5-space contains ordered 5-tuples $(a, b, c, d, e)$. Euclidean $n$-space is usually referred to as $\mathbf{R}^n$. The number $n$ in $\mathbf{R}^n$ refers to how many numbers are needed to describe each location. It is called the **dimension** of $\mathbf{R}^n$. Thus $\mathbf{R}^5$ has 5 dimensions, while $\mathbf{R}^2$ has only two dimensions. Each space will have its origin, the point with respect to which we make our coördinate measurements. As we did in $\mathbf{R}^2$, we will always refer to the origin by the symbol $\mathbf{0}$.

## 10.2 VECTORS

Euclidean spaces are useful for modeling a wide variety of economic phenomena because $n$-tuples of numbers have many useful interpretations. Thus far we have emphasized their interpretation as locations, or points in $n$-space. For example, the point (3, 2) represents a particular location in the plane, found by going 3 units to the right and 2 units up from the origin. This is just the way we use coordinates on a map of a country to find the location of a particular city. We use coordinates to describe locations in exactly the same way in higher dimensions. Many economic applications require us to think of $n$-tuples of numbers as locations. For example, we think of consumption bundles as locations in commodity space.

We can also interpret $n$-tuples as **displacements**. This is a useful way of thinking about vectors for doing calculus. We picture these displacements as arrows in $\mathbf{R}^n$. The displacement (3, 2) means: *move* 3 units to the right and 2 units up from your current location. The tail of the arrow marks the initial location; the head marks the location after the displacement is made. In Figure 10.6, each arrow represents the displacement (3, 2), but in each case the displacement is applied to a different initial location.

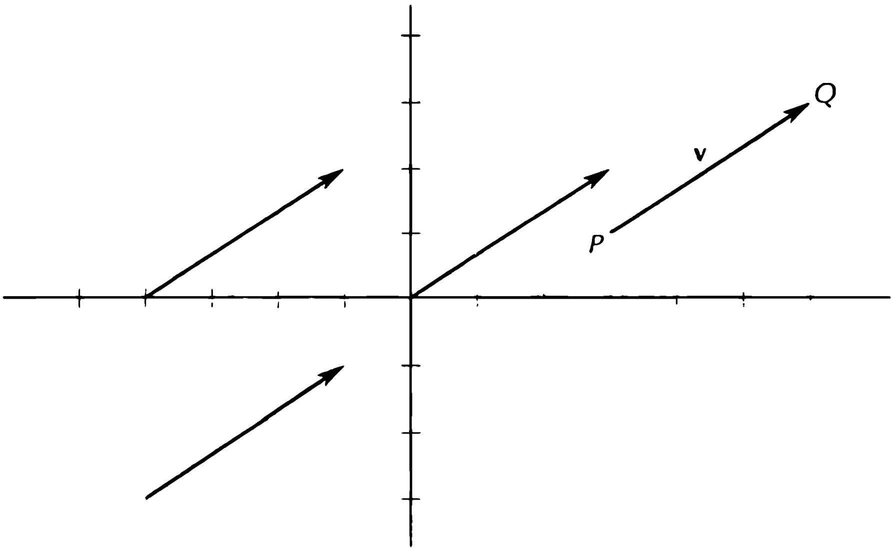



Figure 10.6

*The displacement* (3, 2).

For example, the tail of the displacement labeled $\mathbf{v}$ in Figure 10.6 is at the location (3, 1), and the head is at (6, 3). We will sometimes write $\overrightarrow{PQ}$ for the displacement whose tail is at the point $P$ and head at the point $Q$. Two arrows represent the same displacement if they are parallel and have the same length and direction. For our purposes, two such arrows are equivalent; regardless of their different initial and terminal locations, they both represent the same displacement. The essential ingredients of a displacement are its magnitude and direction.

How do we assign an $n$-tuple to a particular arrow? We measure how far we have to move in each direction to get from the tail to the head of the arrow. For example, consider the arrow $\mathbf{v}$ in Figure 10.6. To get from the tail to the head we have to move 3 units in the $x_1$-direction and 2 units in the $x_2$-direction. Thus $\mathbf{v}$ must represent the displacement (3, 2). More formally, if a displacement goes from the initial location $(a, b)$ to the terminal location $(c, d)$, then the move in the $x_1$-direction is $c - a$, since $a + (c - a) = c$; and the move in the $x_2$-direction is $d - b$, since $b + (d - b) = d$. Thus the displacement is $(c - a, d - b)$. This method of subtracting corresponding coordinates applies to higher dimensions as well. The displacement from the point $\mathbf{p}(a_1, a_2, \ldots, a_n)$ to the point $\mathbf{q}(b_1, b_2, \ldots, b_n)$ in $\mathbf{R}^n$ is written

$$\overrightarrow{\mathbf{pq}} = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n).$$

Figure 10.6 illustrates that there are many (3, 2) displacements. In any given discussion, all the displacements will usually have the same initial location (tail). Often, this initial location will naturally be $\mathbf{0}$, the origin. From this initial location,

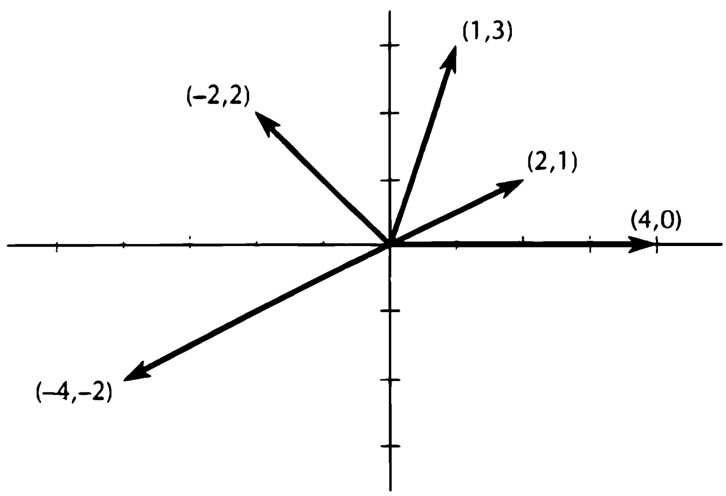



**Figure 10.7** *Some displacements in the plane.*

the *displacement* $(3, 2)$ takes us to the *location* $(3, 2)$. With this "canonical representation" of displacements, we can think of locations as displacements from the origin. Several different displacements are shown in Figure 10.7.

We have just seen that the very different concepts of location and displacement have a common mathematical representation as $n$-tuples of numbers. These concepts act alike mathematically, and so we give them a common name: **vectors.**

Some books distinguish between locations and displacements by writing a location as a row vector $(a, b)$ and a displacement as a column vector $\begin{pmatrix} a \\ b \end{pmatrix}$. This approach is unwieldy and unnecessary. From now on we will use the word "vector" to refer to both locations and displacements. It will either be explicitly mentioned, or clear from the context, whether locations or displacements are meant in any particular discussion.

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## EXERCISES

**10.1** Draw a number line and locate (approximately) the points $1, 3/2, -2, \sqrt{2}, \pi$, and $-\pi/2$.

**10.2** Draw a Cartesian plane and locate on it the following points: $(1, 1)$, $(-1/2, 3/2)$, $(0, 0)$, $(0, -4)$, $(\pi, -\sqrt{2})$.

**10.3** Draw a plane, and show the path you would traverse were you to start at $(-1, 3)$, displace yourself first by the vector $(1, -3)$, and then by the vector $(-1, -3)$.

**10.4** For the points $P$ and $Q$ listed below, draw the corresponding displacement vector $\overrightarrow{PQ}$ and compute the corresponding $n$-tuple for $\overrightarrow{PQ}$:

| | | | | | |
|---|---|---|---|---|---|
| *a*) $P(0,0)$ | and | $Q(2,-1)$, | *b*) $P(3,2)$ | and | $Q(1,1)$, |
| *c*) $P(3,2)$ | and | $Q(5,3)$, | *d*) $P(0,1)$ | and | $Q(3,1)$, |
| *e*) $P(0,0,0)$ | and | $Q(1,2,4)$, | *f*) $P(0,1,0)$ | and | $Q(2,-1,3)$. |

## 10.3 THE ALGEBRA OF VECTORS

There are four basic algebraic operations for the real numbers, $\mathbf{R}^1$: addition, subtraction, multiplication and division. This section introduces the three basic algebraic operations on higher-dimensional Euclidean spaces: vector addition and subtraction and scalar multiplication.

### Addition and Subtraction

We add two vectors just as we add two numbers. We simply add separately the corresponding coordinates of the two vectors. Thus

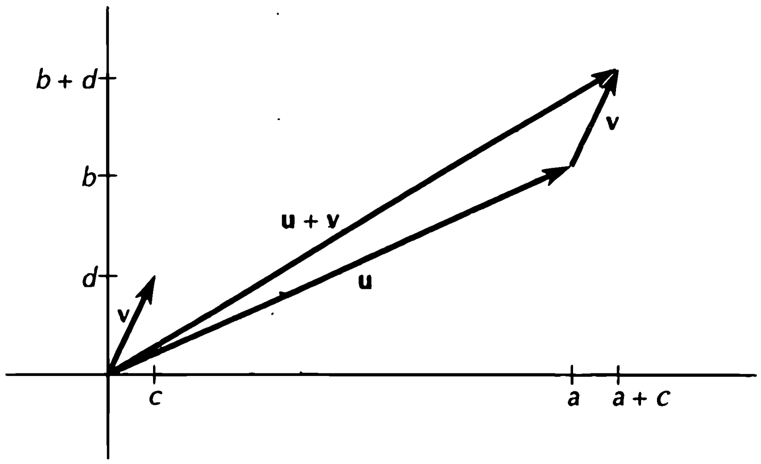
$$(3, 2) + (4, 1) = (7, 3),$$

and

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3).$$

Notice that we can only add together two vectors from the same vector space. The sum $(2, 1) + (3, 4, 1)$ is not defined, since the first vector lives in $\mathbf{R}^2$ while the second vector lives in $\mathbf{R}^3$. Furthermore, the sum of two vectors from $\mathbf{R}^n$ is a vector, and it lives in $\mathbf{R}^n$. When we add $(3, 5, 1, 0) + (0, 0, 0, 1)$ from $\mathbf{R}^4$, we get the vector $(3, 5, 1, 1)$ which is also in $\mathbf{R}^4$.

To develop a geometric intuition for vector addition, it is most natural to think of vectors as displacement arrows. If $\mathbf{u} = (a, b)$ and $\mathbf{v} = (c, d)$ in $\mathbf{R}^2$, then we want $\mathbf{u} + \mathbf{v}$ to represent a displacement of $a + c$ units to the right and $b + d$ units up. Intuitively, we can think of this displacement as follows: Start at some initial location. Apply displacement $\mathbf{u}$. Now apply displacement $\mathbf{v}$ to the terminal location of the displacement $\mathbf{u}$. In other words, move $\mathbf{v}$ until its tail is at the head of $\mathbf{u}$. Then, $\mathbf{u} + \mathbf{v}$ is the displacement from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$, as in Figure 10.8. Verify that $\mathbf{u} + \mathbf{v}$, as drawn, has coordinates $(a + c, b + d)$.




*The sum of two vectors in the plane.* **Figure 10.8**

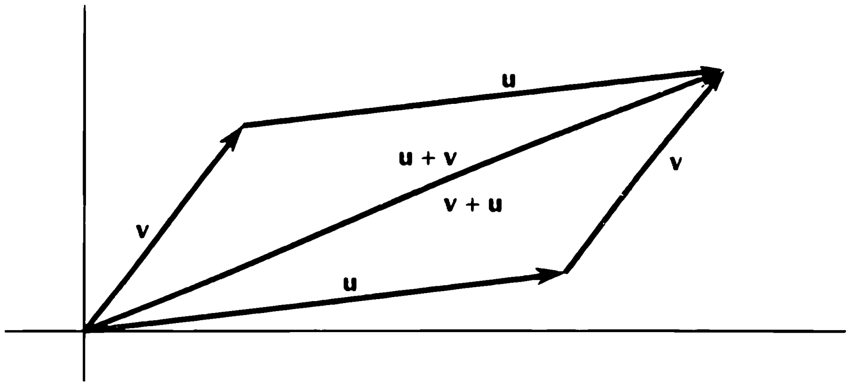



Figure 10.9 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

Figure 10.9 shows that it makes no difference whether we think of $\mathbf{u} + \mathbf{v}$ as displacing first by $\mathbf{u}$ and then by $\mathbf{v}$ or first by $\mathbf{v}$ and then by $\mathbf{u}$. Since the two arrows representing $\mathbf{u}$ in Figure 10.9 are parallel and have the same length and similarly for the two representations of $\mathbf{v}$, the quadrilateral in Figure 10.9 is a parallelogram. Its diagonal represents both $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} + \mathbf{u}$. Formally, Figure 10.9 shows that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$; vector addition, like addition of real numbers, is **commutative**.

One can use the parallelogram in Figure 10.9 to draw $\mathbf{u} + \mathbf{v}$ while keeping the tails of $\mathbf{u}$ and $\mathbf{v}$ at the same point. First, draw the complete parallelogram which has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides, as in Figure 10.9. Then, take $\mathbf{u} + \mathbf{v}$ as the diagonal of this parallelogram with its tail at the common tail of $\mathbf{u}$ and $\mathbf{v}$. Physicists use displacements vectors to represent forces acting at a given point. If vectors $\mathbf{u}$ and $\mathbf{v}$ represent two forces at point $P$, then the vector $\mathbf{u} + \mathbf{v}$ represents the force which results when both forces are applied at $P$ at the same time.

Vector addition obeys the other rules which the addition of real numbers obeys. These are: the associative rule, the existence of a zero (an additive identity), and the existence of an additive inverse. The zero vector is the vector which represents no displacement at all. Analytically we write

$$\mathbf{0} = (0, 0, \ldots, 0).$$

Geometrically, it is a displacement $\overrightarrow{PP}$ having the same terminal point as initial point. Check both algebraically and geometrically that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

If $\mathbf{u} = (a_1, a_2, \ldots, a_n)$, then the negative of $\mathbf{u}$, written $-\mathbf{u}$ and called "minus $\mathbf{u}$", is the vector $(-a_1, -a_2, \ldots, -a_n)$. Geometrically, one interchanges the head and tail of $\mathbf{u}$ to obtain the head and tail of $-\mathbf{u}$. Symbolically, $-\overrightarrow{PQ} = \overrightarrow{QP}$. Check that the algebraic and geometric points of view are consistent and that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

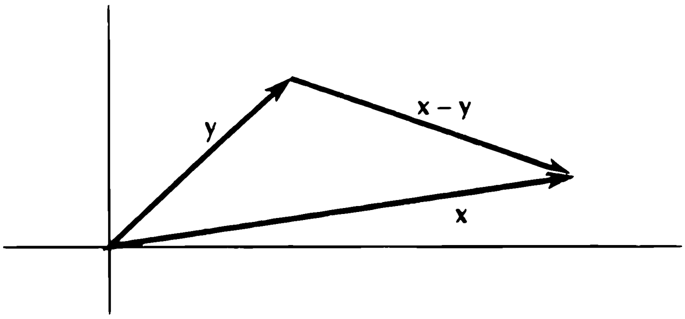
In the real numbers, subtraction is defined by the equation $a - b = a + (-b)$. We can use the same rule to define subtraction for vectors. Thus

$$\begin{aligned}(4, 3, 5) - (1, 3, 2) &= (4, 3, 5) + (-1, -3, -2)\\ &= (4 - 1, 3 - 3, 5 - 2)\\ &= (3, 0, 3).\end{aligned}$$

More generally, for vectors in $\mathbf{R}^n$,

$$(a_1, a_2, \ldots, a_n) - (b_1, b_2, \ldots, b_n) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n).$$

Geometrically we think of subtraction as completing the triangle in Figure 10.8. Given $\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$, find $\mathbf{v}$ to make the diagram work. Put another way, $\mathbf{x} - \mathbf{y}$ is that vector which, when added to $\mathbf{y}$, gives $\mathbf{x}$. Subtraction finds the missing leg of the triangle in Figure 10.10.




*Geometric representation of* $\mathbf{x} - \mathbf{y}$.

