

Let $(X_1, \dots, X_n)$ be independent identically distributed random variables with p.d.f.

$$f(x) = \theta(1-x)^{\theta-1}I_{(0,1)}(x) \quad \theta > 1$$

- 1 Find a sufficient statistics for θ
- 2 Find $\hat{\theta}_{MLE}$ maximum likelihood estimator (MLE) for θ and discuss properties of this estimator.
- 3 Compute the score function and the Fisher information.
- 4 Specify asymptotic distribution of $\hat{\theta}_{MLE}$.

Exercise

Let $(X_1, \dots, X_n)$ be independent identically distributed random variables with p.d.f.

$$f(x) = \theta(1-x)^{\theta-1}I_{(0,1)}(x) \quad \theta > 1$$

- ① Suppose from a random sample of 200 random variables you obtain $\sum_{i=1}^{200} \log(1-x_i) = -110$. Complete ALL of the following questions:
 - ① Find the Likelihood Ratio test statistic for testing $H_0 : \theta = 2$ versus $H_1 : \theta \neq 2$, specify the asymptotic distribution and verify the null hypothesis.
 - ② Find the Wald test statistic for testing $H_0 : \theta = 2$ versus $H_1 : \theta \neq 2$, specify the distribution and verify the null hypothesis.
 - ③ Find the Score test statistic for testing $H_0 : \theta = 2$ versus $H_1 : \theta \neq 2$, specify the distribution and verify the null hypothesis.

$$f(x_1, \dots, x_n) = \prod_i \theta(1 - x_i)^{\theta-1}$$

$$f(x_1, \dots, x_n) = \theta^n \left(\prod_i (1 - x_i)^{\theta-1} \right)$$

Sufficient statistics

$$\prod_i (1 - x_i)$$

$$L(\theta|x_1, \dots, x_n) = \theta^n \left(\prod_i (1 - x_i)^{\theta-1} \right)$$

$$\log L(\theta|x_1, \dots, x_n) = n \log(\theta) + (\theta - 1) \left(\sum_i \log(1 - x_i) \right)$$

$$\frac{\partial \log L(\theta|x_1, \dots, x_n)}{\partial \theta} = \frac{n}{\theta} + \left(\sum_i \log(1 - x_i) \right)$$

$$\frac{\partial^2 \log L(\theta|x_1, \dots, x_n)}{\partial^2 \theta} = -\frac{n}{\theta^2} < 0 \quad \forall \theta$$

Maximum Likelihood Estimator

$$\hat{\theta}_{MLE} = -\frac{n}{\sum_i \log(1 - X_i)}$$

Maximum Likelihood Estimate

1.82

Maximum Likelihood Estimator

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Maximum Likelihood Estimate

1.82

Score Function

$$U(\theta) = \frac{n}{\theta} + \left(\sum_i \log(1 - x_i) \right)$$

Fisher Information

$$I(\theta) = \frac{n}{\theta^2}$$

Score Function

$$U(\theta) = \frac{n}{\theta} + \left(\sum_i \log(1 - x_i) \right)$$

Fisher Information

$$I(\theta) = \frac{n}{\theta^2}$$

Specify asymptotic distribution of $\hat{\theta}_{MLE}$

$$\hat{\theta}_{MLE} \approx N \left(\theta_0, \frac{\theta_0^2}{n} \right)$$

Score Function

$$U(\theta) = \frac{n}{\theta} + \left(\sum_i \log(1 - x_i) \right)$$

Fisher Information

$$I(\theta) = \frac{n}{\theta^2}$$

Specify asymptotic distribution of $\hat{\theta}_{MLE}$

$$\hat{\theta}_{MLE} \approx N \left(\theta_0, \frac{\theta_0^2}{n} \right)$$

$$\hat{\theta} \sim N\left(\theta_0, \frac{1}{nl(\theta_0)}\right)$$
$$\left(\hat{\theta} - \theta_0\right)^2 \times nl(\theta_0) \sim \chi_1$$

Wald test Statistics

$$\left(\hat{\theta} - \theta_0\right)^2 \times nl(\hat{\theta})$$

Wald test Statistics

$$\left(\hat{\theta} - \theta_0\right) \times \sqrt{nl(\hat{\theta})}$$

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Wald test Statistics

$$(\hat{\theta} - \theta_0) \times \sqrt{nI(\hat{\theta})}$$

Wald test Statistics

$$\frac{-\frac{\sum_{i=1}^n \log(1-x_i)}{n} - \theta_0}{\frac{-\frac{\sum_{i=1}^n \log(1-x_i)}{n}}{\sqrt{n}}}$$

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Distribution of test statistics

$$N(0, 1)$$

Wald Test

Wald test Statistics

$$(\hat{\theta} - \theta_0) \times \sqrt{nI(\hat{\theta})}$$

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Distribution of test statistics

$$N(0, 1)$$

Observed value of test statistics

$$\frac{1.82 - 2}{\sqrt{0.0166}} = -1.4 \quad p\text{-value} = 0.16$$

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Decision: ACCEPT

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Decision: ACCEPT

$$\frac{\left(\frac{\partial \log L(\theta_0)}{\partial \theta}\right)^2}{nI(\theta)_0} \sim \chi_1^2$$

Score Test Statistics

$$\frac{\left(\frac{n}{\theta_0} + \sum_{i=1}^n \log(1 - x_i)\right)^2}{\frac{n}{\theta_0^2}}$$

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Score Test Statistics

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Score Test

Score Test Statistics

$$\frac{\frac{n}{\theta_0} + \sum_{i=1}^n \log(1 - x_i)}{\sqrt{\frac{n}{\theta_0^2}}}$$

Distribution of the Score Test Statistics

$$N(0, 1)$$

Observed value of test statistics

$$\begin{aligned} \frac{\delta \log L(\theta_0)}{\delta \theta} &= \frac{200}{2} - 110 = -10 \\ \frac{\log L'(\theta_0)}{\sqrt{nl(\theta_0)}} &= \frac{-10}{\sqrt{50}} = -1.41 \quad pvalue = 0.16 \end{aligned}$$

Score Test

Score Test Statistics

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score=-1.41, p-value=0.16

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Likelihood ratio Test

Log Likelihood

$$\log L(\theta) = n \log \theta + (\theta - 1) \sum_{i=1}^n \log(1 - x_i)$$

Likelihood ratio Test

$$2 \times (\log L(\hat{\theta}) - \log L(\theta_0)) \sim \chi_1^2$$

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Likelihood ratio Test Statistics

$$2 \times (\log L(\hat{\theta}) - \log L(\theta_0))$$
$$2 \times \left(n \log \frac{\hat{\theta}}{\theta_0} + (\hat{\theta} - \theta_0) \sum_{i=1}^n \log(1 - x_i) \right)$$

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Distribution of the Test Statistics

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Observed Test Statistics

$$2 \times (\log L(\hat{\theta}) - \log L(\theta_0)) = 1.88 \quad p\text{-value} = 0.391$$

Likelihood ratio Test

Likelihood ratio Test Statistics

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