

Example: Let X_1, \dots, X_n be iid r.v. distributed as continuous uniform distribution on $[0, \theta]$. The probability distribution function of X_i for each i is:

$$f(x|\theta) = \begin{cases} \theta^{-1}, & 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

Consider $T = \max_i X_i$ an estimator of θ , is T an unbiased estimator for θ ?

Uniform Distribution

Is $T = \max_i X_i$ an unbiased estimator of θ ?

We need to compute $E(T)$, thus we need distribution for T

Let F be the cumulative distribution function for T ,

$$F_T(y) = P(T \leq y) = P(\max_i X_i \leq y)$$

$$P(\max_i X_i \leq y) = P(X_1 \leq y, \dots, X_n \leq y) = P(X \leq y)^n = \left(\frac{y}{\theta}\right)^n$$

$$f_T(t) = \frac{\partial F_T(y)}{\partial y} = n \left(\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}$$

$$E(T) = \int_0^\theta y f_T(y) dy = \int_0^\theta n \left(\frac{y}{\theta}\right)^n dy = \frac{n}{n+1} \theta$$

$$\text{Bias}(T) = E(T) - \theta = \frac{-1}{n+1} \theta$$

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Is $T = \max_i X_i$ a consistent estimator of θ ?

$$f_T(t) = n \left(\frac{y}{\theta} \right)^{n-1} \frac{1}{\theta}$$

$$E(T^2) = \int y^2 f_T(y) dy = \int n \frac{y^{(n+1)}}{\theta^n} dy = \frac{n}{n+2} \theta^2$$

$$\text{Var}(T) = E(T^2) - (E(T))^2 = \frac{n}{n+2} \theta^2 - \left(\frac{n}{n+1} \theta \right)^2$$

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$$\text{Bias}(T) = E(T) - \theta = \frac{-1}{n+1}\theta$$

$$\text{Var}(T) = \frac{n}{(n+2)(n+1)^2}\theta^2$$

$$\text{MSE}(T) = \left(\frac{n}{(n+2)(n+1)^2} \right) \theta^2 + \left(\frac{-1}{n+1} \right)^2 \theta^2$$

$$\text{MSE}(T) = \left(\frac{2\theta^2}{(n+2)(n+1)} \right)$$

$$\lim_{n \rightarrow \infty} \text{MSE}(T_n) = 0$$

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Two estimators for θ

- $\hat{\theta}_{MOM}$
- $\hat{\theta}_{MLE}$

Which estimator is *the best* one?

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$$E(X) = \int xf(x|\theta)dx = \int_0^\theta \frac{x}{\theta} dx = \left[\frac{x^2}{2\theta} \right]_0^\theta = \frac{\theta}{2}$$

$$\bar{X} = \frac{\theta}{2}$$

$$\hat{\theta}_{MOM} = 2\bar{X}$$

Uniform Distribution: $\hat{\theta}_{MOM}$

Is $\hat{\theta}_{MOM}$ an unbiased and a consistent estimator of θ

$$E\left(\hat{\theta}_{MOM}\right) = \theta$$

$$Var\left(\hat{\theta}_{MOM}\right) = 4Var\left(\bar{X}\right) = 4\frac{\theta^2}{12n} = \frac{\theta^2}{3n}$$

$$MSE\left(\hat{\theta}_{MOM}\right) = \frac{\theta^2}{3n}$$

$$\lim_{n \rightarrow \infty} MSE(\hat{\theta}_{MLE}) = 0$$

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Comparison between two estimators: $\hat{\theta}_{MLE}$ and $\hat{\theta}_{MOM}$

Which estimator is *the best* one?

$$\begin{aligned}MSE\left(\hat{\theta}_{MLE}\right) &\leq Var\left(\hat{\theta}_{MOM}\right) \\ \frac{1}{(n+1)(n+2)}\theta^2 &\leq \frac{\theta^2}{3n} \\ \frac{2}{(n+1)(n+2)} &\leq \frac{1}{3n} \\ \frac{2}{(n+1)(n+2)} &\leq \frac{1}{3n} \\ \frac{n}{(n+1)(n+2)} &\leq \frac{2}{3}\end{aligned}$$

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Let (X_1, \dots, X_n) be independent identically distributed random variables with p.d.f.

$$f(x) = \theta^2 x \exp(-\theta x) \quad x > 0$$

Is $T(X_1, \dots, X_n) = 1/X_1$ an unbiased estimator of θ ?

Let (X_1, \dots, X_n) be independent identically distributed random variables with p.d.f.

$$f(x) = \theta^2 x \exp(-\theta x) \quad x > 0$$

$$E(T(X_1, \dots, X_n)) = \int_{(x_1, x_2, \dots, x_n)} \frac{1}{x_1} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$$

$$E(T(X_1, \dots, X_n)) = \int_0^\infty \frac{1}{x_1} \theta^2 x_1 \exp(-\theta x_1) dx_1$$

$$E\left(\frac{1}{X_1}\right) = \theta$$

Let (X_1, \dots, X_n) be independent identically distributed random variables with $E(X) = \mu$, $Var(X) = \sigma^2$, Are the following estimators unbiased estimator for σ^2 ?

$$T_1(X_1, \dots, X_n) = \frac{(X_1 - X_2)^2}{2}$$

$$T_2(X_1, \dots, X_n) = \frac{(X_1 + X_2)^2}{2} - X_1 X_2$$

Let (X_1, \dots, X_n) be a random sample of i.i.d. random variables with expected value μ and variance σ^2 . Consider the following estimator of μ :

$$T_n(a) = a \times X_n + (1 - a) \times \bar{X}_{n-1}$$

where X_n is the n -th observed random variable and \bar{X}_{n-1} is the sample mean based on $n - 1$ observations.

- 1 Find value of a such that $T_n(a)$ is an unbiased estimator for μ
- 2 Find value of a^* such that $T_n(a^*)$ is the most efficient estimator for μ within the class $T_n(a)$?
- 3 Define concept of efficiency

Example:

Let X be distributed as a Poisson;

$$f(x; \lambda) = \frac{\lambda^x \exp(-\lambda)}{x!}$$

Compare the following estimator for $\exp(-\lambda)$:

$$T_1 = \exp(-\bar{X}) \quad T_2 = \frac{\sum_{i=1}^n I(X_i = 0)}{n}$$

Poisson distribution