**Figure 10.10**

### Scalar Multiplication

It is generally not possible to multiply two vectors in a nice way so as to generalize the multiplication of real numbers. For example, coordinatewise multiplication does not satisfy the basic properties that the multiplication of real numbers satisfies. For one thing, the coordinatewise product of two nonzero vectors, such as (1, 0) and (0, 1), could be the zero vector. When this happens, division, the inverse operation to multiplication, cannot be defined. However, there is a vector space operation which corresponds to statements like, "go twice as far" or "you are halfway there." This operation is called **scalar multiplication.** In it we multiply a vector, coordinatewise, by a real number, or **scalar**. If $r$ is a scalar and $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector, then their product is

$$r \cdot \mathbf{x} = (rx_1, \ldots, rx_n).$$

For example, $2 \cdot (1, 1) = (2, 2)$, and $\frac{1}{2} \cdot (-4, 2) = (-2, 1)$.

Geometrically, scalar multiplication of a displacement vector $\mathbf{x}$ by a nonnegative scalar $r$ corresponds to stretching or shrinking $\mathbf{x}$ by the factor $r$ without changing its direction, as in Figure 10.11. Scalar multiplication by a negative scalar causes not only a change in the length of a vector but also a reverse in direction.

In the algebra of the real numbers, addition and multiplication are linked by the **distributive laws**:

$$a \cdot (b + c) = ab + ac \quad \text{and} \quad (a + b) \cdot c = ac + bc.$$

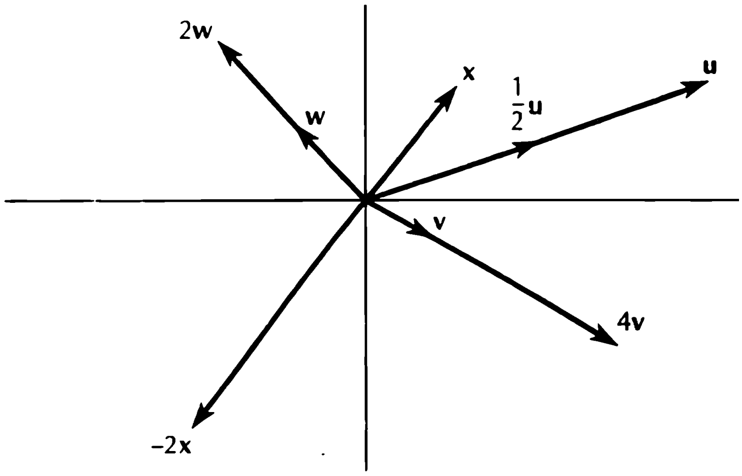



**Figure 10.11** *Scalar multiplication in the plane.*

There are distributive laws in Euclidean spaces as well. It is easy to see that vector addition distributes over scalar multiplication and that scalar multiplication distributes over vector addition:

(*a*) $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$ for all scalars $r, s$ and vectors $\mathbf{u}$.
(*b*) $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$ for all scalars $r$ and vectors $\mathbf{u}, \mathbf{v}$.

Any set of objects with a vector addition and scalar multiplication which satisfies the rules we have outlined in this section is called a **vector space**. The elements of the set are called **vectors**. (The operations of vector addition and scalar multiplication are the operations of matrix addition and scalar multiplication of matrices, respectively, applied to $1 \times n$ or $n \times 1$ matrices, as defined in Section 1 of Chapter 8. The scalar product of the next section will also correspond to a matrix operation.)

## EXERCISES

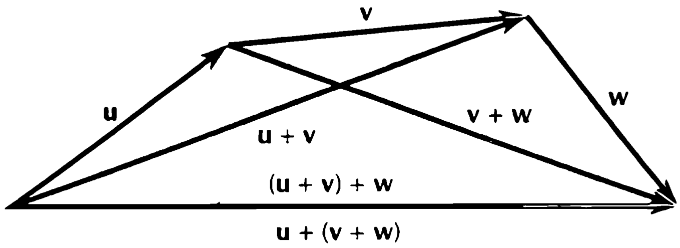
**10.5** Let $\mathbf{u} = (1, 2)$, $\mathbf{v} = (0, 1)$, $\mathbf{w} = (1, -3)$, $\mathbf{x} = (1, 2, 0)$, and $\mathbf{z} = (0, 1, 1)$. Compute the following vectors, whenever they are defined: $\mathbf{u} + \mathbf{v}$, $-4\mathbf{w}$, $\mathbf{u} + \mathbf{z}$, $3\mathbf{z}$, $2\mathbf{v}$, $\mathbf{u} + 2\mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $3\mathbf{x} + \mathbf{z}$, $-2\mathbf{x}$, $\mathbf{w} + 2\mathbf{x}$.

**10.6** Carry out all of the possible operations in Exercise 10.5 *geometrically*.

**10.7** Show that $-\mathbf{u} = (-1)\mathbf{u}$.

**10.8** Prove the distributive laws for vectors in $\mathbf{R}^n$.

**10.9** Use Figure 10.12 to give a geometric proof of the **associative law** for vector addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.




$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$

**Figure 10.12**

## 10.4 LENGTH AND INNER PRODUCT IN $\mathbf{R}^n$

Among the key geometric concepts that guide our analysis of two-dimensional economic models are length, distance and angle. In this section, we describe the $n$-dimensional analogues of these concepts which we will use for more complex, higher dimensional economic models.

When we build mathematical models of economic phenomena in Euclidean spaces, we will often be interested in the geometric properties of these spaces, for example, the distance between two points or the angle between two vectors. In this section we develop the analytical tools needed to study these properties. In fact, all the geometrical results of planar (that is, two-dimensional) Euclidean geometry can be derived using purely analytical techniques. Furthermore, these analytic techniques are all we have for generalizing the results of plane geometry to higher-dimensional Euclidean spaces.

### Length and Distance

The most basic geometric property is distance or length. If $P$ and $Q$ are two points in $\mathbf{R}^n$, we write $\overline{PQ}$ for the line segment joining $P$ to $Q$ and $\overrightarrow{PQ}$ for the vector from $P$ to $Q$.

**Notation** The **length** of line segment $\overline{PQ}$ is denoted by the symbol $\|\overline{PQ}\|$. The vertical lines draw attention to the analogy of length in the plane with absolute value in the line.

We now develop a formula for $\|\overline{PQ}\|$, or equivalently, for the **distance** between points $P$ and $Q$. First, consider the case where $P$ and $Q$ lie in the plane $\mathbf{R}^2$ and have the same $x_2$-coordinate. We have pictured this situation in Figure 10.13, where $P$ has coordinates $(a_1, b)$ and $Q$ has coordinates $(a_2, b)$. The length of this line is clearly the length of the line segment connecting $a_1$ and $a_2$ on the $x_1$-axis. Since length is always a positive number, the length of this segment on the $x_1$-axis is simply $|a_2 - a_1|$. We conclude that $\|\overline{PQ}\| = |a_2 - a_1|$, as in Figure 10.13.

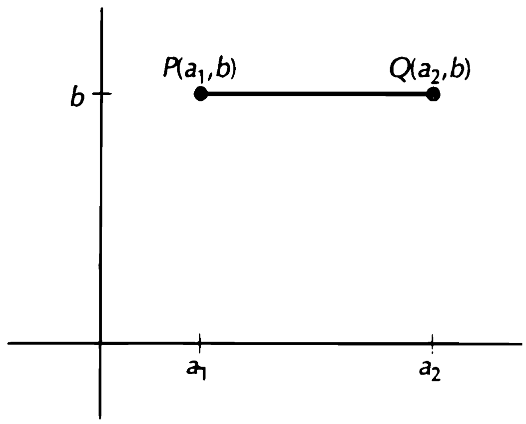



**Figure 10.13** $\|\overline{PQ}\| = |a_2 - a_1|$.

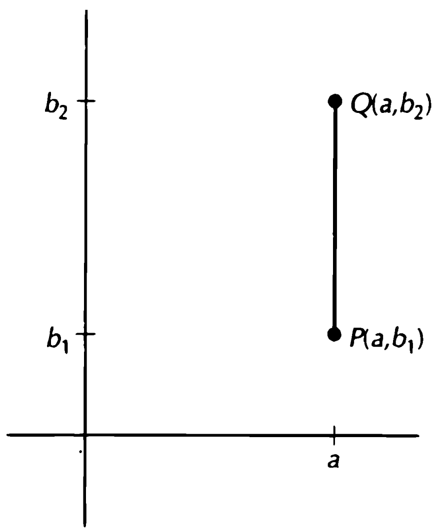



**Figure 10.14** $\|\overline{PQ}\| = |b_2 - b_1|$.

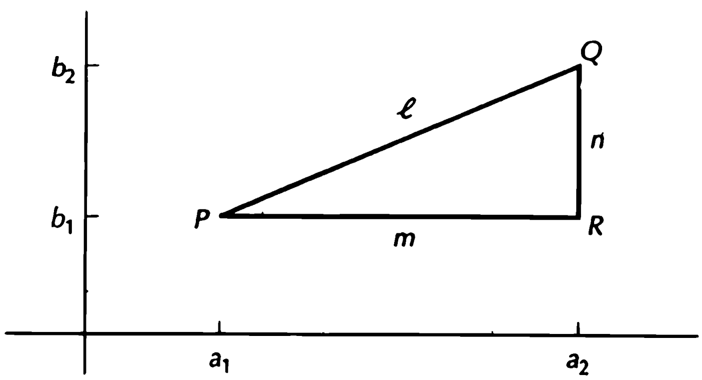



**Figure 10.15** *Computing $\|\overline{PQ}\|$ in the plane.*

Next, consider the case where $P$ and $Q$ have the same $x_1$-component. Say $P$ is $(a, b_1)$ and $Q$ is $(a, b_2)$, as in Figure 10.14. Here, the distance is naturally $|b_2 - b_1|$.

Finally, we consider the general case, as pictured in Figure 10.15. To compute the length of line $\ell$ joining points $P(a_1, b_1)$ and $Q(a_2, b_2)$, mark the intermediate point $R(a_2, b_1)$. Let $m$ be the (horizontal) line segment from $P(a_1, b_1)$ to $R(a_2, b_1)$ and let $n$ be the (vertical) line segment from $Q(a_2, b_2)$ to $R(a_2, b_1)$. The corresponding triangle $PRQ$ is a right triangle whose hypotenuse is the line segment $\ell$.

Apply the Pythagorean Theorem to deduce the length of $\ell$:

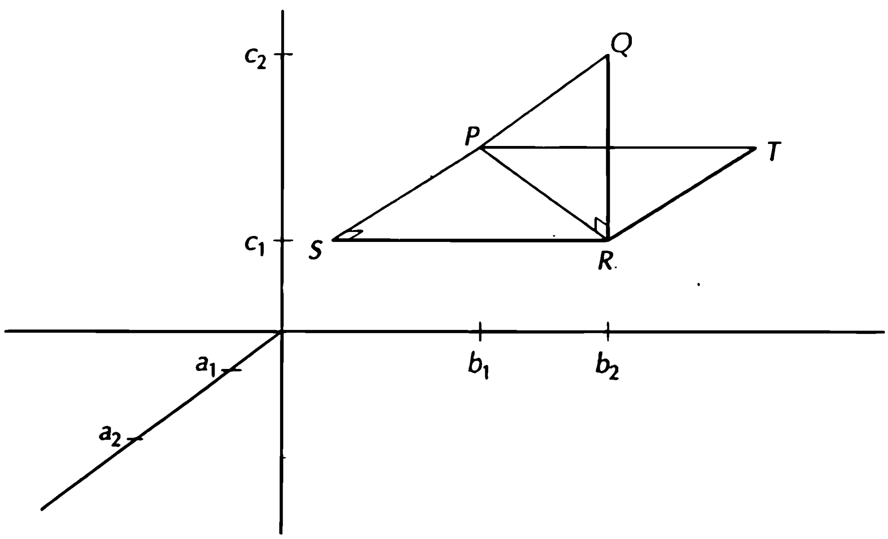
$$\begin{aligned}(\text{length}\,\ell)^2 &= (\text{length}\,m)^2 + (\text{length}\,n)^2\\ &= |a_1 - a_2|^2 + |b_1 - b_2|^2.\end{aligned}$$

Taking the square root of both sides of this equation gives

$$\|\overline{PQ}\| = \text{length}\,\ell = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}. \tag{1}$$

We can apply this argument to higher dimensions, as pictured in Figure 10.16. To find the distance from $P(a_1, b_1, c_1)$ to $Q(a_2, b_2, c_2)$ in $\mathbf{R}^3$, we use the point $R(a_2, b_2, c_1)$, which has the same $x_3$-coordinate as $P$ and the same $x_1$- and $x_2$-coordinates as $Q$. Since $P$ and $R$ have the same $x_3$-coordinate, the segment $PR$ lies on the $x_3 = c_1$ plane, which is parallel to the $x_1x_2$-plane ($x_3 = 0$). Since $Q$ and $R$ have the same $x_1$- and $x_2$-coordinates, segment $QR$ is parallel to the $x_3$-axis and therefore perpendicular to the segment $PR$. Therefore, $\Delta PRQ$ is a right triangle with hypotenuse $PQ$. By the Pythagorean Theorem,

$$\|PQ\|^2 = \|PR\|^2 + \|RQ\|^2. \tag{2}$$




*Computing the length of line PQ in* $\mathbf{R}^3$.

**Figure 10.16**

Since $RQ$ is parallel to the $x_3$-axis, its length is simply $|c_2 - c_1|$. To find the length of $PR$, we work in the two-dimensional plane through $PR$ parallel to the $x_1x_2$-plane. Note that if $S = (a_2, b_1, c_1)$, $PS$ is parallel to the $x_1$-axis and therefore has length $|a_2 - a_1|$, and $SR$ is parallel to the $x_2$-axis with length $|b_2 - b_1|$. Applying the Pythagorean Theorem to right triangle $PSR$ yields:

$$\begin{aligned}||PR||^2 &= ||PS||^2 + ||SR||^2 \\ &= |a_2 - a_1|^2 + |b_2 - b_1|^2.\end{aligned}$$

Substituting this into Equation (2) yields:

$$||PQ||^2 = |a_2 - a_1|^2 + |b_2 - b_1|^2 + |c_2 - c_1|^2.$$

Therefore, the distance from $P$ to $Q$ is

$$||PQ|| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}. \qquad (3)$$

Formulas (1) and (3) generalize readily to points in higher dimensional Euclidean spaces. If $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are the coordinates of **x** and **y**, respectively, in Euclidean $n$-space, then the **distance** between **x** and **y** is

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

We will use this same formula whether we think of **x** and **y** as points or as displacement vectors. Recall that $\mathbf{x} - \mathbf{y}$ is the vector joining points **x** and **y** and its length $||\mathbf{x} - \mathbf{y}||$ is the same as the distance between these two points. Thus, it is natural to write

$$||\mathbf{x} - \mathbf{y}|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

In particular, if we take **y** to be **0**, then the distance from the point $\mathbf{x} = (x_1, \ldots, x_n)$ to the origin or the **length** of the vector **x** is

$$||\mathbf{x}|| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

We can now make more precise the effect of scalar multiplication on the length of a vector **v**. If $r$ is a positive scalar, the length of $r\mathbf{v}$ is $r$ times the length of **v**. If $r$ is a negative scalar, the length of $r\mathbf{v}$ is $|r|$ times the length of **v**. This can be summarized as follows.

**Theorem 10.1** $||r\mathbf{v}|| = |r| \cdot ||\mathbf{v}||$ for all $r$ in $\mathbf{R}^1$ and **v** in $\mathbf{R}^n$.

*Proof*

$$\begin{aligned}\|r(v_1, \ldots, v_n)\| &= \|(rv_1, rv_2, \ldots, rv_n)\| \\ &= \sqrt{(rv_1)^2 + \cdots + (rv_n)^2} \\ &= \sqrt{r^2(v_1^2 + \cdots + v_n^2)} \\ &= |r|\sqrt{v_1^2 + \cdots + v_n^2}, \quad \text{since} \quad \sqrt{r^2} = |r|. \quad \blacksquare\end{aligned}$$

Given a non-zero displacement vector **v**, we will occasionally need to find a vector **w** which points in the same direction as **v**, but has length 1. Such a vector **w** is called the **unit vector** in the direction of **v**, or sometimes simply the **direction** of **v**. To achieve such a vector **w**, simply premultiply **v** by the scalar $r = \frac{1}{\|\mathbf{v}\|}$, because

$$\left\|\frac{1}{\|\mathbf{v}\|} \cdot \mathbf{v}\right\| = \left|\frac{1}{\|\mathbf{v}\|}\right| \cdot \|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|} \cdot \|\mathbf{v}\| = 1.$$

*Example 10.1* For example, the length of $(1, -2, 3)$ in $\mathbf{R}^3$ is

$$\|(1, -2, 3)\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}.$$

It follows that

$$\frac{1}{\sqrt{14}}(1, -2, 3) = \left(\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

is a vector which points in the same direction as $(1, -2, 3)$ but has length 1.

## The Inner Product

We have learned how to add and subtract two vectors and how to compute the distance between them. In this section we introduce another operation on pairs of vectors, the Euclidean inner product. This operation assigns a number to each pair of vectors. We will see that it is connected to the notion of "angle between two vectors," and therefore is useful for discussing geometric problems.

**Definition** Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ be two vectors in $\mathbf{R}^n$. The **Euclidean inner product** of **u** and **v**, written as $\mathbf{u} \cdot \mathbf{v}$, is the *number*

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Because of the dot in the notation, the Euclidean inner product is often called the **dot product**. To emphasize that the result of the operation is a scalar, the Euclidean inner product is also called the **scalar product**. In the exercises to this section, we introduce the outer product or cross product as a way of multiplying two vectors in $\mathbf{R}^3$ to obtain another vector in $\mathbf{R}^3$.

*Example 10.2* If $\mathbf{u} = (4, -1, 2)$ and $\mathbf{v} = (6, 3, -4)$, then

$$\mathbf{u} \cdot \mathbf{v} = 4 \cdot 6 + (-1) \cdot 3 + 2 \cdot (-4) = 13.$$

The following theorem summarizes the basic analytical properties of the inner product — properties that we will use often in this text. Its proof is straightforward and is left as an exercise. Work out the relationships in this theorem to build up a working knowledge of inner product.

**Theorem 10.2** Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be arbitrary vectors in $\mathbf{R}^n$ and let $r$ be an arbitrary scalar. Then,

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,
(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,
(*c*) $\mathbf{u} \cdot (r\mathbf{v}) = r(\mathbf{u} \cdot \mathbf{v}) = (r\mathbf{u}) \cdot \mathbf{v}$,
(*d*) $\mathbf{u} \cdot \mathbf{u} \geq 0$,
(*e*) $\mathbf{u} \cdot \mathbf{u} = 0$ implies $\mathbf{u} = \mathbf{0}$, and
(*f*) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v}$.

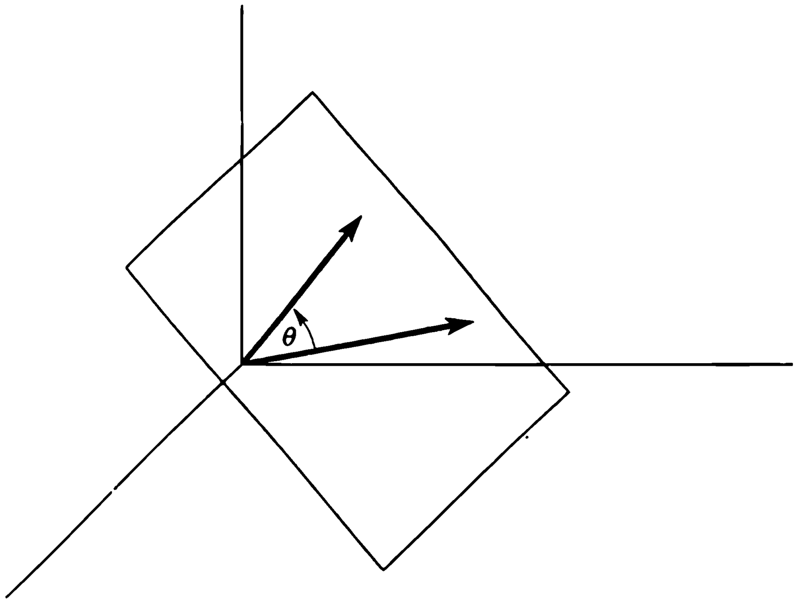
The Euclidean inner product is closely connected to the Euclidean length of a vector. Since

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \cdots + u_n^2 \quad \text{and} \quad \|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2},$$
$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}.$$

Consequently, the distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ can be written in terms of the inner product as

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}.$$

Any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$ determine a plane, as illustrated in Figure 10.17. In that plane we can measure the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$. The inner product yields an important connection between the lengths of $\mathbf{u}$ and $\mathbf{v}$ and the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$.

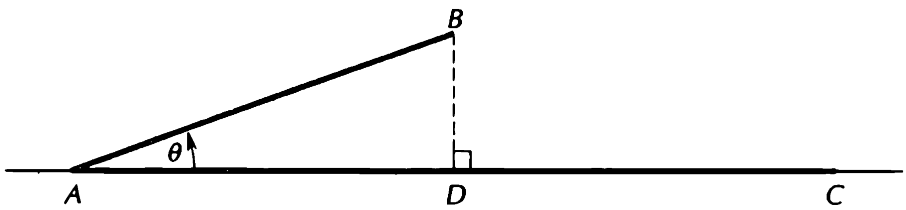



*The angle between two vectors in* $\mathbf{R}^n$. Figure 10.17

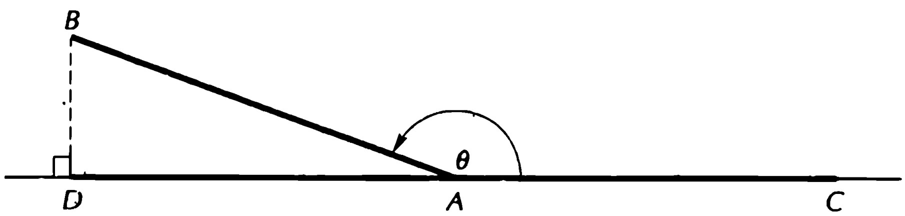
**Theorem 10.3** Let **u** and **v** be two vectors in $\mathbf{R}^n$. Let $\theta$ be the angle between them. Then,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta.$$

**Remark** Recall that to measure the cosine of an angle $\theta = \angle BAC$ as in Figure 10.18, draw the perpendicular from $B$ to a point $D$ on the line containing $A$ and $C$. Then, in the right triangle $BAD$, the cosine of $\theta$ is the length of adjacent side $AD$ divided by the length of hypotenuse $AB$. See the Appendix of this book for more details. If $\theta$ is an *obtuse* angle (between 90 degrees and 270 degrees), then Figure 10.19 is the relevant diagram and the cosine of $\theta$ is the *negative* of $\|AD\|/\|AB\|$.




$\cos \theta = \|AD\|/\|AB\|$. Figure 10.18




*Computing the cosine of an obtuse angle:* $\cos \theta = -\|AD\|/\|AB\|$. Figure 10.19

In either case, cosine $\theta$ lies between $-1$ and $+1$ since a leg of a right triangle can never be longer than the hypotenuse. For us, the important properties of $\cos\theta$ are:

$$\begin{aligned} \cos\theta &> 0 && \text{if } \theta \text{ is acute,} \\ \cos\theta &< 0 && \text{if } \theta \text{ is obtuse,} \\ \cos\theta &= 0 && \text{if } \theta \text{ is a right angle.} \end{aligned}$$

*Proof of Theorem 10.3* The following proof is a bit more complex than the other proofs we have seen. It uses the Pythagorean Theorem again. Without loss of generality, we can work with $\mathbf{u}$ and $\mathbf{v}$ as vectors with tails at the origin $\mathbf{0}$; say $\mathbf{u} = \overrightarrow{OP}$ and $\mathbf{v} = \overrightarrow{OQ}$. Let $\ell$ be the line through the vector $\mathbf{v}$, that is, the line through the points $\mathbf{0}$ and $Q$. Draw the perpendicular line segment $m$ from the point $P$ (the head of $\mathbf{u}$) to the line $\ell$, as in Figure 10.20. Let $R$ be the point where $m$ meets $\ell$. Since $R$ lies on $\ell$, $\overrightarrow{OR}$ is a scalar multiple of $\mathbf{v} = \overrightarrow{OQ}$. Write $\overrightarrow{OR} = t\mathbf{v}$. Since $\mathbf{u}$, $t\mathbf{v}$, and the segment $m$ are the three sides of the right triangle $OPR$, we can write $m$ as the vector $\mathbf{u} - t\mathbf{v}$. Since $\mathbf{u}$ is the hypotenuse of this right triangle,

$$\cos\theta = \frac{\|t\mathbf{v}\|}{\|\mathbf{u}\|} = \frac{t\|\mathbf{v}\|}{\|\mathbf{u}\|}. \tag{4}$$

On the other hand, by the Pythagorean Theorem and Theorem 10.2, the square of the length of the hypotenuse is:

$$\begin{aligned} \|\mathbf{u}\|^2 &= \|t\mathbf{v}\|^2 + \|\mathbf{u} - t\mathbf{v}\|^2 \\ &= t^2\|\mathbf{v}\|^2 + (\mathbf{u} - t\mathbf{v}) \cdot (\mathbf{u} - t\mathbf{v}) \\ &= t^2\|\mathbf{v}\|^2 + \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot (t\mathbf{v}) + (t\mathbf{v}) \cdot (t\mathbf{v}) \\ &= t^2\|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2t(\mathbf{u} \cdot \mathbf{v}) + t^2\|\mathbf{v}\|^2, \end{aligned}$$

or

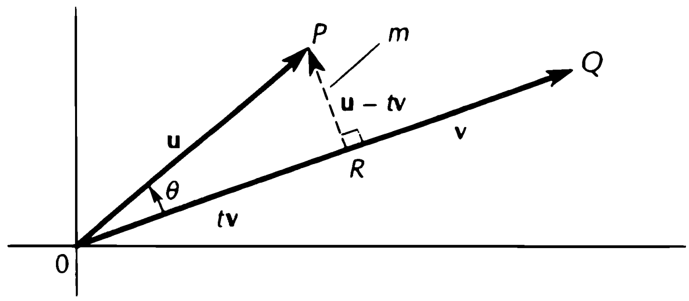
$$2t(\mathbf{u} \cdot \mathbf{v}) = 2t^2\|\mathbf{v}\|^2.$$

It follows that

$$t = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}. \tag{5}$$

Plugging equation (5) into equation (4) yields

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}. \qquad \blacksquare$$




*Choose t so that* **v** *and* **u** − *t***v** *are perpendicular.*

**Figure 10.20**

The following example illustrates how one can use the inner product to compute angles explicitly.

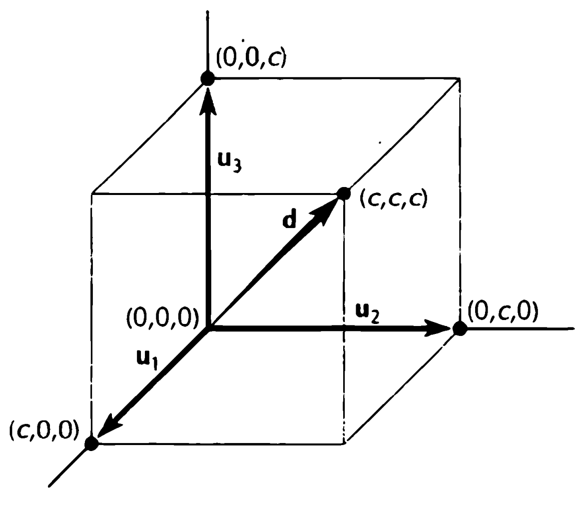
*Example 10.3* We will use the inner product to compute the angle between the diagonal of a cube and one of its sides. Consider a cube in $\mathbf{R}^3$ with each side of length $c$. Position this cube in $\mathbf{R}^3$ in the most natural manner, i.e., with vertices at $O(0,0,0)$, $P_1(c,0,0)$, $P_2(0,c,0)$, and $P_3(0,0,c)$, as in Figure 10.21. Write $\mathbf{u}_i$ for the vector $\overrightarrow{OP_i}$ for $i = 1,2,3$. Then, the diagonal $\mathbf{d}$ is $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$, which is the vector $(c,c,c)$.

The angle $\theta$ between $\mathbf{u}_1$ and $\mathbf{d}$ satisfies

$$\cos\theta = \frac{\mathbf{u}_1 \cdot \mathbf{d}}{\|\mathbf{u}_1\|\,\|\mathbf{d}\|} = \frac{(c,0,0)\cdot(c,c,c)}{c\cdot\sqrt{c^2+c^2+c^2}}$$

$$= \frac{c^2}{c\sqrt{3c^2}} = \frac{1}{\sqrt{3}}.$$

Using a trig table or calculator, one finds that $\cos\theta = 1/\sqrt{3}$ implies that $\theta \approx 54°44'$.




*Cube of side c.*

**Figure 10.21**

Rarely do we care to know that the angle between two vectors is 71° or $3\pi/7$ radians. More often, we are interested in whether the angle is acute, obtuse, or a right angle. Since $\cos\theta$ is positive when $\theta$ is acute, negative when $\theta$ is obtuse, and zero when $\theta$ is a right angle, the dot product tells us the information we want by Theorem 10.3.

**Theorem 10.4** The angle between vectors **u** and **v** in $\mathbf{R}^n$ is

(*a*) acute, if $\mathbf{u} \cdot \mathbf{v} > 0$,
(*b*) obtuse, if $\mathbf{u} \cdot \mathbf{v} < 0$,
(*c*) right, if $\mathbf{u} \cdot \mathbf{v} = 0$.

When this angle is a right angle, we say that **u** and **v** are **orthogonal**. So, vectors **u** and **v** are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + \cdots + u_nv_n = 0$, a simple check indeed.

We have taken some liberties with the case where one of the vectors is zero. When this occurs, $\theta$ is not defined. However, we will run into no difficulties with the concept of orthogonality if we simply watch for zero vectors.

Finally, we use Theorem 10.3 to derive a basic property of length or norm — the **triangle inequality**. This rule states that any side of a triangle is shorter than the sum of the lengths of the other two sides. Intuitively, it follows from the fact that the straight line segment gives the shortest path between any two points in $\mathbf{R}^n$. In vector notation, we want to prove that

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \quad \text{for all } \mathbf{u}, \mathbf{v} \text{ in } \mathbf{R}^n.$$

Figure 10.22 illustrates the equivalence of this analytic formulation with the above statement about triangles.

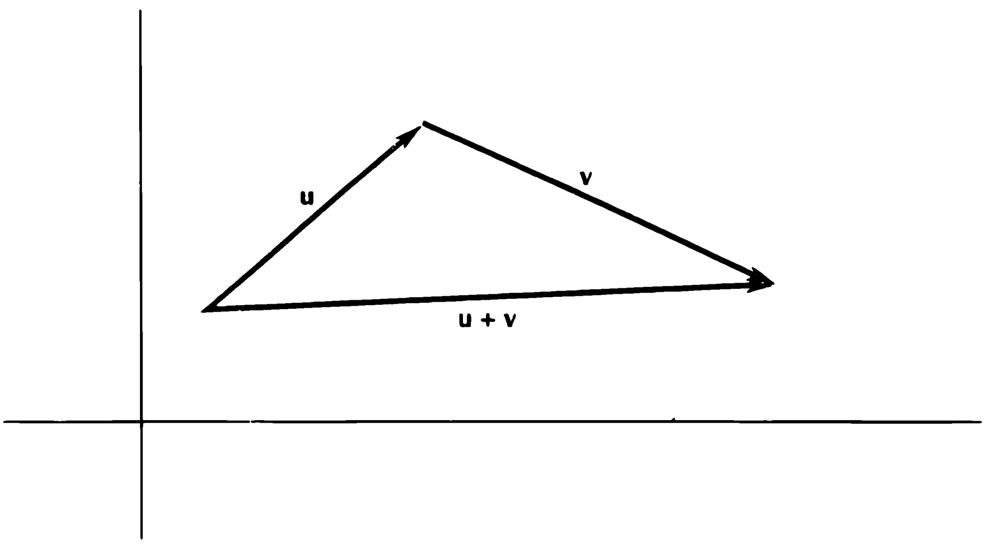



**Figure 10.22** **u**, **v**, *and* **u** + **v** *are three sides of a triangle.*

**Theorem 10.5** For any two vectors **u**, **v** in $\mathbf{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|. \tag{6}$$

*Proof* Recall that

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \cos\theta \leq 1$$

by Theorem 10.3. Therefore,

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &\leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|, \\
\|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 &\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2, \\
\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} &\leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2, \\
(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &\leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2, \\
\|\mathbf{u} + \mathbf{v}\|^2 &\leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2, \\
\|\mathbf{u} + \mathbf{v}\| &\leq \|\mathbf{u}\| + \|\mathbf{v}\|. \quad ■
\end{aligned}$$

We will use the triangle inequality (6) over and over again in our study of Euclidean spaces. Just about every mathematical statement involving an inequality requires the triangle inequality in its proof. The next theorem presents a variant of the triangle inequality which we will also use frequently in our analysis, especially when we want to derive a lower bound for some expression. To understand this result more fully, you should test it on pairs of real numbers, especially pairs with opposite signs.

**Theorem 10.6** For any two vectors **x** and **y** in $\mathbf{R}^n$,

$$\big|\,\|\mathbf{x}\| - \|\mathbf{y}\|\,\big| \leq \|\mathbf{x} - \mathbf{y}\|.$$

*Proof* Apply Theorem 10.5 with $\mathbf{u} = \mathbf{x} - \mathbf{y}$ and $\mathbf{v} = \mathbf{y}$ in (6), to obtain the inequality $\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|$, or

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|. \tag{7}$$

Now apply Theorem 10.5 with $\mathbf{u} = \mathbf{y} - \mathbf{x}$ and $\mathbf{v} = \mathbf{x}$ in (6) to obtain the inequality $\|\mathbf{y}\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x}\|$, or

$$\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\| = \|\mathbf{x} - \mathbf{y}\|. \tag{8}$$

Inequalities (7) and (8) imply that

$$\big|\,\|\mathbf{x}\| - \|\mathbf{y}\|\,\big| \leq \|\mathbf{x} - \mathbf{y}\|. \qquad \blacksquare$$

The three basic properties of Euclidean length are:

(1) $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ only when $\mathbf{u} = \mathbf{0}$.
(2) $\|r\mathbf{u}\| = |r|\|\mathbf{u}\|$
(3) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Any assignment of a real number to a vector that satisfies these three properties is called a **norm**. Exercise 10.16 lists other norms that arise naturally in applications. We will say more about norms in the last section of Chapter 29.

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## EXERCISES

**10.10** Find the length of the following vectors. Draw the vectors for *a* through *g*:

*a*) (3, 4), *b*) (0, −3), *c*) (1, 1, 1), *d*) (3, 3), *e*) (−1, −1),
*f*) (1, 2, 3), *g*) (2, 0), *h*) (1, 2, 3, 4), *i*) (3, 0, 0, 0, 0).

**10.11** Find the distance from $P$ to $Q$, drawing the picture wherever possible:

*a*) $P(0, 0)$, $Q(3, -4)$; *b*) $P(1, -1)$, $Q(7, 7)$;
*c*) $P(5, 2)$, $Q(1, 2)$; *d*) $P(1, 1, -1)$, $Q(2, -1, 5)$;
*e*) $P(1, 2, 3, 4)$, $Q(1, 0, -1, 0)$.

**10.12** For each of the following pairs of vectors, first determine whether the angle between them is acute, obtuse, or right and then calculate this angle:

*a*) $\mathbf{u} = (1, 0)$, $\mathbf{v} = (2, 2)$; *b*) $\mathbf{u} = (4, 1)$, $\mathbf{v} = (2, -8)$;
*c*) $\mathbf{u} = (1, 1, 0)$, $\mathbf{v} = (1, 2, 1)$; *d*) $\mathbf{u} = (1, -1, 0)$, $\mathbf{v} = (1, 2, 1)$;
*e*) $\mathbf{u} = (1, 0, 0, 0, 0)$, $\mathbf{v} = (1, 1, 1, 1, 1)$.

**10.13** For each of the following vectors, find a vector of length 1 which points in the same direction. *a*) (3, 4), *b*) (6, 0), *c*) (1, 1, 1), *d*) (−1, 2, −3).

**10.14** For each of the vectors in the last exercise, find a vector of length five which points in the opposite direction.

**10.15** Prove that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$

**10.16** *a*) In view of the last paragraph in this section, prove that each of the following is a norm in $\mathbf{R}^2$:

$$|||(u_1, u_2)||| = |u_1| + |u_2|,$$

$$|||(u_1, u_2)||| = \max\{|u_1|, |u_2|\}.$$

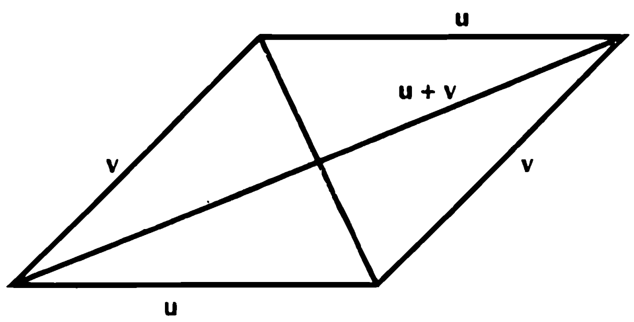
*b*) What are the analogous norms in $\mathbf{R}^n$?

**10.17** Provide a complete and careful proof of Theorem 10.2.

**10.18** Fill in all the details in the proof of Theorem 10.3.

**10.19** For a rectangular $2' \times 3' \times 4'$ box, find the angle that the longest diagonal makes with the $4'$-side.

**10.20** Use vector notation to prove that the diagonals of a rhombus are orthogonal to each other. See Figure 10.23.




*If* $\|\mathbf{u}\| = \|\mathbf{v}\|$, *this quadrilateral is a rhombus.*

**Figure 10.23**

**10.21** Prove the following identities.

*a*) $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$,

*b*) $\mathbf{u}\cdot\mathbf{v} = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$.

**10.22** Prove that if **u** and **v** are orthogonal vectors, then $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$. Explain why this statement is called the general version of the Pythagorean Theorem.

**10.23** The **cross product** is a commonly-used multiplication of vectors in $\mathbf{R}^3$, for which the product of two vectors in $\mathbf{R}^3$ is another *vector* in $\mathbf{R}^3$. It is defined as follows, using the determinant notation introduced in Chapter 9:

$$(u_1, u_2, u_3) \times (v_1, v_2, v_3) = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$
$$= \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right).$$

Prove the following properties of the cross-product:

*a*) $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$,

*b*) $\mathbf{u}\times\mathbf{v}$ is perpendicular to **u**,

*c*) $\mathbf{u}\times\mathbf{v}$ is perpendicular to v,

*d*) $(r\mathbf{u})\times\mathbf{v} = r(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(r\mathbf{v})$,

*e*) $(\mathbf{u}_1+\mathbf{u}_2)\times\mathbf{v} = (\mathbf{u}_1\times\mathbf{v}) + (\mathbf{u}_2\times\mathbf{v})$,

*f*) $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$,

*g*) $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$ (use item *f* and Theorem 10.3),

*h*) $\mathbf{u}\times\mathbf{u} = \mathbf{0}$,

*i*) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$.

**10.24** Show that the cross-product can be represented **symbolically** as

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Treat the $\mathbf{e}_i$'s as points or symbols in the expansion of the determinant.

10.25 Use the cross product to find a vector perpendicular to both **u** and **v**:

$$a)\ \mathbf{u} = (1, 0, 1) \quad \mathbf{v} = (1, 1, 1),$$
$$b)\ \mathbf{u} = (1, -1, 2) \quad \mathbf{v} = (0, 5, -3).$$

10.26 Consider the parallelogram determined by the vectors **u** and **v** in $\mathbf{R}^3$, as in Figure 10.24.

*a*) Show that the area of this parallelogram is $\|\mathbf{u} \times \mathbf{v}\|$. [Hint: Express height $h$ in terms of **u**, **v**, and $\theta$.]

*b*) Find the area of the triangle in $\mathbf{R}^3$ whose vertices are $(1, -1, 2)$, $(0, 1, 3)$, and $(2, 1, 0)$.

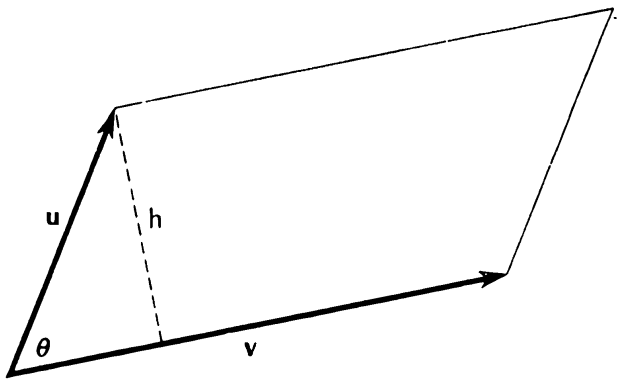



**Figure 10.24** *The parallelogram spanned by* **u** *and* **v**.

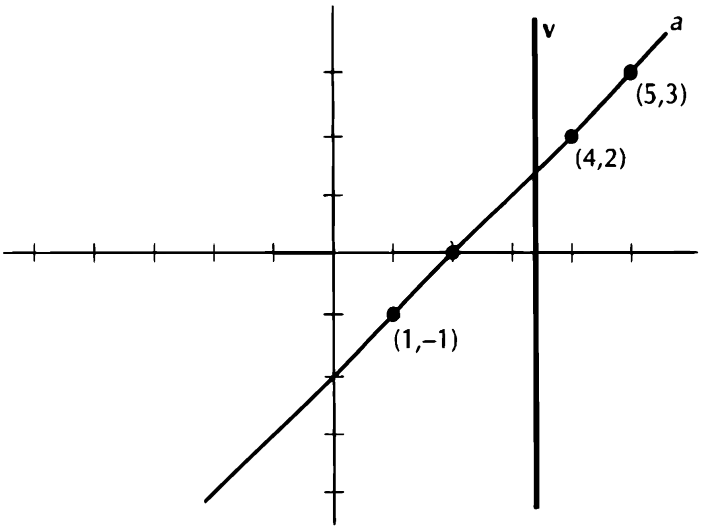
## 10.5 LINES

The fundamental objects of Euclidean geometry are points, lines, and planes. These next two sections show how to describe lines and planes and their higher-dimensional analogues.

First, we will work with lines in $\mathbf{R}^2$. In high school algebra, we learn that straight lines have an equation of the form

$$x_2 = mx_1 + b. \tag{9}$$

The coefficient $m$ is the slope of the line and the coefficient $b$ is the $y$-intercept. This algebraic representation of the line is convenient for solving equations. However, it is not the most useful equation for representing geometric objects. What is the equation of line $v$ in Figure 10.25? We cannot solve for $x_2$ in terms of $x_1$. More important than the awkwardness of this special case is the need for an algebraic representation which clearly expresses the geometry of the line. We will often find a *parametric representation* of the line more useful.



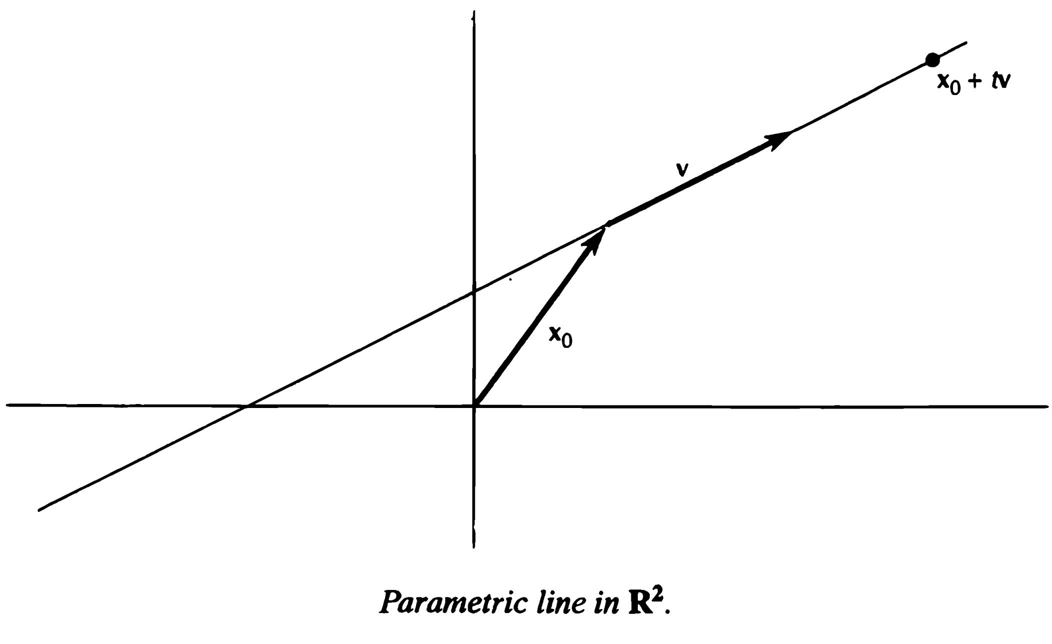

*Lines in* $\mathbf{R}^2$.

**Figure 10.25**

A parametric representation of a point on a line uses a parameter $t$ in the coordinate expression of the point; more formally, a **parametric representation** is an expression $(x_1(t), x_2(t))$ with parameter $t$ in $\mathbf{R}^1$. The point $\mathbf{x} = (x_1, x_2)$ is on the line if and only if $\mathbf{x} = (x_1(t^*), x_2(t^*))$ for some value $t^*$ of the parameter $t$. To make matters concrete, you might think of $t$ as representing time, and the parameterization as describing the transversal of a path. The coordinates $(x_1(t), x_2(t))$ describe the particular location which is reached at time $t$.

A line is completely determined by two things: a point $\mathbf{x}_0$ on the line and a direction $\mathbf{v}$ in which to move from $\mathbf{x}_0$. Geometrically, to describe motion in the direction $\mathbf{v}$ from the point $\mathbf{x}_0$, we simply add scalar multiples of $\mathbf{v}$ to $\mathbf{x}_0$ as in Figure 10.26. The result is the parametric representation

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}. \tag{10}$$




*Parametric line in* $\mathbf{R}^2$.

**Figure 10.26**

*Example 10.4* For example, line $a$ in Figure 10.25 is the line which goes through the point (4, 2) and moves directly to the northeast — in the direction (1, 1). It is described by the parameterization

$$\begin{aligned} \mathbf{x}(t) &= (x_1(t), x_2(t)) \\ &= (4, 2) + t(1, 1) \\ &= (4 + t \cdot 1,\ 2 + t \cdot 1), \end{aligned}$$

or

$$x_1 = 4 + t \cdot 1 \tag{11}$$

$$x_2 = 2 + t \cdot 1. \tag{12}$$

Figure 10.25 shows that (5, 3) and (1, −1) are on line $a$. The first point is reached when $t = 1$, and the second when $t = -3$.

Note that the same line can be described by different parametric equations. For example, we can also view line $a$ in Figure 10.25 as the line through the point (1, −1) in the direction (2, 2). This yields the parameterization

$$(x_1(t), x_2(t)) = (1, -1) + t(2, 2) = (1 + 2t, -1 + 2t).$$

With this parameterization, the line passes through (4, 2) when $t = 1.5$ and through (5, 3) when $t = 2$.

Of course, the parameterization (10) works in all dimensions. For example, the line in $\mathbf{R}^3$ through the point $\mathbf{x}_0 = (2, 1, 3)$ in the direction $\mathbf{v} = (4, -2, 5)$ has the parameterization

$$\begin{aligned} \mathbf{x}(t) &= (x_1(t), x_2(t), x_3(t)) \\ &= (2, 1, 3) + t(4, -2, 5) \\ &= (2 + 4t,\ 1 - 2t,\ 3 + 5t). \end{aligned}$$

Another way to determine a line is to identify two points on the line. Suppose that $\mathbf{x}$ and $\mathbf{y}$ lie on a line $\ell$. Then, $\ell$ can be viewed as the line which goes through $\mathbf{x}$ and points in the direction $\mathbf{y} - \mathbf{x}$. Thus, a parameterization for the line is

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \\ &= \mathbf{x} + t\mathbf{y} - t\mathbf{x} \\ &= (1 - t)\mathbf{x} + t\mathbf{y}. \end{aligned} \tag{13}$$

When $t = 0$, we are at point $\mathbf{x}$; and when $t = 1$, we are at point $\mathbf{y}$. When $t$ lies between 0 and 1, we are at points between $\mathbf{x}$ and $\mathbf{y}$. Consequently, we parameterize the **line segment joining x to y** as

$$\ell(\mathbf{x}, \mathbf{y}) = \{(1 - t)\mathbf{x} + t\mathbf{y} : 0 \le t \le 1\}.$$

Given two points $\mathbf{x} = (a, b)$ and $\mathbf{y} = (c, d)$ on a line $\ell$ in the plane, one can write the parameterized equation of $\ell$ as (13) or the nonparameterized equation of $\ell$ as

$$x_2 - b = \frac{d-b}{c-a}(x_1 - a).$$

One can use these two expressions to pass from the parameterized equation of a line in the plane to the nonparameterized equation, and vice versa, by first finding two points on the line from the given equation and using these points to find the new equation. One can also pass *directly* from form (10) to form (9) by solving the equations in (10) for $t$ and then setting the new equations equal to each other. For example, in equations (11, 12)

$$t = \frac{x_1 - 4}{1} \quad \text{and} \quad t = \frac{x_2 - 2}{1}.$$

So,

$$\frac{x_1 - 4}{1} = \frac{x_2 - 2}{1}, \quad \text{or} \quad x_2 = x_1 - 2.$$

To go the other way, just note that equation (9) is the equation of the line through the point $(0, b)$ in the direction $(1, m)$.

## EXERCISES

**10.27** Show that the midpoint of $\ell(\mathbf{x}, \mathbf{y})$ occurs where $t = \frac{1}{2}$. In other words, if $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$, show that $\|\mathbf{x} - \mathbf{z}\| = \|\mathbf{y} - \mathbf{z}\|$.

**10.28** For each of the following pairs of points $\mathbf{p}_1$, $\mathbf{p}_2$, write the parametric equation of the line through $\mathbf{p}_1$ and $\mathbf{p}_2$, find the midpoint of $\ell(\mathbf{p}_1, \mathbf{p}_2)$, and sketch the line.

*a)* $\mathbf{p}_1 = (3, 0)$, $\mathbf{p}_2 = (5, 0)$;
*b)* $\mathbf{p}_1 = (1, 0)$, $\mathbf{p}_2 = (0, 1)$;
*c)* $\mathbf{p}_1 = (1, 0, 1)$, $\mathbf{p}_2 = (2, 1, 0)$.

**10.29** Is the point $\begin{pmatrix} 11 \\ 14 \\ 17 \\ 18 \end{pmatrix}$ on the line $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}$?

**10.30** Transform each of the following parameterized equations into the form (9):

*a)* $\begin{aligned} x_1 &= 4 - 2t \\ x_2 &= 3 + 6t; \end{aligned}$ *b)* $\begin{aligned} x_1 &= 3 + t \\ x_2 &= 5 - t; \end{aligned}$ *c)* $\begin{aligned} x_1 &= 3 + t \\ x_2 &= 5. \end{aligned}$

**10.31** Transform each of the following nonparameterized equations into the form (10):

*a)* $2x_2 = 3x_1 + 5$; *b)* $x_2 = -x_1 + 7$; *c)* $x_1 = 6$.

## 10.6 PLANES

### Parametric Equations

A line is one-dimensional. Intuitively, the dimension of the line is reflected in the fact that it can be described using only one parameter. Planes are two-dimensional, and so it stands to reason that they are described by expressions with two parameters.

To be more concrete, let $\mathcal{P}$ be a plane in $\mathbf{R}^3$ through the origin. Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in $\mathcal{P}$, as shown in Figure 10.27. Choose $\mathbf{v}$ and $\mathbf{w}$ so that they point in different directions, in other words, so that neither is a scalar multiple of the other. In this case, we say that $\mathbf{v}$ and $\mathbf{w}$ are **linearly independent**, a topic to be discussed in more detail in the next chapter. For any scalars $s$ and $t$, the vector $s\mathbf{v} + t\mathbf{w}$ is called a **linear combination** of $\mathbf{v}$ and $\mathbf{w}$. By our geometric interpretation of scalar multiplication and vector addition, it should be clear that all linear combinations of $\mathbf{v}$ and $\mathbf{w}$ lie on the plane $\mathcal{P}$. In fact, if we take every linear combination of $\mathbf{v}$ and $\mathbf{w}$, we recover the entire plane $\mathcal{P}$. The equation

$$\mathbf{x} = s\mathbf{v} + t\mathbf{w}$$

or

$$\begin{aligned} x_1 &= sv_1 + tw_1 \\ x_2 &= sv_2 + tw_2 \\ x_3 &= sv_3 + tw_3, \end{aligned}$$

provides a parameterization of the plane $\mathcal{P}$.

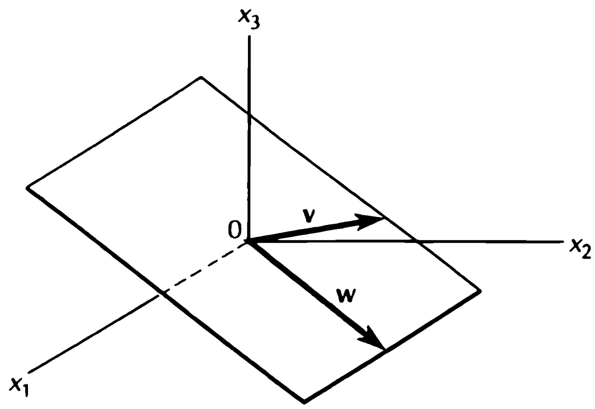
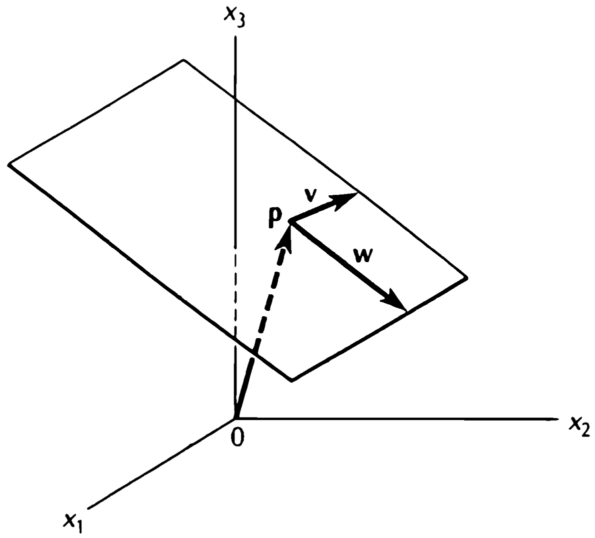



**Figure 10.27** *A plane $\mathcal{P}$ through the origin.*

If the plane does not pass through the origin but through the point $\mathbf{p} \neq \mathbf{0}$ and if $\mathbf{v}$ and $\mathbf{w}$ are linearly independent direction vectors from $\mathbf{p}$ which still lie in the plane, then as indicated in Figure 10.28, we can use the above method to parameterize the plane as

$$\mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}, \qquad s, t \text{ in } \mathbf{R}^1. \tag{14}$$




*A plane not through the origin.*

**Figure 10.28**

Just as two points determine a line, three (non-collinear) points determine a plane. To find the parametric equation of the plane containing the points **p**, **q**, and **r**, note that we can picture **q** − **p** and **r** − **p** as displacement vectors from **p** which lie on the plane. So, one parameterization of the plane is

$$\begin{aligned} \mathbf{x}(s,t) &= \mathbf{p} + s(\mathbf{q}-\mathbf{p}) + t(\mathbf{r}-\mathbf{p}) \\ &= (1-s-t)\mathbf{p} + s\mathbf{q} + t\mathbf{r}. \end{aligned} \tag{15}$$

Compare (15) with the corresponding parameterized equation (13) of a line. From equation (15), we see that a plane is the set of those linear combinations of three fixed vectors whose coefficients sum to 1:

$$\mathbf{x} = t_1\mathbf{p} + t_2\mathbf{q} + t_3\mathbf{r}, \qquad t_1 + t_2 + t_3 = 1. \tag{16}$$

If we further restrict the scalars $t_i$ in (16) so that they are nonnegative, we obtain the (filled-in) triangle in $\mathbf{R}^3$ whose vertices are **p**, **q**, and **r** — the darkened region in Figure 10.29. The numbers $(t_1, t_2, t_3)$ are called the **barycentric coordinates** of a point in this triangle. For example, the barycentric coordinates of the vertex **p** are (1, 0, 0) since $t_1 = 1$, $t_2 = 0$, and $t_3 = 0$ in expression (16) yield the point **p**. Similarly, the barycentric coordinates of the vertices **q** and **r** are (0, 1, 0) and (0, 0, 1), respectively. The center of mass or centroid of this triangle is the point

$$\mathbf{x} = \frac{1}{3}\mathbf{p} + \frac{1}{3}\mathbf{q} + \frac{1}{3}\mathbf{r},$$

whose barycentric coordinates are (1/3, 1/3, 1/3).

Equations such as (14) and (15) give a parameterization of a two-dimensional plane in any Euclidean space, not just $\mathbf{R}^3$. For example, the two-dimensional plane

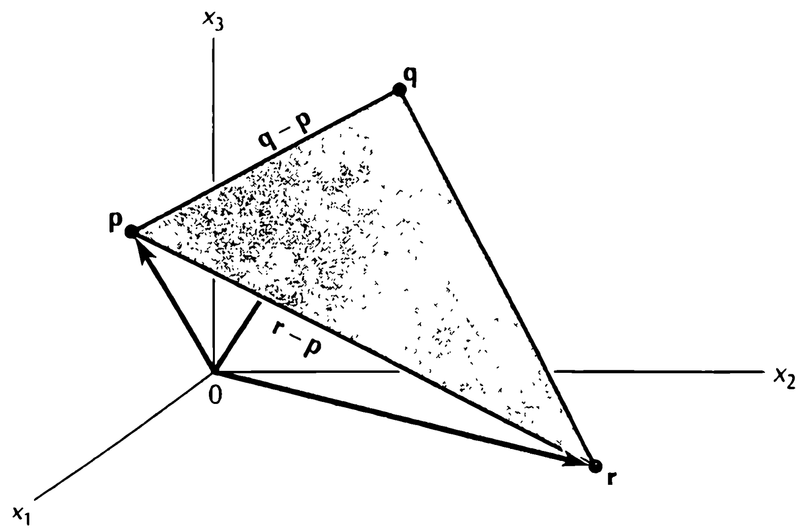



**Figure 10.29** *Triangle with vertices* **p**, **q**, *and* **r**.

through the points (1, 2, 3, 4), (5, 6, 7, 8), and (9, 0, 1, 2) in $\mathbf{R}^4$ has the parametric equations

$$\begin{aligned} x_1 &= 1r + 5s + 9t \\ x_2 &= 2r + 6s + 0t \\ x_3 &= 3r + 7s + 1t \\ x_4 &= 4r + 8s + 2t, \qquad \text{where } r + s + t = 1. \end{aligned}$$

## Nonparametric Equations

We turn now to the *nonparametric equations of a plane* in $\mathbf{R}^3$. Just as with a line in $\mathbf{R}^2$, a plane in $\mathbf{R}^3$ is completely described by giving its inclination and a point on it. We usually express its inclination by specifying a vector $\mathbf{n}$, called a **normal vector**, which is perpendicular to the plane. Suppose we want to write the equation for the plane through the point $\mathbf{p} = (x_0, y_0, z_0)$ and having the normal vector $\mathbf{n} = (a, b, c)$. If $\mathbf{x} = (x, y, z)$ is an arbitrary point on the plane, then $\mathbf{x} - \mathbf{p}$ will be a vector in the plane and consequently will be perpendicular to $\mathbf{n}$, as in Figure 10.30.

Recalling that two vectors are perpendicular if and only if their dot product is zero, we write

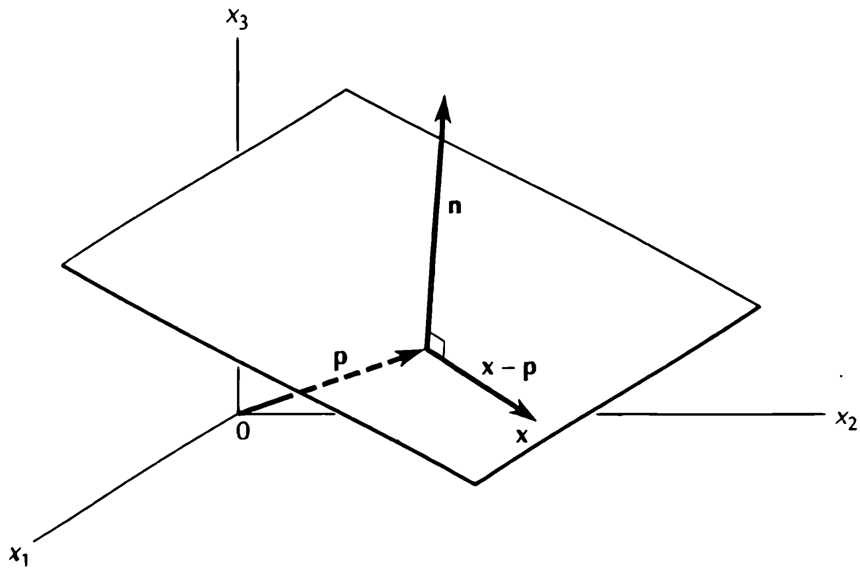
$$0 = \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = (a, b, c) \cdot (x - x_0, y - y_0, z - z_0),$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{17}$$

Form (17) is called the **point-normal equation** of the plane. It is sometimes written as

$$ax + by + cz = d, \tag{18}$$




*Plane through* **p** *with normal* **n**.

**Figure 10.30**

where, in this case, $d = ax_0 + by_0 + cz_0$. Conversely, one can see that equation (18) is the equation of the plane which has normal vector $(a, b, c)$ and which contains each of the points $(0, 0, d/c)$, $(0, d/b, 0)$, and $(d/a, 0, 0)$.

*Example 10.5* The equation of the plane through the point (1, 2, 3) with normal vector (4, 5, 6) is

$$4(x-1) + 5(y-2) + 6(z-3) = 0$$

or

$$4x + 5y + 6z = 32.$$

*Example 10.6* The equation $3x - y + 4z = 12$ is a nonparametric equation of the plane through the point (4, 0, 0) (or (0, 0, 3) or (0, −12, 0) or (5, 7, 1)) with normal vector $\mathbf{n} = (3, -1, 4)$.

To go from a nonparametric equation (18) of a plane to a parametric one, just use (18) to find three points on the plane and then use equation (15). It is more difficult to go from a parametric representation to a nonparametric one, because we need to find a normal **n** to the plane given vectors **v** and **w** parallel to the plane. There are two ways to compute such an **n**. First, one may use the exercises in the last section and take **n** to be the cross product $\mathbf{v} \times \mathbf{w}$. Alternatively, given **v** and **w**, one can solve the system of equations $\mathbf{n} \cdot \mathbf{v} = 0$ and $\mathbf{n} \cdot \mathbf{w} = 0$ explicitly for **n**

*Example 10.7* To find the point-normal equation of the plane $\mathcal{P}$ which contains the points

$$\mathbf{p} = (2, 1, 1), \quad \mathbf{q} = (1, 0, -3), \quad \text{and} \quad \mathbf{r} = (0, 1, 7),$$

note that vectors

$$\mathbf{v} \equiv \mathbf{q} - \mathbf{p} = (-1, -1, -4) \quad \text{and} \quad \mathbf{u} \equiv \mathbf{r} - \mathbf{p} = (-2, 0, 6)$$

both lie on $\mathcal{P}$. To find a normal $\mathbf{n}=(n_1, n_2, n_3)$ to $\mathcal{P}$, solve the system

$$\begin{aligned} \mathbf{n} \cdot \mathbf{v} &= -n_1 - n_2 - 4n_3 = 0 \\ \mathbf{n} \cdot \mathbf{u} &= -2n_1 + 0n_2 + 6n_3 = 0, \end{aligned}$$

say by Gaussian elimination, to compute that $\mathbf{n}$ is any multiple of $(3, -7, 1)$. Finally, use $\mathbf{n}$ and $\mathbf{p}$ to write out the point-normal form

$$3(x - 2) - 7(y - 1) + 1(z - 1) = 0$$

or

$$3x - 7y + z = 0.$$

## Hyperplanes

A line in $\mathbf{R}^2$ and a plane in $\mathbf{R}^3$ are examples of sets described by a single linear equation in $\mathbf{R}^n$. Such spaces are often called **hyperplanes**. A line in $\mathbf{R}^2$ can be written as

$$a_1x_1 + a_2x_2 = d,$$

and a plane in $\mathbf{R}^3$ can be written in point-normal form as

$$a_1x_1 + a_2x_2 + a_3x_3 = d.$$

Similarly, a hyperplane in $\mathbf{R}^n$ can be written in point-normal form as

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = d. \tag{19}$$

The hyperplane described by equation (19) can be thought of as the set of all vectors with tail at $(0, \ldots, 0, d/a_n)$ which are perpendicular to the vector $\mathbf{n} = (a_1, \ldots, a_n)$. We continue to call $\mathbf{n}$ a normal vector to the hyperplane.

---

### EXERCISES

**10.32** Does the point $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ lie on the plane $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$?

**10.33** Derive parametric and nonparametric equations for the lines which pass through each of the following pairs of points in $\mathbf{R}^2$:

*a*) (1, 2) and (3, 6); *b*) (1, 1) and (4, 10); *c*) (3, 0) and (0, 4).

**10.34** Write the parametric equations for each of the following lines and planes:

*a*) $x_2 = 3x_1 - 7$; *b*) $3x_1 + 4x_2 = 12$;
*c*) $x_1 + x_2 + x_3 = 3$; *d*) $x_1 - 2x_2 + 3x_3 = 6$.

**10.35** Write nonparametric equations for each of the following lines and planes:
*a*) $x = 3 - 4t, \quad y = 1 + 2t$;
*b*) $x = 2t, \quad y = 1 + t$;
*c*) $x = 1 + s + t, \quad y = 2 + 3s + 4t, \quad z = s - t$;
*d*) $x = 2 - 3s + t, \quad y = 4, \quad z = 1 + s + t$.

**10.36** Derive parametric and nonparametric equations for the planes through each of the following triplets of points in $\mathbf{R}^3$:
*a*) (6, 0, 0), (0, −6, 0), (0, 0, 3);
*b*) (0, 3, 2), (3, 3, 1), (2, 5, 0).

**10.37** Nonparametric equations of a line in $\mathbf{R}^3$ are equations of the form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{20}$$

These are called **symmetric equations** of the line. They can be derived from the parametric equations by eliminating $t$, just as one does in the plane.
*a*) What are the parametric equations which correspond to the symmetric equations (20)?
*b*) In form (20), one can view the line as the intersection of which two planes?
*c*) Find the symmetric equations of the following two lines in $\mathbf{R}^3$:

*i*) $x_1 = 2 - t$, $x_2 = 3 + 4t$, $x_3 = 1 + 5t$; *ii*) $x_1 = 1 + 4t$, $x_2 = 2 + 5t$, $x_3 = 3 + 6t$.

*d*) For each line in part *c*, find the equations of two planes whose intersection is that line.

**10.38** Determine whether the following pairs of planes intersect;

*a*) $x + 2y - 3z = 6$ and $x + 3y - 2z = 6$;
*b*) $x + 2y - 3z = 6$ and $-2x - 4y + 6z = 10$.

**10.39** Find a nonparametric equation of the plane:
*a*) through the point (1, 2, 3) and normal to the vector (1, −1, 0),
*b*) through the point (1, 1, −1) and perpendicular to the line $(x_1, x_2, x_3) = (4 - 3t, 2 + t, 6 + 5t)$,
*c*) whose intercepts are $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ with $a$, $b$, and $c$ all nonzero.

**10.40** Find the intersection of the plane $x + y + z = 1$ and the line $x = 3 + t, y = 1 - 7t, z = 3 - 3t$.

**10.41** Use Gaussian elimination to find the equation of the line which is the intersection of the planes $x + y - z = 4$ and $x + 2y + z = 3$.

## 10.7 ECONOMIC APPLICATIONS

### Budget Sets in Commodity Space

An important application of Euclidean spaces in economic theory is the notion of a commodity space. In an economy with $n$ commodities, let $x_i$ denote the amount of commodity $i$. Assume that each commodity is completely divisible so that $x_i$ can be any nonnegative number. The vector

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

which assigns a nonnegative quantity to each of the $n$ commodities is called a **commodity bundle**. Since we are dealing only with nonnegative quantities, the set of all commodity bundles is the **positive orthant** of $\mathbf{R}^n$

$$\{(x_1, \ldots, x_n) : x_1 \geq 0, \ldots, x_n \geq 0\}$$

and is called a **commodity space**.

Let $p_i > 0$ denote the price of commodity $i$. Then, the cost of purchasing commodity bundle $\mathbf{x} = (x_1, \ldots, x_n)$ is

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n = \mathbf{p} \cdot \mathbf{x}.$$

A consumer with income $I$ can purchase only bundles $\mathbf{x}$ such that $\mathbf{p} \cdot \mathbf{x} \leq I$. This subset of commodity space is called the consumer's **budget set**. It is bounded above by the hyperplane $\mathbf{p} \cdot \mathbf{x} = I$, whose normal vector is just the price vector $\mathbf{p}$. We have drawn the usual two-dimensional picture for this situation in Figure 10.31.

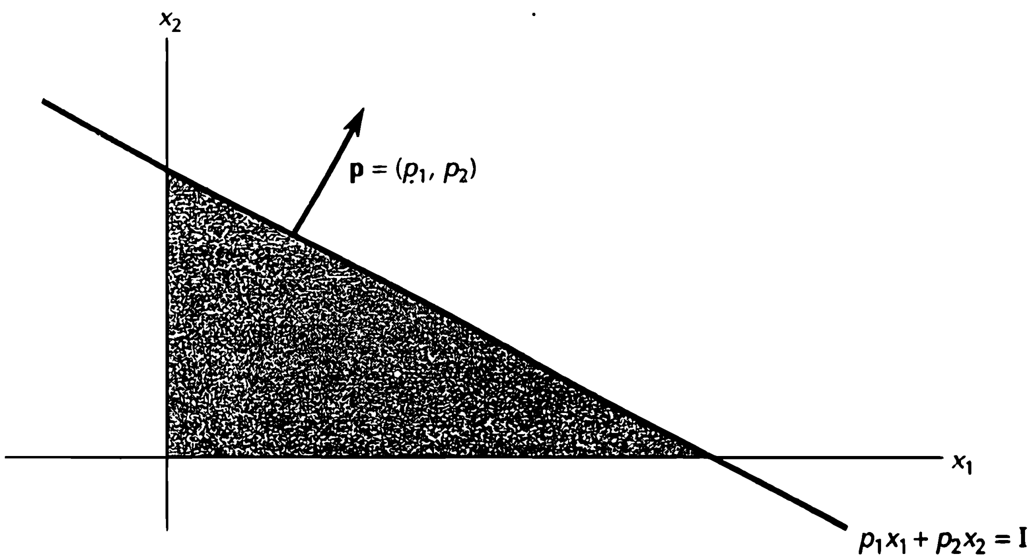



**Figure 10.31** *A consumer's budget set, $\mathbf{p} \cdot \mathbf{x} \leq I$, in commodity space.*

### Input Space

A similar situation exists for a production process which uses $n$ inputs. If $x_i$ denotes an amount of input $i$, then $\mathbf{x} = (x_1, \ldots, x_n)$ is an **input vector in input space**, which is also the positive orthant in $\mathbf{R}^n$. If $w_i$ denotes the cost per unit of input $i$ and $\mathbf{w} = (w_1, \ldots, w_n)$, then the cost of purchasing input bundle $\mathbf{x}$ is $\mathbf{w} \cdot \mathbf{x}$. The set of all input bundles which have a total cost $C$, an isocost set, is that part of the hyperplane $\mathbf{w} \cdot \mathbf{x} = C$ which lies in the positive orthant. The price vector $\mathbf{w}$ is normal to this hyperplane. If we fix $\mathbf{w}$ and let $C$ vary, we obtain isocost hyperplanes which are parallel to each other.

Depending on the situation under study, we sometimes write inputs as *negative* numbers. In this case, input space would be the negative orthant in $\mathbf{R}^n$.

### Probability Simplex

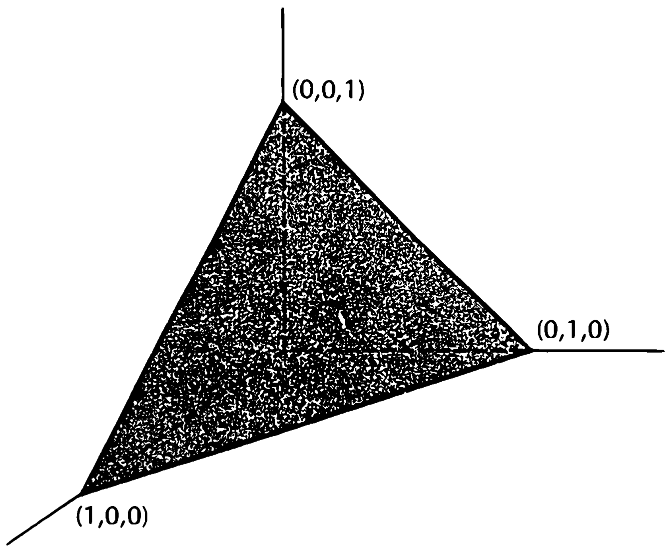
A hyperplane that arises frequently in applications is the space of **probability vectors**

$$P_n = \{(p_1, \ldots, p_n) : p_i \geq 0 \text{ and } p_1 + p_2 + \cdots + p_n = 1\},$$

which we call a **probability simplex**. In these applications there are $n$ mutually exclusive states of the world and $p_i$ is the probability that state $i$ occurs. Since one of these $n$ states must occur, the $p_i$'s sum to 1. The probability simplex $P_n$ is part of a hyperplane in $\mathbf{R}^n$ whose normal vector is $\mathbf{1} = (1, 1, \ldots, 1)$; $P_3$ is pictured in Figure 10.32.

One can also consider $P_n$ as the set of barycentric coordinates with respect to the points

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_n = (0, 0, \ldots, 0, 1).$$




*The probability simplex for $n = 3$.* **Figure 10.32**

## The Investment Model

The portfolio analysis introduced in Example 5 of Chapter 6 fits naturally into the geometric framework of this chapter.

Suppose that an investor is choosing the fraction $x_i$ of his or her wealth to invest in asset $i$. If there are $A$ different investment opportunities, a **portfolio** is an $A$-tuple $\mathbf{x} = (x_1, \dots, x_A)$. Since the $x_i$'s represent fractions of total wealth, they must sum to 1. Therefore the budget constraint is

$$x_1 + x_2 + \cdots + x_A = 1.$$

However, since we allow short positions, $x_i$ may be negative. In this case, the budget set is the entire hyperplane

$$\mathbf{x} \cdot \mathbf{1} = 1$$

normal to the vector $\mathbf{1} = (1, 1, \dots, 1)$. Figure 10.32 shows the intersection of this hyperplane with the positive orthant of $\mathbf{R^n}$ (for $n = 3$).

Suppose that there are $S$ possible financial climates or "states of nature" in the coming investment period. Let $r_{si}$ denote the return on asset $i$ if state $s$ occurs. Form the **state** $s$ **return vector**

$$\mathbf{r}_s = (r_{s1}, r_{s2}, \dots, r_{sA}).$$

Then, the return to the investor of portfolio $\mathbf{x} = (x_1, \dots, x_A)$ is $\mathbf{r}_s \cdot \mathbf{x}$. A portfolio $\mathbf{x}$ is riskless if it returns the same return in every state of nature:

$$\mathbf{r}_1 \cdot \mathbf{x} = \mathbf{r}_2 \cdot \mathbf{x} = \cdots = \mathbf{r}_S \cdot \mathbf{x}.$$

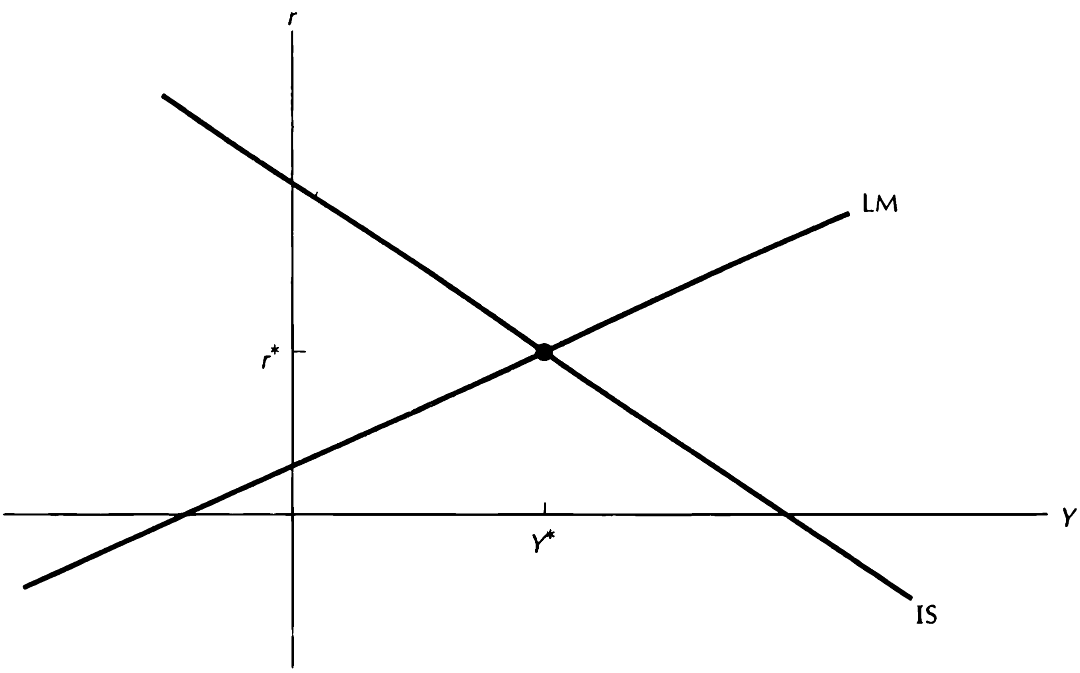
## IS-LM Analysis

We have discussed a linear Keynesian macroeconomic model and Hicks' **IS-LM** interpretation of it in Chapter 6 and again in Chapter 9. In Exercise 9.18, we examined a more or less complete version of this model in five linear equations which could be combined into two equations as

$$\begin{aligned} [1 - c_1(1 - t_1) - a_0]Y + (a + c_2)r &= c_0 - c_1 t_0 + I^* + G \\ mY - hr &= M_s - M^*. \end{aligned}$$

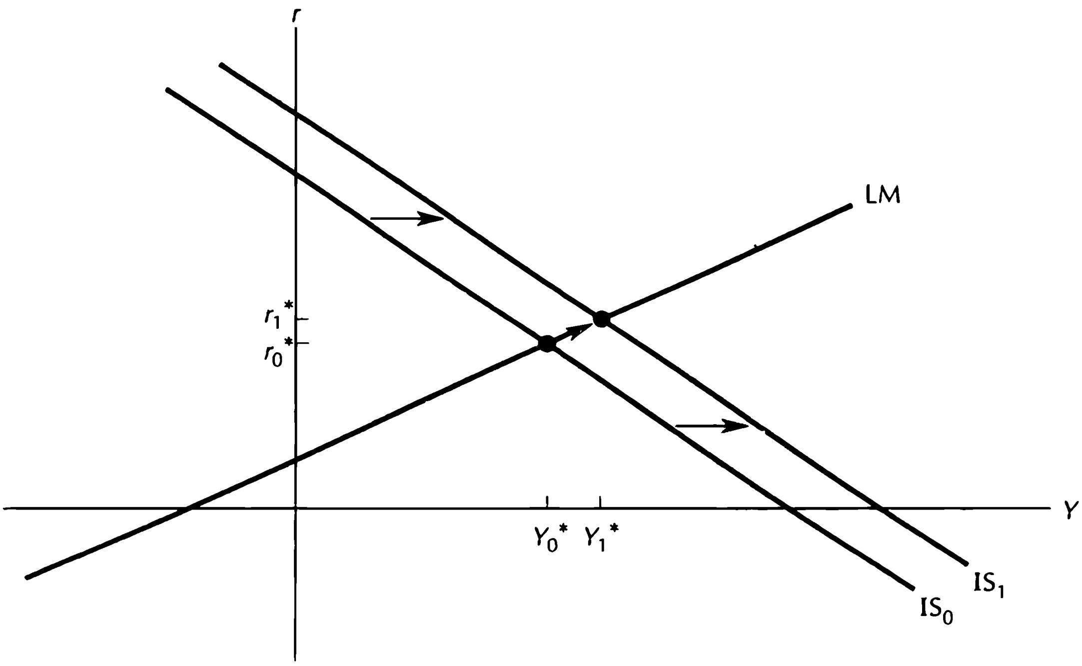
The first equation represents the production equilibrium and is called the IS (investment-savings) equation. The second represents the money market equilibrium and is called the LM (liquidity-money) equation. In intermediate macroeconomics courses, one studies this system graphically by drawing the IS-line and the LM-line in the plane, as in Figure 10.33. The normal vector to the IS-line is

$$(1 - c_1(1 - t_1) - a_0,\ a + c_2).$$




*The graphs of the IS- and LM-lines.*

**Figure 10.33**




*The effect of an increase in G or I*.*

**Figure 10.34**

The parameters $c_1$, $t_1$, and $a_0$ are naturally between 0 and 1. It is usually assumed that $0 < c_1(1 - t_1) + a_0 < 1$, so that the normal vector points northeast and the IS-line has negative slope

$$-\frac{a + c_2}{1 - c_1(1 - t_1) - a_0}.$$

The normal vector to the LM-line is $(m, -h)$ which points southeast, and so the LM-line has a positive slope $h/m$.

Using these diagrams, one can use geometry to study the effects of changes in parameters or in exogenous variables, just as we did analytically in the exercises in Section 9.3. For example, if $G$ or $I^*$ increases or if $t_0$ decreases, then the right hand side of the IS-equation increases and the IS-line shifts outward as in Figure 10.34. The result is an increase in the equilibrium $Y$ and $r$, just as we found in Exercise 9.15. Note that this result would hold even if the slope of the IS-line were positive, as long as it was less than the slope of the LM-line.

## EXERCISES

**10.42** Use the diagram in Figure 10.33 to find the effect on $Y$ and $r$ of an increase in each of the variables $I^*$, $M_s$, $m$, $h$, $a_0$, $a$, $c_0$, and $t_1$.

CHAPTER 11

# Linear Independence

Many economic problems deal with number or size. How many equilibria does a model of an economy or a game have? How large is the production possibility set? Since these sets are often described as solutions of a system of equations, questions of size often reduce to questions about the size of the set of solutions to a particular system of equations. If there are *finitely* many solutions, the exact number of solutions gives a satisfactory answer. But if there are *infinitely* many solutions, the size of the solution set is best captured by its *dimension*. We have a good intuition about the difference between a one-dimensional line and a two-dimensional plane. In this chapter, we will give a precise definition of "dimension" for linear spaces. The key underlying concept is that of linear independence.

The most direct relevant mathematical question is the size, that is, the dimension, of the set of solutions of a system of linear equations $A\mathbf{x} = \mathbf{b}$. Chapter 27 presents a sharp answer to this question via the *Fundamental Theorem of Linear Algebra*: the dimension of the solution set of $A\mathbf{x} = \mathbf{b}$ is the number of variables minus the rank of $A$. Chapter 27 also investigates the size of the set of right-hand sides $\mathbf{b}$ for which a given system $A\mathbf{x} = \mathbf{b}$ has a solution; and we present an in-depth description of the dimension of an abstract vector space. Chapter 28 presents applications of these concepts to portfolio analysis, voting paradoxes, and activity analysis. Those who have the time are encouraged to read Chapters 27 and 28 between Chapters 11 and 12.

Linear independence is defined and characterized in Section 11.1. The complementary notion of span is the focus of Section 11.2. The concept of a basis for Euclidean space is introduced in Section 11.3.

## 11.1 LINEAR INDEPENDENCE

In Section 10.5, we noted that the set of all scalar multiples of a nonzero vector $\mathbf{v}$ is a straight line through the origin. In this chapter, we denote this set by $\mathcal{L}[\mathbf{v}]$:

$$\mathcal{L}[\mathbf{v}] \equiv \{r\mathbf{v} : r \in \mathbf{R}\},$$

and call it the line *generated* or *spanned* by $\mathbf{v}$. See Figure 11.1. For example, if $\mathbf{v} = (1, 0, \ldots, 0)$, then $\mathcal{L}[\mathbf{v}]$ is the $x_1$-axis in $\mathbf{R}^n$. If $\mathbf{v} = (1, 1)$ in $\mathbf{R}^2$, then $\mathcal{L}[\mathbf{v}]$ is the diagonal line pictured in Figure 11.1.

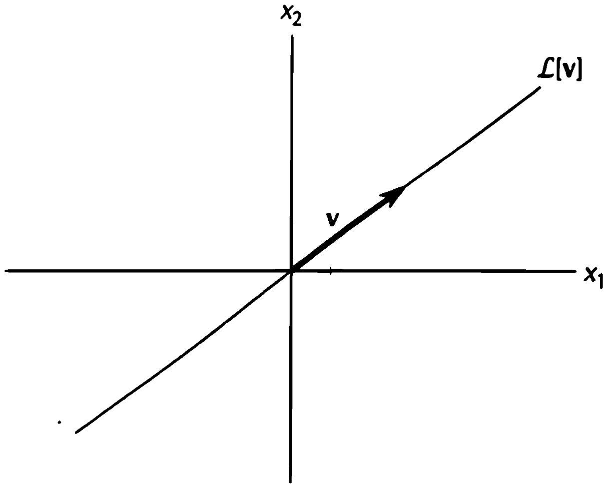



Figure 11.1 *The line $\mathcal{L}[\mathbf{v}]$ spanned by vector* **v**.

## Definition

If we start with two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ (considered as vectors with their tails at the origin), we can take all possible *linear combinations* of $\mathbf{v}_1$ and $\mathbf{v}_2$ to obtain the set *spanned* by $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2] \equiv \{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 : r_1 \in \mathbf{R} \text{ and } r_2 \in \mathbf{R}\}.$$

If $\mathbf{v}_1$ is a multiple of $\mathbf{v}_2$, then $\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2] = \mathcal{L}[\mathbf{v}_2]$ is simply the line spanned by $\mathbf{v}_2$, as in Figure 11.2. However, if $\mathbf{v}_1$ is not a multiple of $\mathbf{v}_2$, then together they generate a two-dimensional plane $\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2]$, which contains the lines $\mathcal{L}[\mathbf{v}_1]$ and $\mathcal{L}[\mathbf{v}_2]$, as in Figure 11.3.

If $\mathbf{v}_1$ is a multiple of $\mathbf{v}_2$, or vice versa, we say that $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly dependent**. Otherwise, we say that $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly independent**. We now develop a precise way of expressing these two concepts. If $\mathbf{v}_1$ is a multiple of $\mathbf{v}_2$,

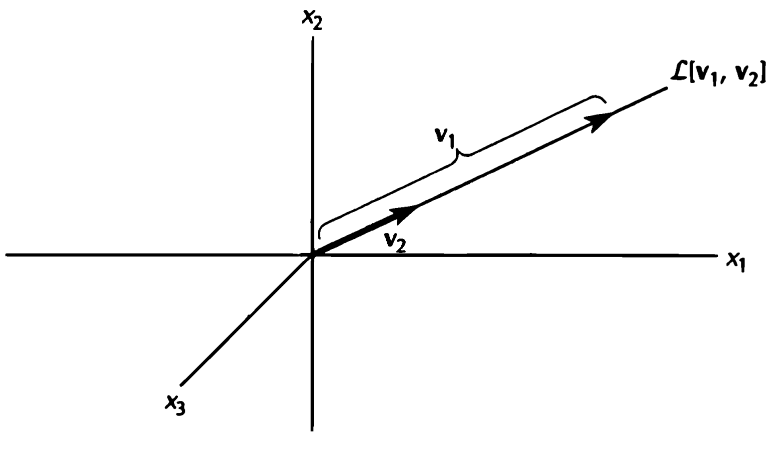
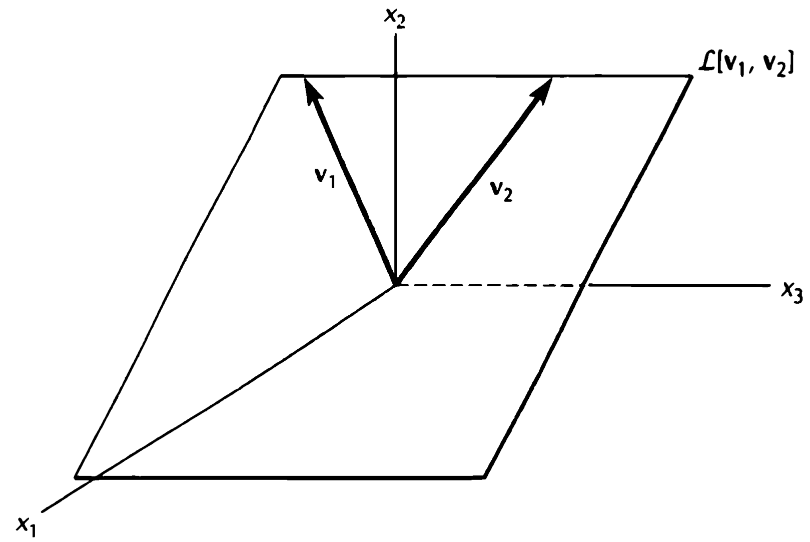



Figure 11.2 *If $\mathbf{v}_1$ is a multiple of $\mathbf{v}_2$, $\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2] = \mathcal{L}[\mathbf{v}_2]$, a line.*




*If* $\mathbf{v}_1$ *is not a multiple of* $\mathbf{v}_2$, *then the set* $\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2]$ *is a plane.*

**Figure 11.3**

we write

$$\mathbf{v}_1 = r_2\mathbf{v}_2 \quad \text{or} \quad \mathbf{v}_1 - r_2\mathbf{v}_2 = \mathbf{0} \tag{1}$$

for some scalar $r_2$. If $\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$, we write

$$\mathbf{v}_2 = r_1\mathbf{v}_1 \quad \text{or} \quad r_1\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0} \tag{2}$$

for some scalar $r_1$. We can combine statements (1) and (2) by defining $\mathbf{v}_1$ and $\mathbf{v}_2$ to be **linearly dependent** if there exist scalars $c_1$ and $c_2$, *not both zero*, so that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}, \qquad c_1 \text{ or } c_2 \text{ nonzero.} \tag{3}$$

In Exercise 11.1 below, you are asked to show that (3) is an equivalent definition to (1) and (2).

From this point of view, we say that $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly independent** if there are no scalars $c_1$ and $c_2$, at least one nonzero, so that (3) holds. A working version of this definition is the following:

$$\begin{gathered}\text{vectors } \mathbf{v}_1 \text{ and } \mathbf{v}_2 \text{ are } \textbf{linearly independent} \text{ if} \\ c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \implies c_1 = c_2 = 0.\end{gathered} \tag{4}$$

This process extends to larger collections of vectors. The set of all **linear combinations** of three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$,

$$\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \equiv \{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 : r_1, r_2, r_3 \in \mathbf{R}\},$$

yields a three-dimensional space, provided that no one of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is a linear combination of the other two. If, say, $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, that is, $\mathbf{v}_3 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2$, while $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, then $\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2]$ is a plane and $\mathbf{v}_3$ lies on this plane; so all combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, $\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, yield just the plane $\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2]$, as pictured in Figure 11.4. As before, we say that $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent if one of them can be written as a linear combination of the other two. The working version of this definition is that some *nonzero* combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ yields the **0**-vector:

$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are **linearly dependent** if and only if there exist scalars $c_1, c_2, c_3$, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. (5)

Conversely, we say:

$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are **linearly independent** if and only if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \quad \Longrightarrow \quad c_1 = c_2 = c_3 = 0. \tag{6}$$

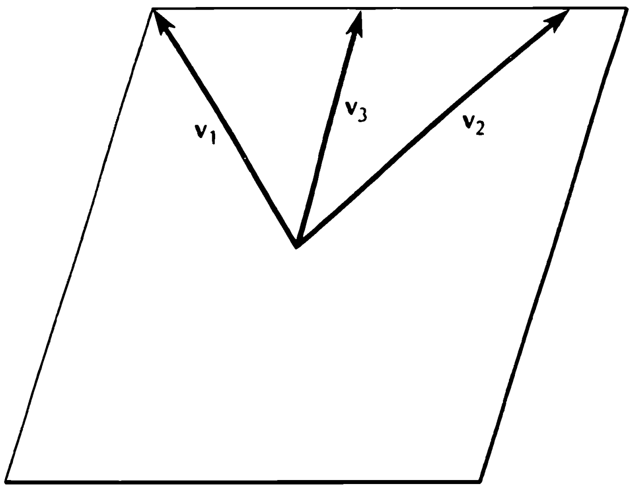



**Figure 11.4** $\mathcal{L}[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ *is a plane if* $\mathbf{v}_3$ *is a linear combination of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$.

It is straightforward now to generalize the concepts of linear dependence and linear independence to arbitrary finite collections of vectors in $\mathbf{R}^n$ by extending definitions (5) and (6) in the natural way.

**Definition** Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbf{R}^n$ are **linearly dependent** if and only if there exist scalars $c_1, c_2, \ldots, c_k$, *not all zero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbf{R}^n$ are **linearly independent** if and only if $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ for scalars $c_1, \ldots, c_k$ implies that $c_1 = \cdots = c_k = 0$.

*Example 11.1* The vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

in $\mathbf{R}^n$ are linearly independent. because if $c_1, \ldots, c_n$ are scalars such that $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n = \mathbf{0}$.

$$c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The last vector equation implies that $c_1 = c_2 = \cdots = c_n = 0$.

*Example 11.2* The vectors

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad \text{and} \quad \mathbf{w}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

are linearly dependent in $\mathbf{R}^3$, since

$$1 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2 \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 1 \cdot \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

as can easily be verified.

## Checking Linear Independence

How would one decide whether or not $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ in Example 11.2 are linearly independent starting from scratch? To use definition (5), start with the equation

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + c_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{7}$$

and solve this system for all possible values of $c_1$, $c_2$, and $c_3$. Multiplying system (7) out yields

$$\begin{aligned} 1c_1 + 4c_2 + 7c_3 &= 0 \\ 2c_1 + 5c_2 + 8c_3 &= 0 \\ 3c_1 + 6c_2 + 9c_3 &= 0, \end{aligned} \tag{8}$$

a system of linear equations in the variables $c_1$, $c_2$, and $c_3$. The matrix formulation of system (8) is

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

Note that the coefficient matrix in (9) is simply the matrix whose columns are the original three vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$. So, the question of the linear independence of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ reduces to a consideration of the coefficient matrix whose columns are $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$. In that case, we reduce the coefficient matrix to its row echelon form:

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix},$$

and conclude that, because its row echelon form has a row of zeros, the coefficient matrix in (9) is singular and therefore that system (9) has a nonzero solution (in fact, infinitely many). One such solution is easily seen to be

$$c_1 = 1, \quad c_2 = -2, \quad \text{and} \quad c_3 = 1,$$

the coefficients we used in Example 11.2. We conclude that $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ are linearly dependent.

The analysis with $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ in the previous example can easily be generalized to prove the following theorem by substituting general $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in steps (7) to (9) for $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ in Example 11.2.

**Theorem 11.1** Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbf{R}^n$ are linearly dependent if and only if the linear system

$$A \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \mathbf{0}$$

has a nonzero solution $(c_1, \ldots, c_k)$, where $A$ is the $n \times k$ matrix whose columns are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ under study:

$$A = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_k).$$

The following is a restatement of Theorem 11.1 for the case $k = n$, using the fact that a square matrix is nonsingular if and only if its determinant is not zero.

**Theorem 11.2** A set of $n$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbf{R^n}$ is linearly independent if and only if

$$\det(\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n) \neq 0.$$

For example, the matrix whose columns are the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbf{R^n}$ in Example 11.1 is the identity matrix, whose determinant is one. We conclude from Theorem 11.2 that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a linearly independent set of $n$-vectors.

We can use Theorem 11.1 to derive a basic result about linear independence. It generalizes the fact that any two vectors on a line are linearly dependent and any three vectors in a plane are linearly dependent.

**Theorem 11.3** If $k > n$, any set of $k$ vectors in $\mathbf{R^n}$ is linearly dependent.

*Proof* Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be $k$ vectors in $\mathbf{R^n}$ with $k > n$. By Theorem 11.1, the $\mathbf{v}_i$'s are linearly dependent if and only if the system

$$A\mathbf{c} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_k)\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \mathbf{0}$$

has a nonzero solution $\mathbf{c}$. But by Fact 7.6 in Section 7.4, any matrix $A$ with more columns than rows will have a free variable and therefore $A\mathbf{c} = \mathbf{0}$ will have infinitely many solutions, all but one of which are nonzero. ■

## EXERCISES

**11.1** Show that if (1) or (2) holds, then (3) holds and, if (3) holds, then (1) or (2) holds.

**11.2** Which of the following pairs or triplets of vectors are linearly independent?

*a*) (2, 1), (1, 2); *b*) (2, 1), (−4, −2);

*c*) (1, 1, 0), (0, 1, 1); *d*) (1, 1, 0), (0, 1, 1), (1, 0, 1).

**11.3** Determine whether or not each of the following collections of vectors in $\mathbf{R}^4$ are linearly independent:

*a*) $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$; *b*) $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

**11.4** Prove that if (4) holds, then $\mathbf{v}_1$ is not a multiple of $\mathbf{v}_2$ and $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$.

**11.5** *a*) Show that if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not satisfy (5), they satisfy (6), and vice versa.
*b*) Show that (5) is equivalent to the statement that one of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is a linear combination of the other two.

**11.6** Prove that any collection of vectors that includes the zero-vector cannot be linearly independent.

**11.7** Prove Theorem 11.1.

**11.8** Prove Theorem 11.2.

## 11.2 SPANNING SETS

Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be a fixed set of $k$ vectors in $\mathbf{R}^n$. In the last section, we spoke of the set of all **linear combinations** of $\mathbf{v}_1, \dots, \mathbf{v}_k$,

$$\mathcal{L}[\mathbf{v}_1, \dots, \mathbf{v}_k] \equiv \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k : c_1, \dots, c_k \in \mathbf{R}\},$$

and called it the set **generated** or **spanned** by $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Suppose that we are given a subset $V$ of $\mathbf{R}^n$. It is reasonable to ask whether or not there exists $\mathbf{v}_1, \dots, \mathbf{v}_k$ in $\mathbf{R}^n$ such that every vector in $V$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$:

$$V = \mathcal{L}[\mathbf{v}_1, \dots, \mathbf{v}_k]. \tag{10}$$

When (10) occurs, we say that $\mathbf{v}_1, \dots, \mathbf{v}_k$ **span** $V$.

*Example 11.3* Every line through the origin is the span of a nonzero vector on the line. For example, the $x_1$-axis is the span of $\mathbf{e}_1 = (1, 0, \dots, 0)$, and the diagonal line

$$\Delta \equiv \{(a, a, \dots, a) \in \mathbf{R}^n : a \in \mathbf{R}\}$$

is the span of the vector $(1, 1, \dots, 1)$.

*Example 11.4* The $x_1x_2$-plane in $\mathbf{R}^3$ is the span of the unit vectors $\mathbf{e}_1 = (1, 0, 0)$ and $\mathbf{e}_2 = (0, 1, 0)$, because any vector $(a, b, 0)$ in this plane can be written as

$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

*Example 11.5* The $n$-dimensional Euclidean space itself is spanned by the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ of Example 11.1. For, if $(a_1, \dots, a_n)$ is an arbitrary vector in $\mathbf{R}^n$, then we can write

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + a_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

*Example 11.6* Different sets of vectors can span the same space. For example, each of the following sets of vectors spans $\mathbf{R}^2$:

$$a) \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$b) \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$c) \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

$$d) \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

$$e) \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Theorem 11.1 presented a matrix criterion for checking whether a given set of vectors is linearly independent. The following theorem carries out the analogous task for checking whether a set of vectors spans.

**Theorem 11.4** Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a set of $k$ vectors in $\mathbf{R}^n$. Form the $n \times k$ matrix whose columns are these $\mathbf{v}_j$'s:

$$A = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_k). \tag{11}$$

Let $\mathbf{b}$ be a vector in $\mathbf{R}^n$. Then, $\mathbf{b}$ lies in the space $\mathcal{L}[\mathbf{v}_1, \ldots, \mathbf{v}_k]$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_k$ if and only if the system $A\mathbf{c} = \mathbf{b}$ has a solution $\mathbf{c}$.

*Proof* Write $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in coordinates as

$$\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ \vdots \\ v_{1n} \end{pmatrix}, \ldots, \mathbf{v}_k = \begin{pmatrix} v_{k1} \\ \vdots \\ v_{kn} \end{pmatrix}$$

Then, $\mathbf{b}$ is in $\mathcal{L}[\mathbf{v}_1, \ldots, \mathbf{v}_k]$ if and only if we can find $c_1, \ldots, c_k$ such that

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{b},$$

or

$$c_1\begin{pmatrix} v_{11} \\ \vdots \\ v_{1n} \end{pmatrix} + \cdots + c_k\begin{pmatrix} v_{k1} \\ \vdots \\ v_{kn} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

or

$$\begin{aligned} c_1v_{11} + \cdots + c_kv_{k1} &= b_1 \\ \vdots \qquad \vdots \qquad & \quad \vdots \\ c_1v_{1n} + \cdots + c_kv_{kn} &= b_n, \end{aligned}$$

or

$$\begin{pmatrix} v_{11} & \cdots & v_{k1} \\ \vdots & \ddots & \vdots \\ v_{1n} & \cdots & v_{kn} \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}. \tag{12}$$

So, $\mathbf{b} \in \mathcal{L}[\mathbf{v}_1, \ldots, \mathbf{v}_k]$ if and only if system (12) has a solution $\mathbf{c}$. ■

The following corollary of Theorem 11.4 provides a simple criterion for whether or not a given set of vectors spans all of $\mathbf{R}^n$. Its proof is left as a simple exercise.

**Theorem 11.5** Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a collection of vectors in $\mathbf{R}^n$. Form the $n \times k$ matrix $A$ whose columns are these $\mathbf{v}_j$'s, as in (11). Then, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span $\mathbf{R}^n$ if and only if the system of equations $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ for every right-hand side $\mathbf{b}$,

In Example 11.5, we found $n$ vectors that span $\mathbf{R}^n$. In Example 11.6, we listed various collections of two or three vectors that span $\mathbf{R}^2$. Clearly, it takes at least two vectors to span $\mathbf{R}^2$. The next theorem, which follows easily from Theorem 11.5, states that one needs at least $n$ vectors to span $\mathbf{R}^n$.

**Theorem 11.6** A set of vectors that spans $\mathbf{R}^n$ must contain at least $n$ vectors.

*Proof* By Theorem 11.5, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span $\mathbf{R}^n$ if and only if system (12) has a solution $\mathbf{c}$ for every right-hand side $\mathbf{b} \in \mathbf{R}^n$. Fact 7.7 tells us that if system (12) has a solution for each right-hand side, then the rank of the coefficient matrix equals the number of rows, $n$. Fact 7.1 states that the rank of the coefficient matrix is always less than or equal to the number of columns, $k$. Therefore, if $k$ vectors span $\mathbf{R}^n$, then $n \leq k$. ■

## EXERCISES

**11.9** *a*) Write (2, 2) as a linear combination of (1, 2) and (1, 4).

*b*) Write (1, 2, 3) as a linear combination of (1, 1, 0), (1, 0, 1), and (0, 1, 1).

**11.10** Do $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}4\\5\\12\end{pmatrix}$, and $\begin{pmatrix}0\\8\\0\end{pmatrix}$ span $\mathbf{R}^3$? Explain.

**11.11** Prove Theorem 11.5.

## 11.3 BASIS AND DIMENSION IN $\mathbf{R}^n$

If we have a spanning set of vectors, we can always throw in $\mathbf{0}$ or any linear combination of the vectors in the spanning set to create a larger spanning set. But what we would really like to do is to go the other way and find an efficient spanning set.

*Example 11.7* Let $W$ be the set of all linear combinations of $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, -1, -1)$, and $\mathbf{v}_3 = (2, 0, 0)$ in $\mathbf{R}^3$: $W = \mathcal{L}[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. Note that $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$. Thus, any vector which is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ can be written as a linear combination of just $\mathbf{v}_1$ and $\mathbf{v}_2$, because if $\mathbf{w} \in W$, then there are scalars $a$, $b$, and $c$ such that

$$\begin{aligned}\mathbf{w} &= a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3\\ &= a\mathbf{v}_1 + b\mathbf{v}_2 + c(\mathbf{v}_1 + \mathbf{v}_2)\\ &= (a + c)\mathbf{v}_1 + (b + c)\mathbf{v}_2.\end{aligned}$$

The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a more "efficient" spanning set than is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

For the sake of efficiency, if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span $V$, we would like to find the *smallest* possible subset of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ that spans $V$. However, this is precisely the role of the concept of linear independence that we considered in Section 11.1. If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, no one of these vectors is a linear combination of the others and therefore no proper subset of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ spans $\mathcal{L}[\mathbf{v}_1, \ldots, \mathbf{v}_k]$. The set $\mathbf{v}_1, \ldots, \mathbf{v}_k$ spans $\mathcal{L}[\mathbf{v}_1, \ldots, \mathbf{v}_k]$ most efficiently. In this case, we call $\mathbf{v}_1, \ldots, \mathbf{v}_k$ a *basis* of $\mathcal{L}[\mathbf{v}_1, \ldots, \mathbf{v}_k]$. Since $\mathcal{L}[\mathbf{v}_1, \ldots, \mathbf{v}_k]$ can be spanned by different sets of vectors, as illustrated in Example 11.6, we define a basis more generally as any set of *linearly independent* vectors that *span* $\mathcal{L}[\mathbf{v}_1, \ldots, \mathbf{v}_k]$.

**Definition** Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a fixed set of $k$ vectors in $\mathbf{R}^n$. Let $V$ be the set $\mathcal{L}[\mathbf{v}_1, \ldots, \mathbf{v}_k]$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Then, if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called a **basis** of $V$. More generally, let $\mathbf{w}_1, \ldots, \mathbf{w}_m$ be a collection of vectors in $V$. Then, $\mathbf{w}_1, \ldots, \mathbf{w}_m$ forms a **basis** of $V$ if:

(*a*) $\mathbf{w}_1, \ldots, \mathbf{w}_m$ span $V$, and
(*b*) $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are linearly independent.

*Example 11.8* We conclude from Examples 11.1 and 11.5 that the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

form a basis of $\mathbf{R}^n$. Since this is such a natural basis, it is called the **canonical basis** of $\mathbf{R}^n$.

*Example 11.9* Example 11.6 presents five collections of vectors that span $\mathbf{R}^2$. By Theorem 11.3, collections $c$ and $e$ are not linearly independent since each contains more than two vectors. However, the collections in $a$, $b$ and $d$ are linearly independent (exercise), and therefore, each forms a basis of $\mathbf{R}^2$.

Notice that each basis in $\mathbf{R}^2$ singled out in Example 11.9 is composed of *two* vectors. This is natural since $\mathbf{R}^2$ is a plane and two linearly independent vectors span a plane. The following theorem generalizes this result to $\mathbf{R}^n$.

**Theorem 11.7** Every basis of $\mathbf{R}^n$ contains $n$ vectors.

*Proof* By Theorem 11.3, a basis of $\mathbf{R}^n$ cannot contain more than $n$ elements; otherwise, the set under consideration would not be linearly independent. By Theorem 11.6, a basis of $\mathbf{R}^n$ cannot contain fewer than $n$ elements; otherwise, the set under consideration would not span $\mathbf{R}^n$. It follov's that a basis of $\mathbf{R}^n$ must have exactly $n$ elements. ■

We can combine Theorems 11.1, 11.2, and 11.5 and the fact that a square matrix is nonsingular if and only if its determinant is nonzero to achieve the following equivalence of the notions of linear independence, spanning, and basis for $n$ vectors in $\mathbf{R}^n$.

**Theorem 11.8** Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a collection of $n$ vectors in $\mathbf{R}^n$. Form the $n \times n$ matrix $A$ whose columns are these $\mathbf{v}_j$'s: $A = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n)$. Then, the following statements are equivalent:

(*a*) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent,
(*b*) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span $\mathbf{R}^n$,
(*c*) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of $\mathbf{R}^n$, and
(*d*) the determinant of $A$ is nonzero.

## Dimension

The fact that every basis of $\mathbf{R}^n$ contains exactly $n$ vectors tells us that there are $n$ independent directions in $\mathbf{R}^n$. We express this when we say that $\mathbf{R}^n$ is $n$-dimensional. We can use the idea of basis to extend the concept of dimension to other subsets of $\mathbf{R}^n$. In particular, let $V$ be the set $\mathcal{L}[\mathbf{v}_1, \ldots, \mathbf{v}_k]$ generated by the set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, they form a basis of $V$. In Chapter 27, we prove that *every* basis of $V$ has exactly $k$ vectors — the analogue of Theorem 11.7 for proper subsets of $\mathbf{R}^n$. This number $k$ of vectors in every basis of $V$ is called the dimension of $V$.

### EXERCISES

**11.12** Which of the following are bases of $\mathbf{R}^2$?

$$a)\ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-2\\-2\end{pmatrix} \qquad b)\ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\-2\end{pmatrix} \qquad c)\ \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-2\\2\end{pmatrix} \qquad d)\ \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$$

**11.13** Show that the collections in $a$, $b$ and $d$ in Example 11.6 form a basis of $\mathbf{R}^2$.

**11.14** Which of the following are bases in $\mathbf{R}^3$?

$$a)\ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}; \qquad b)\ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}; \qquad c)\ \begin{pmatrix}6\\3\\9\end{pmatrix}, \begin{pmatrix}5\\2\\8\end{pmatrix}, \begin{pmatrix}4\\1\\7\end{pmatrix};$$

$$d)\ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}; \qquad e)\ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$$

**11.15** Prove Theorem 11.8.

## 11.4 EPILOGUE

This completes our introduction to linear independence, spanning, and dimension. You may want to delve more deeply into these topics before going on to the study of nonlinear functions in Part 3. If so, the following chapters of more advanced material would fit in naturally here:

**Chapter 27: Subspaces Attached to a Matrix** As the continuation of Chapter 11, this chapter defines an abstract vector space and its subspaces and carries the notion of dimension to such spaces. As important examples, it studies three subspaces attached to any matrix: the row space, the column space, and the nullspace.

It concludes with a complete characterization of the size, that is, dimension, of the set of solutions to a system of linear equation $A\mathbf{x} = \mathbf{b}$.

**Chapter 28: Applications of Linear Independence** This chapter presents applications of the material in Chapters 11 and 27 to portfolio analysis, activity analysis, and voting paradoxes.